Geometry and analysis of SL(2,R)

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1 The group $\mathrm{SL}(2,\mathbb{R})$.

In this section we will investigate some basic properties of the group $\mathrm{SL}(2,\mathbb{R})$, which is the group of real 2×2 matrices of determinant 1. Thus, $\mathrm{SL}(2,\mathbb{R})$ consists of the matrices $g = g_{a,b,c,d}$ of the form

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{1}$$

with $\det(g) = ad - bc = 1$. It is readily checked that $\mathrm{SL}(2,\mathbb{R})$, equipped with matrix multiplication, is a group. Furthermore, the set $\mathrm{M}(2,\mathbb{R})$ of all 2×2 matrices with real entries, equipped with entry-wise addition and scalar multiplication is a real linear space. Via the entries we may identify this linear space with $\mathbb{R}^4$.

The determinant map $\det : \mathrm{M}(2,\mathbb{R}) \to \mathbb{R}$ is continuous. Moreover, $\mathrm{SL}(2,\mathbb{R})$ equals preimage $\det^{-1}(\{1\})$ in $\mathrm{M}(2,\mathbb{R})$ of the closed subset $\{1\}$ of $\mathbb{R}$. It follows from these remarks that $\mathrm{SL}(2,\mathbb{R})$ is a closed subset of $\mathrm{M}(2,\mathbb{R})$.

We can strengthen these statements as follows. The determinant map $\det : \mathrm{M}(2,\mathbb{R}) \to \mathbb{R}$ is C^∞ differentiable, and its total derivative at a matrix $g \in \mathrm{SL}(2,\mathbb{R})$ is given by

$$D\det(g)X = \frac{d}{dt}\Big|_{t=0} \det(g+tX) = \frac{d}{dt}\Big|_{t=0} \det(g)\det(I + tg^{-1}X) = \det(g)\,\mathrm{tr}(g^{-1}X).$$

In particular it follows that $D\det(g)$ is a surjective linear map $\mathrm{M}(2,\mathbb{R}) \to \mathbb{R}$ for every $g \in \mathrm{SL}(2,\mathbb{R})$. By application of the submersion theorem it thus follows that $\mathrm{SL}(2,\mathbb{R})$ is 3-dimensional submanifold of $\mathrm{M}(2,\mathbb{R})$. Furthermore, the group multiplication map $m : (x,y) \mapsto xy$ is the restriction of a bilinear map $\mathrm{M}(2,\mathbb{R}) \times \mathrm{M}(2,\mathbb{R}) \to \mathrm{M}(2,\mathbb{R})$ hence C^∞. The inversion map $i : \mathrm{SL}(2,\mathbb{R}) \mapsto \mathrm{SL}(2,\mathbb{R}),\ x \mapsto x^{-1}$ is also C^∞, since it is the restriction of the linear endomorphism of $\mathrm{M}(2,\mathbb{R})$ given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

This makes that $\mathrm{SL}(2,\mathbb{R})$ is a Lie group.

Definition 1.1 A Lie group is a group G equipped with a manifold structure such that the group operation $G \times G \to G, (x,y) \mapsto xy$ and the inversion map $G \to G, x \mapsto x^{-1}$ are smooth maps.

In a similar fashion, $\mathrm{SL}(2,\mathbb{C})$ is defined to be the set of matrices $g = g_{a,b,c,d}$ as in (1), with $a, b, c, d \in \mathbb{C}$ such that $\det g = 1$. By a similar argumentation as above, but with differentiation replaced by complex differentiation, it follows that $\mathrm{SL}(2,\mathbb{C})$ is a three dimensional complex submanifold of $\mathrm{M}(2,\mathbb{C})$, the complex linear space of complex 2×2-matrices. Matrix multiplication induces a group structure on $\mathrm{SL}(2,\mathbb{C})$ for which it becomes a complex Lie group, i.e., a group with a complex manifold structure such that the group operation and the inversion map are complex differentiable maps. We note that $\mathrm{SL}(2,\mathbb{R})$ is a subgroup and submanifold of $\mathrm{SL}(2,\mathbb{C})$ (viewed as a real Lie group).

