Lie derivatives, tensors and forms

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Linear maps and tensors

The purpose of these notes is to give conceptual proofs of a number of results on Lie derivatives of tensor fields and differential forms. We start with some remarks on the effect of linear maps on tensors. In what follows, U, V, W will be finite dimensional real vector spaces.

A linear map $A : V \to W$ induces a linear map

$$A_* = \otimes^k A : \otimes^k V \to \otimes^k W.$$

Indeed, let $a : V^k \to \otimes^k W$ be the multilinear map given by $a(v_1, \ldots, v_k) \mapsto Av_1 \otimes \cdots Av_k$. Then A_* is just the map $\bar{a}$ from the universal property of the tensor product. In particular, on elementary tensors A_* is given by

$$A_*(v_1 \otimes \cdots \otimes v_k) = Av_1 \otimes \cdots \otimes Av_k.$$

On the other hand, A also induces the adjoint linear map $A^* : W^* \to V^*$, $\eta \mapsto \eta \circ A$. This map in turn induces a linear map

$$A^* = \otimes^k (A^*) : \otimes^k W^* \to \otimes^k V^*.$$

Note that the assignment $A \mapsto A_*$ preserves the direction of arrows, whereas $A \mapsto A^*$ reverses directions. Therefore, it is in general impossible to define a natural induced map between the spaces of mixed tensors of type (r, s), $V_{r,s}$ and $W_{r,s}$. Here we have used the notation of Warner's book.

We recall that there exists a natural isomorphism

$$V_{0,s} = \otimes^s V^* \simeq M_s(V),$$

where $M_s(V)$ denotes the space of k-multilinear forms on V. Via this isomorphism, a tensor of the form $\xi_1 \otimes \cdots \otimes \xi_s$ corresponds with the multilinear form

$$(v_1, \ldots, v_s) \mapsto \xi_1(v_1) \cdots \xi_s(v_s).$$

From this we see that the induced map $A^* : V_{0,s} \to W_{s,0}$ corresponds with the map $M_s(W) \to M_s(V)$, $\mu \mapsto \mu \circ A^s$, which is the genuine pull-back of multi-linear forms by A.

As in Warner's book, let $C(V)$ denote the (graded) tensor algebra $\oplus_{k \geq 0} V_{k,0}$, where $V_{0,0} := \mathbb{R}$. Then a linear map $A : V \to W$ induces a linear map $A_* : C(V) \to C(W)$ which is readily seen to be a homomorphism of graded algebras. Also, it is clear that A_* maps $I(V)$ into $I(W)$, hence induces an algebra homomorphism $A_* : \wedge V \to \wedge W$.

Similarly, the adjoint $A^* : W^* \to V^*$ induces an algebra homomorphism

$$(A^*)_* : \wedge W^* \to \wedge V^*,$$

which we briefly denote by A^*. Again, this is a homomorphism of graded algebras.

We recall that we introduced a particular linear isomorphism

$$\wedge^k V^* \simeq A_k(V).$$

Under this isomorphism, an element $\xi_1 \wedge \cdots \wedge \xi_k \in \wedge^k V$ corresponds with the alternating k-form given by

$$(v_1, \ldots, v_k) \mapsto \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma) \xi_{\sigma 1}(v_1) \cdots \xi_{\sigma k}(v_k) = \det(\xi_i(v_j)). \tag{1}$$

Identifying the elements of $\wedge^k V^*$ with alternating k-forms in this fashion, we see that A^* becomes the ordinary pull-back map

$$A_k(W) \to A_k(V), \ \ \mu \mapsto \mu \circ A^k.$$

As $A_k(V) \subset M_k(V)$, we may use the above isomorphisms to identify $\wedge^k V^*$ with a subspace of $V_{0,k}$. From (1) we then see that

$$\xi_1 \wedge \cdots \wedge \xi_k = \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma) \xi_{\sigma 1} \otimes \cdots \otimes \xi_{\sigma k}.$$

Finally, if A is a linear isomorphism of V onto W, we do have an induced map

$$A_* := \otimes^r A \otimes \otimes^s A^{-1*} : V_{r,s} \to W_{r,s},$$

In this situation we agree to also write

$$A^* := A_*^{-1} = \otimes^r A^{-1} \otimes \otimes^s A^* : W_{r,s} \to V_{r,s}.$$