2 Fractional linear transformations

The group $\mathrm{SL}(2,\mathbb{C})$ acts on $\mathbb{C}^2$ by matrix multiplication,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} az_1 + bz_2 \\ cz_1 + dz_2 \end{pmatrix}$$

Clearly, the complement $\mathbb{C}^2 \setminus \{0\}$ of the origin is an invariant subset for this action. Let $\mathbb{P}^1(\mathbb{C})$ denote one dimensional complex projective space and let $\pi : \mathbb{C}^2 \setminus \{0\} \to \mathbb{P}^1(\mathbb{C})$ be the natural map $z = (z_1, z_2) \mapsto \mathbb{C}z$. We will write $[z_1 : z_2]$ for the line $\mathbb{C}z$. The action of $\mathrm{SL}(2,\mathbb{C})$ on $\mathbb{C}^2 \setminus \{0\}$ preserves the fibers of π, hence induces the action on $\mathbb{P}^1(\mathbb{C})$ given by

$$g[z_1 : z_2] = [az_1 + bz_2 : cz_1 + dz_2].$$

Let $\varphi : \mathbb{C} \mapsto \mathbb{P}^1(\mathbb{C})$ be the embedding given by $\varphi(z) = [z : 1]$. It is easy to see that the image of φ has a complement consisting of the single point $[1 : 0]$. Writing $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ (disjoint union), we see that φ has a unique extension to a bijection $\widehat{\varphi} : \widehat{\mathbb{C}} \mapsto \mathbb{P}^1(\mathbb{C})$; it maps ∞ to $[1 : 0]$. We equip $\widehat{\mathbb{C}}$ with the structure of complex manifold by requiring that $\widehat{\varphi}$ is a bi-holomorphic isomorphism. The resulting manifold $\widehat{\mathbb{C}}$ is called the Riemann sphere. Under $\widehat{\varphi}$, the action of $\mathrm{SL}(2,\mathbb{C})$ on $\mathbb{P}^1(\mathbb{C})$ transfers to an action on $\widehat{\mathbb{C}}$ by bi-holomorphic transformations.

Lemma 2.1 *Let $g = g_{a,b,c,d} \in \mathrm{SL}(2,\mathbb{C})$ be as in (1). Then the biholomorphic transformation $T_g : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, $z \mapsto g \cdot z$ is given by the following rules.*

(a) *If $z \in \mathbb{C}$ and $cz + d \neq 0$, then*
$$T_g(z) = \frac{az + b}{cz + d}.$$

(b) *If $z \in \mathbb{C},\ cz + d = 0$ then $T_g(z) = \infty$.*

(c) *$T_g(\infty) = c^{-1}a$; in particular, if $c = 0$, then $T_g(\infty) = \infty$.*

Proof. Let $z \in \mathbb{C}$; then

$$\widehat{\varphi}(T_g(z)) = \widehat{\varphi}(g \cdot z) = g\widehat{\varphi}(z) = g[z : 1] = [az + b : cz + d]. \tag{2}$$

Thus, if $cz + d \neq 0$ then the expression at the extreme right side of (2) equals

$$[\frac{az+b}{cz+d} : 1] = \widehat{\varphi}(\frac{az+b}{cz+d}).$$

Since $\widehat{\varphi}$ is bijective, this implies (a).

If $cz + d = 0$ then the expression at the extreme right side of (2) equals

$$[az + b : 0] = [1 : 0] = \widehat{\varphi}(\infty)$$

and (b) follows. Finally,

$$\widehat{\varphi}(T_g(\infty)) = \widehat{\varphi}(g \cdot \infty) = g[1 : 0] = [a : c] = [c^{-1}a : 1] = \widehat{\varphi}(c^{-1}a)$$

and the final assertion follows. In particular, since $(a, c) \neq (0, 0)$, $c = 0$ implies that $g \cdot \infty = \infty$. □

The biholomorphic transformations T_g, for $g \in \mathrm{SL}(2, \mathbb{C})$, are generally known as fractional linear transformations.

Exercise 2.2 Show that the fractional linear transformations T_g for $g = g_{a,b,c,d}$, $ad - bc \neq 0$ form a group $\mathcal{G}$ of bijective transformations of $\widehat{\mathbb{C}}$. Show that the map $\mathrm{SL}(2, \mathbb{C}) \to \mathcal{G}$ given by $g \mapsto T_g$ is a surjective group homomorphism onto $\mathcal{G}$ with kernel $\{\pm I\}$. Determine the kernel of the similar homomorphism $\mathrm{GL}(2, \mathbb{C}) \to \mathcal{G}$. *Remark:* The group $\mathcal{G}$ is also denoted by $\mathrm{PGL}(2, \mathbb{C})$.

Definition 2.3 By a circle in $\widehat{\mathbb{C}}$ we mean a subset which is either a circle C of the real Euclidean space $\mathbb{C} \simeq \mathbb{R}^2$ or a set of the form $L \cup \{\infty\}$ with L an affine real line in $\mathbb{C} \simeq \mathbb{R}^2$.

We will show that the fractional linear transformations of $\widehat{\mathbb{C}}$ preserves the collection its circles. Before doing so, we need a suitable description of them. For this we use the standard sesquilinear inner product $\langle\, \cdot\,, \cdot\, \rangle$ on $\mathbb{C}^2$ given by

$$\langle\, (z_1, z_2), (w_1, w_2)\, \rangle = z_1\overline{w}_1 + z_2\overline{w}_2.$$

Let $\alpha \in \mathbb{C}$ and $r > 0$, then the circle $C_{\alpha,r}$ in $\mathbb{C}$ of center α and radius r is given by the equation

$$|z - \alpha|^2 = r^2.$$

By (sesquilinear) homogenization this equation may be written as

$$(z_1 - \alpha z_2)\overline{(z_1 - \alpha z_2)} = rz_2\overline{rz_2}$$

with the requirement that $(z_1, z_2) = (z, 1)$. The above homogeneous form may be rewritten as

$$\langle\, z, H_{\alpha,r} z\, \rangle = 0, \qquad z = (z_1, z_2), \tag{3}$$

where

$$H_{\alpha,r} = \begin{pmatrix} 1 & -\alpha \\ 0 & 0 \end{pmatrix}^* \begin{pmatrix} 1 & -\alpha \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & r^2 \end{pmatrix} = \begin{pmatrix} 1 & -\alpha \\ -\bar{\alpha} & |\alpha|^2 - r^2 \end{pmatrix}.$$

Here the star indicates that the Hermitian conjugate of the matrix is taken. Obviously, $\det H_{\alpha,r} = -r^2$.

Clearly, the equation (3) determines a subset of $\mathbb{P}^1(\mathbb{C})\setminus\{[0:1]\}$. The corresponding image in $\widehat{\mathbb{C}}$ equals precisely $C_{\alpha,r}$. This observation motivates the following definition. For H a Hermitian matrix with $\det H < 0$ we define the subset $C_H \subset \mathbb{P}^1(\mathbb{C})$ by

$$[z_1 : z_2] \in C_H \iff \langle\, z, Hz \,\rangle = 0.$$

Furthermore we define $\widehat{C}_H := \widehat{\varphi}^{-1}(C_H)$.

Lemma 2.4 *The collection of circles on* $\widehat{\mathbb{C}}$ *is the collection of subsets of the form* $\widehat{C}_H$, *for* $H \in M(2,\mathbb{C})$ *a Hermitean matrix with* $\det H < 0$. *Furthermore,*

$$\widehat{C}_H \ni \infty \iff H_{11} = 0.$$

Proof. We have already shown that any circle not containing ∞ is of the form C_H with $H_{11} = 1$. Conversely, let $H_{11} \neq 0$. Then $H' := H_{11}^{-1}H$ is Hermitean of negative determinant with $H'_{11} = 1$, and $C_H = C_{H'}$. Hence C_H is a circle not containing ∞.

Let now C be a circle in $\widehat{\mathbb{C}}$ containing ∞. Then $C = L \cup \{\infty\}$ with L an affine real line in $\mathbb{C} \simeq \mathbb{R}^2$. There exists $\zeta \in \mathbb{C}$ with $|\zeta| = 1$ and $s \in \mathbb{R}$ such that $L = \zeta(i\mathbb{R} + s)$. If $z \in \mathbb{C}$ then

$$z \in L \iff \bar{\zeta}z \in i\mathbb{R} + s \iff \mathrm{Re}(\bar{\zeta}z) = s \iff \bar{\zeta}z + \zeta\bar{z} = 2s.$$