For later purposes, we need the notion of contraction of tensors. First of all, the natural pairing $V \times V^* \to \mathbb{R}$ corresponds to a linear map $V \otimes V^* \to \mathbb{R}$, called contraction, and denoted by $\mathcal{C}$. Note that

$$\mathcal{C}(v \otimes \xi) = \xi(v).$$

If $A : V \to W$ is a linear isomorphism, then

$$\mathcal{C}_V \circ A^* = \mathcal{C}_W.$$

This is readily seen from

$$\begin{aligned} \mathcal{C}_V \circ A^*(w \otimes \eta) &= \mathcal{C}_V(A^{-1}w \otimes A^*\eta) \\ &= A * \eta(A^{-1}w) = \eta(AA^{-1}w) \\ &= \mathcal{C}_W(w \otimes \eta). \end{aligned}$$

More generally, if $r, s \geq 1$ and $1 \leq i \leq r$ and $1 \leq j \leq s$, we may a define a contraction $C_{i,j} : V_{r,s} \to V_{r-1,s-1}$ on the i-th contravariant slot and the j-th covariant slot. More precisely, $C_{i,j}$ is the linear map induced by the multi-linear map

$$(v_1, \ldots, v_r, \xi_1 \ldots \xi_s) \mapsto$$

$$\mathcal{C}(v_i \otimes \xi_j) v_1 \otimes \cdots \otimes \widehat{v_i} \otimes \cdots \otimes v_r \otimes \xi_1 \otimes \cdots \otimes \widehat{\xi_j} \otimes \cdots \otimes \xi_s.$$

The above formula is now readily seen to generalize to

$$A^* \circ \mathcal{C}_{W,i,j} = \mathcal{C}_{V,i,j} \circ A^*.$$

Note that $A^* = I$ on $V_{0,0} = \mathbb{R}$.

Maps and tensor fields

We will now describe the effect of maps on tensor fields. Let $\varphi : M \to N$ be a smooth map between manifolds. Then for every $m \in M$ we have an induced linear map $d_m\varphi : T_m M \to T_{\varphi(m)} N$ which in turn induces an algebra homomorphism

$$(d_m\varphi)^* : \wedge T^*_{\varphi(m)} N \to \wedge T^*_m M.$$

Accordingly, we have the induced map

$$\varphi^* : E(N) \to E(M)$$

given by

$$(\varphi^*\omega)_m = (d_m\varphi)^* \omega_{\varphi(m)}, \qquad (\omega \in E(N)).$$

As the wedge product of forms is defined pointwise, the map φ^* thus defined is a homomorphism of graded algebras.

Note that $E_0(M) \simeq \Gamma^\infty(\mathbb{R}) = C^\infty(M)$. Moreover, for $f \in C^\infty(M)$ the function $\varphi^* f$ is given by the usual pull-back $f \circ \varphi$.

Similar remarks can be made for the spaces of covariant tensor fields: a smooth map $\varphi : M \to N$ naturally induces a linear map

$$\varphi^* : \Gamma T_{0,s} N \to \Gamma T_{0,s} M.$$

By the identifications of the previous sections, we have a natural embedding of vector bundles

$$\wedge^k T^* M \hookrightarrow T_{0,k} M,$$

and accordingly an embedding

$$E_k(M) \hookrightarrow \Gamma T_{0,k} M,$$

identifying k-differential forms with alternating tensor fields. The embedding is compatible with the definitions of φ^* given above.

Lemma 1 *Let $\varphi : M \to N$ be a smooth map of manifolds. Then for every differential form $\omega \in E(N)$,*

$$\varphi^*(d\omega) = d\varphi^*\omega.$$

Proof: We will first check the formula for $\omega = f \in C^\infty(M)$. Then, for every $m \in M$,

$$d(\varphi^* f)_m = d_m(f \circ \varphi) = d_{\varphi(m)} f \circ d_m\varphi.$$

by the chain rule. The latter expression equals

$$(df)_{\varphi(m)} \circ d_m\varphi = (d_m\varphi)^*(df)_{\varphi(m)} = (\varphi^* df)_m.$$

The formula for f follows.

In general, let U be a coordinate patch in N and observe that the restriction of φ to $\varphi^{-1}(U)$ is a smooth map $\varphi^{-1}(U) \to U$. It suffices to prove the formula for a k-form $\omega \in E_k(U)$. By using local coordinates, we see that in such a patch $\omega \in E_k(U)$ can be expressed as a sum of k-forms of type

$$\lambda = f dg^1 \wedge \cdots \wedge dg^k,$$

with $f, g^j \in C^\infty(U)$. Hence, it suffices to prove the formula for such a k-form λ.

As d is an anti-derivation, and $d^2 = 0$,

$$d\lambda = df \wedge dg^1 \wedge \cdots \wedge dg^k$$

so that

$$\begin{aligned}
\varphi^*(d\lambda) &= \varphi^*(df) \wedge \varphi^*(dg^1) \wedge \cdots \wedge \varphi^*(dg^k) \\
&= d\varphi^* f \wedge d\varphi^* g^1 \wedge \cdots \wedge d\varphi^* g^k \\
&= d[(\varphi^* f) d\varphi^* g^1 \wedge \cdots \wedge d\varphi^* g^k] \\
&= d[(\varphi^* f) \varphi^* dg^1 \wedge \cdots \wedge \varphi^* dg^k] \\
&= d[\varphi^* \lambda].
\end{aligned}$$

Here we have been using that φ^* is an algebra homomorphism $E(U) \to E(\varphi^{-1}(U))$. □

Lie derivatives

If φ is a (local) diffeomorphism $M \to N$, we may define a pull-back map $\varphi^* : \Gamma T_{r,s} N \to \Gamma T_{r,s} M$ on mixed tensor fields as follows. For $T \in \Gamma T_{r,s} N$ we define

$$\varphi^*(T)_m := (d_m \varphi)^* T_{\varphi(m)}.$$

This definition facilitates the definition of Lie derivative of tensors with respect to a given smooth vector field.

Let $X \in \mathfrak{X}(M)$ be a smooth vector field. Then for every $m \in M$ we denote by $t \mapsto \varphi_X^t(m)$ the (maximal) integral curve for X with initial point m. The domain of this integral curve is an open interval $I_{X,m}$ containing 0. Let $T \in \Gamma T_{r,s} M$. Then we define the Lie derivative of T with respect to X by

$$(\mathcal{L}_X T)_m := \left.\frac{d}{dt}\right|_{t=0} [(\varphi_X^t)^* T]_m.$$

Here we note that φ_X^t is a diffeomorphism from a neighborhood of m onto a neighborhood of $\varphi_X^t(m)$. Accordingly, the expression

$$[(\varphi_X^t)^* T]_m$$

is a well-defined element of $(T_m M)_{r,s}$ which depends smoothly on t (in a neighborhood of 0). Accordingly, $(\mathcal{L}_X T)_m$ defines a tensor in $(T_m M)_{r,s}$. Moreover, by smoothness of the flow of the vector field X it follows that the section $\mathcal{L}_X T$ of the tensor bundle $T_{r,s} M$ thus defined is smooth. In other words, we have defined a linear map

$$\mathcal{L}_X : \Gamma T_{r,s} M \to \Gamma T_{r,s} M,$$

called the Lie derivative. In a similar way it is seen that the Lie derivative defines a linear map $\mathcal{L}_X : E_k(M) \to E_k(M)$.

Lemma 2 *Let $f \in C^\infty(M)$. Then $\mathcal{L}_X f = Xf$.*

Proof: By definition and application of the chain rule,

$$\begin{aligned}(\mathcal{L}_X f)(m) &= \left.\frac{d}{dt}\right|_{t=0} (\varphi_X^t)^* f(m) \\ &= \left.\frac{d}{dt}\right|_{t=0} f(\varphi_X^t(m)) \\ &= d_m f \frac{d}{dt}\varphi_X^t(m)|_{t=0} \\ &= d_m f X_m = (Xf)_m.\end{aligned}$$

□

We have the following Leibniz rule with respect to tensor products

Lemma 3 *Let $S \in \Gamma T_{r,s}M$ and $T \in \Gamma T_{u,v}M$. Then*

$$\mathcal{L}_X(S \otimes T) = \mathcal{L}_X S \otimes T + S \otimes \mathcal{L}_X T.$$

Proof: We note that

$$(\varphi_X^t)^*(S \otimes T)_m = ((\varphi_X^t)^* S)_m \otimes ((\varphi_X^t)^* T)_m.$$

Now differentiate at $t = 0$ and apply the lemma below. □

Lemma 4 *Let I be an open interval containing 0 and let $f : I^n \to M$ be a smooth map. Then*

$$\begin{aligned}&\left.\frac{d}{dt}\right|_{t=0} f(t, t, \ldots, t) = \\ &= \left.\frac{d}{dt}\right|_{t=0} f(t, 0, \ldots, 0) + \left.\frac{d}{dt}\right|_{t=0} f(0, t, \ldots, 0) + \left.\frac{d}{dt}\right|_{t=0} f(0, 0, \ldots, t).\end{aligned}$$

Proof: We will prove this for $n = 2$. The general case is proved similarly.