Homogenization of the above equation leads to the equation

$$\bar{\zeta}z_1\bar{z}_2 + \zeta z_2\bar{z}_1 - 2sz_1\bar{z}_2 = 0,$$

or, equivalently $[z_1 : z_2] \in C_{H_{\zeta,s}}$, where

$$H_{\zeta,s} = \begin{pmatrix} 0 & \bar{\zeta} \\ \zeta & -2s \end{pmatrix}.$$

The point $[1:0]$ belongs to $C_{H_{\zeta,s}}$, so that

$$\widehat{C}_{H_{\zeta,s}} = L \cup \{\infty\}.$$

Thus, every circle in $\widehat{C}$ containing ∞ is of the required form. Finally, let H be Hermitian with $H_{11} = 0$ and $\det H < 0$. Then $|H_{12}|^2 = H_{12}H_{21} = -\det(H) > 0$, so that $H' = |H_{12}|^{-1}H$ is of the form

$$H' = H_{\zeta,s},$$

with $|\zeta| = 1$ and $s \in \mathbb{R}$. It follows that

$$\widehat{C}_H = \widehat{C}_{H'} = \widehat{C}_{H_{\zeta,s}}$$

which by the above equals the circle C in $\widehat{\mathbb{C}}$ containing ∞ and with $C \setminus \{\infty\} = \zeta(i\mathbb{R} + s)$. $\square$

Now that we have given a precise description of the set of circles on $\widehat{\mathbb{C}}$ in terms of linear algebra, we can prove the following result.

Lemma 2.5 *Let H be a Hermitean 2×2 matrix of negative determinant. Then for each $g \in \mathrm{SL}(2, \mathbb{C})$,*

$$g \cdot C_H = C_{g^{-1*}Hg^{-1}}.$$

In particular, for every $g \in \mathrm{SL}(2, \mathbb{C})$, the transformation T_g maps all circles of $\widehat{\mathbb{C}}$ to circles of $\widehat{\mathbb{C}}$.

Proof. Let H be as asserted. Then the image of $C_H \subset \mathbb{P}^1(\mathbb{C})$ under g consists of the points $[z_1 : z_2] \in \mathbb{P}^1(\mathbb{C})$ such that

$$g^{-1}[z_1 : z_2] \in C_H,$$

or, equivalently,

$$0 = \langle\, g^{-1}z, Hg^{-1}z \,\rangle.$$

As the latter expression may be rewritten as $\langle\, z, g^{-1*}Hg^{-1}z \,\rangle = 0$ we see that

$$g \cdot C_H = C_{H'},$$

where $H' = g^{-1*}Hg$. It is readily verified that H' is Hermitian and that $\det H' = \det H < 0$. This establishes the first assertion. By applying $\widehat{\varphi}^{-1}$ we obtain

$$T_g(\widehat{C}_H) = \widehat{C}_{g^{-1*}Hg^{-1}},$$

and the final assertion follows. $\square$

Exercise 2.6 Let $\mathcal{C}$ denote the collection of circles on $\widehat{\mathbb{C}}$. Let $\mathcal{H}$ be the collection of Hermitian 2×2 matrices of determinant -1.

(a) Show that the map $\mathcal{H} \to \mathcal{C}$, $H \mapsto \widehat{C}_H$ is surjective.

(b) Show that the action of $\mathrm{SL}(2, \mathbb{C})$ on $\mathcal{C}$ given by $(g, C) \mapsto T_g(C)$ is transitive.

(c) Show that the action of $\mathrm{SL}(2, \mathbb{C})$ on $\mathcal{H}$ given by $(g, H) \mapsto g^{-1*}Hg^{-1}$ is transitive.

(d) Show that the stabilizer of $\widehat{\mathbb{R}}$ in $\mathrm{SL}(2, \mathbb{C})$, denoted

$$\mathrm{SL}(2, \mathbb{R})_{\widehat{\mathbb{R}}} = \{g \in \mathrm{SL}(2, \mathbb{R}) \mid T_g(\widehat{\mathbb{R}}) = \widehat{\mathbb{R}}\}$$

is given by

$$\mathrm{SL}(2, \mathbb{R})_{\widehat{\mathbb{R}}} = \mathrm{SL}(2, \mathbb{R}) \cup \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \mathrm{SL}(2, \mathbb{R}).$$

(e) Show that the stabilizer in $\mathrm{SL}(2,\mathbb{C})$ of the matrix

$$M := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

equals $\mathrm{SL}(2,\mathbb{R})$, i.e., show that

$$\mathrm{SL}(2,\mathbb{R}) = \{g \in \mathrm{SL}(2,\mathbb{C}) \mid g^* M g = M\}.$$

(f) Show that the map of (a) is 2 - 1. More precisely, show that for $H, H' \in \mathcal{H}$ we have $\widehat{C}_H = \widehat{C}_{H'} \iff H = \pm H'$.