Consider the diagonal map $\delta : I \to I^2, t \mapsto (t, t)$. Then

$$\begin{aligned}\left.\frac{d}{dt}\right|_{t=0} f(t, t) &= \left.\frac{d}{dt}\right|_{t=0} f \circ \delta(t) \\ &= d_m f \cdot \delta'(0) = d_m f \cdot (1, 1) \\ &= d_m f \cdot (1, 0) + d_m f \cdot (0, 1) \\ &= \left.\frac{d}{dt}\right|_{t=0} f(t, 0) + \left.\frac{d}{dt}\right|_{t=0} f(0, t).\end{aligned}$$

□

Taking the Lie derivative commutes with contractions. More precisely, if $r \geq 1$, $s \geq 1$, and $1 \leq i \leq r$, $1 \leq j \leq s$, we may define a contraction map

$$\mathcal{C}_{i,j} : \Gamma T_{r,s} M \to \Gamma T_{r-1,s-1} M$$

by point-wise contraction:

$$(\mathcal{C}_{i,j} S)_m := \mathcal{C}_{T_m M,i,j}(S_m).$$

Let $\varphi : M \to N$ be diffeomorphism. Then it is readily seen that

$$\varphi^* \circ \mathcal{C}_{N,i,j} = \mathcal{C}_{M,i,j} \circ \varphi^* \qquad \text{on} \quad \Gamma T_{r,s} N.$$

Lemma 5 *Let X be a smooth vector field on M. Then*

$$\mathcal{L}_X \circ \mathcal{C}_{i,j} = \mathcal{C}_{i,j} \circ \mathcal{L}_X$$

on $\Gamma T_{r,s} M$.

Proof: Let $S \in \Gamma T_{r,s} M$ and $m \in M$. Then

$$\begin{aligned}
(\mathcal{L}_X \circ \mathcal{C}_{i,j} S)_m &= \frac{d}{dt}\Big|_{t=0} ((\varphi^t)^* \mathcal{C}_{i,j} S)_m \\
&= \frac{d}{dt}\Big|_{t=0} \mathcal{C}_{T_m M,i,j} (\varphi^t)^* S)_m \\
&= \mathcal{C}_{T_m M,i,j} \frac{d}{dt}\Big|_{t=0} (\varphi^t)^* S)_m \\
&= (\mathcal{C}_{i,j} \mathcal{L}_X S)_m.
\end{aligned}$$

Here the interchange of d/dt and $\mathcal{C}_{T_m M,i,j}$ is allowed by linearity of the latter map. □

Lemma 6 *The Lie derivative $\mathcal{L}_X$ defines a derivation of order 0 of the graded algebra $E(M)$ which commutes with the exterior differentiation d.*

Proof: We observed already that $\mathcal{L}_X$ maps the subspace $E_k(M)$ to itself, for each k. Let $\omega, \eta \in E(M)$. Then, for $m \in M$,

$$\begin{aligned}
\mathcal{L}_X(\omega \wedge \eta)_m &= \frac{d}{dt}\Big|_{t=0} (\varphi^t)^* (\omega \wedge \nu)_m \\
&= \frac{d}{dt}\Big|_{t=0} [((\varphi^t)^* \omega)_m \wedge ((\varphi^t)^* \nu)_m].
\end{aligned}$$

Now apply Lemma 4 to see that $\mathcal{L}_X$ is a derivation.

Let now $\omega \in E(M)$. Then we must show that $\mathcal{L}_X d\omega = d\mathcal{L}_X \omega$. We first assume that $\omega = f \in C^\infty(M)$. Fix $m \in M$ and $Y_m \in T_m M$ and extend Y_m to a smooth vector field on M. Then it suffices to show that

$$(\mathcal{L}_X df)_m Y_m = d(\mathcal{L}_X f)_m Y_m.$$

The expression on the left-hand side equals

$$\begin{aligned} \left.\frac{d}{dt}\right|_{t=0} (\varphi^{t*} df)_m Y_m &= \left.\frac{d}{dt}\right|_{t=0} (d\varphi^{t*} f)_m Y_m \\ &= \left.\frac{\partial}{\partial t}\right|_{t=0} \left.\frac{\partial}{\partial s}\right|_{s=0} \varphi^{t*} f(\psi^s(m)), \end{aligned}$$

where $\psi^s := \varphi^s_Y$. In the last expression the derivatives with respect to s and t may be interchanged. From this we see that the expression equals

$$\left.\frac{d}{ds}\right|_{s=0} (\mathcal{L}_X f)(\psi^s(m)) = (d\mathcal{L}_X f)_m Y_m,$$

and the result for $\omega = f$ follows.

For general ω we may now obtain the result by applying the method of the proof of Lemma 1. □

Lemma 7 *Let X, Y be smooth vector fields on M. Then $\mathcal{L}_X Y = [X, Y]$.*

Proof: Let $f \in C^\infty(M)$. Then $Yf = df(Y)$ equals the contraction $\mathcal{C}_{1,1}$ of $Y \otimes df$. It follows that

$$\begin{aligned} XYf = \mathcal{L}_X(Yf) &= \mathcal{L}_X \mathcal{C}_{1,1}(Y \otimes df) \\ &= \mathcal{C}_{1,1} \mathcal{L}_X (Y \otimes df) \\ &= \mathcal{C}_{1,1}[(\mathcal{L}_X Y) \otimes df + Y \otimes d(\mathcal{L}_X f)] \\ &= (\mathcal{L}_X Y) f + Y(Xf). \end{aligned}$$

The result follows. □

Lemma 8 (Cartan's formula) *Let X be a smooth vector field on M. Then on $E(M)$,*

$$\mathcal{L}_X = i(X) \circ d + d \circ i(X).$$

Proof: As in Warner, it is seen that the right-hand side of the expression is a derivation of $E(X)$ of order 0, which commutes with d. The same was seen to be true for the operator on the left-hand side. It follows that the equality needs only be checked when applied to a function $f \in C^\infty(M)$. Now $i(X)f = 0$ and

$$\mathcal{L}_X f(m) = Xf(m) = d_m f \cdot X_m = (i(X)df)_m$$

so that the result follows. □

Lemma 9 *Let $\omega \in E_k(M)$ and let $X_0, \dots, X_k$ be smooth vector fields on M. Then*

$$X_0[\omega(X_1, \dots, X_k)] = \mathcal{L}_{X_0}\omega(X_1, \dots, X_k) + \sum_{j=1}^{k} \omega(X_1, \dots, [X_0, X_j], \dots X_k).$$

Proof: Viewing ω as an alternating tensor field in $\Gamma T_{0,k} M$, we observe that

$$\omega(X_1, \dots, X_k) = \mathcal{C}_{1,1}\mathcal{C}_{1,1} \cdots \mathcal{C}_{1,1} \omega \otimes X_1 \otimes \cdots \otimes X_k.$$

The result now follows by applying Lemmas 2, 5 and 7. □

Lemma 10 *Let* $\omega \in E_k(M)$ *and let* $X_0, \dots, X_k$ *be smooth vector fields on* M. *Then*

$$d\omega(X_0, \dots, X_k) = \sum_{j=0}^{k} (-1)^j X_j \omega(X_0, \dots, \widehat{X_j}, \dots, X_k)$$

$$+ \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_0, \dots, \widehat{X_i}, \dots, \widehat{X_j}, \dots, X_k).$$

Proof: First of all, if $k = 0$, then ω is a function and the equation is obvious. Note that in this situation the second sum on the right-hand side equals zero. We now proceed by induction. Thus, let $k > 0$ and assume the result has been established for strictly smaller values of k. Let $\omega \in E_k(M)$ and let $X_0, \dots, X_k$ be smooth vector fields. Then

$$\begin{aligned} d\omega(X_0, \dots, X_k) &= ([i(X_0) \circ d]\omega)(X_1, \dots, X_k)) \\ &= ([\mathcal{L}_{X_0} - d \circ i(X_0)]\omega)(X_1, \dots, X_k) \\ &= X_0 \omega(X_1, \dots, X_k) - \sum_{j=1}^{k} \omega(X_1, \dots, [X_0, X_j], \dots X_k) + \\ &\qquad -[d(i(X_0)\omega)](X_1, \dots, X_k). \end{aligned}$$

Now apply the induction hypothesis to $i(X_0)\omega$ to complete the proof. □