3 Orbits for the action of $\mathrm{SL}(2,\mathbb{C})$

We will now investigate the action of $\mathrm{SL}(2,\mathbb{C})$ on $\mathbb{C}$ in some further detail. Recall that $\mathrm{SU}(2)$ is the group of unitary matrices in $\mathrm{SL}(2,\mathbb{C})$, i.e., $g \in \mathrm{SL}(2,\mathbb{C})$ such that $g^* = g^{-1}$. For $g = g_{a,b,c,d}$ the equation becomes $a = \bar{d}$ and $b = -\bar{c}$, hence $\mathrm{SU}(2)$ consists of the matrices

$$\begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix}, \qquad \alpha, \beta \in \mathbb{C},\ |\alpha|^2 + |\beta|^2 = 1.$$

For a given $\varphi \in \mathbb{R}$ we write

$$t_\varphi = \begin{pmatrix} e^{i\varphi} & 0 \\ 0 & e^{-i\varphi} \end{pmatrix}.$$

Then $T = \{t_\varphi \mid \varphi \in \mathbb{R}\}$ is a subgroup of $\mathrm{SU}(2)$. We denote by B the group of upper triangular matrices $g = g_{a,b,c,d}$, $c = 0$ with $\det(g) = ad = 1$, and by $\bar{B}$ the group of lower triangular matrices $g_{a,b,c,d}$, $b = 0$ with $\det(g) = ad = 1$. Then both B and $\bar{B}$ are subgroups of $\mathrm{SL}(2,\mathbb{C})$.

Lemma 3.1 *The actions of* $\mathrm{SU}(2)$ *and* $\mathrm{SL}(2,\mathbb{C})$ *on* $\widehat{\mathbb{C}}$ *are transitive. The stabilizer of* 0 *in* $\mathrm{SL}(2,\mathbb{C})$ *equals* $\bar{B}$ *and the stabilizer of* 0 *in* $\mathrm{SU}(2)$ *equals* T. *The inclusion map* $\mathrm{SU}(2) \to \mathrm{SL}(2,\mathbb{C})$ *and the action map* $g \mapsto T_g(0)$ *induces bijections*

$$\mathrm{SU}(2)/T \simeq \mathrm{SL}(2,\mathbb{C})/\bar{B} \simeq \widehat{\mathbb{C}}.$$

Proof. For $\varphi \in \mathbb{R}$ we write

$$r_\varphi = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}.$$

Then $R = \{r_\varphi \mid \varphi \in \mathbb{R}\}$ is readily seen to be a subgroup of $\mathrm{SU}(2)$. Furthemore, $r_\varphi \cdot 0 = -\tan\varphi$, from which we see that $R \cdot 0 = \widehat{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$.

On the other hand, $t_\varphi \cdot z = e^{2i\varphi} z$, so that $Tz = \{w \in \mathbb{C} \mid |w| = |z|\}$. We thus see that $TR \cdot 0 = \widehat{\mathbb{C}}$.

Since $\mathrm{SL}(2,\mathbb{C})$ contains $\mathrm{SU}(2)$, the action of $\mathrm{SL}(2,\mathbb{C})$ on $\widehat{\mathbb{C}}$ is transitive as well. An element $g = g_{a,b,c,d} \in \mathrm{SL}(2,\mathbb{C})$ stabilizes 0 iff $b = 0$, or, equivalently $g \in \bar{B}$. It is readily seen that $B \cap \mathrm{SU}(2) = T$, so that T is the stabilizer of 0 in $\mathrm{SU}(2)$. It now readily follows that the sequence of maps $\mathrm{SU}(2) \to \mathrm{SL}(2,\mathbb{C}) \to \mathbb{C}$ induces the required sequence of bijections. $\square$

4 Orbits for the action of $\mathrm{SL}(2,\mathbb{R})$

To prepare for this section, we start with the following useful lemma, which uses that the matrices of $\mathrm{SL}(2,\mathbb{R})$ have real entries.

Lemma 4.1 *Let $z \in \mathbb{C}$ and $\mathrm{Im}(z) \neq 0$. Then for $g = g_{a,b,c,d} \in \mathrm{SL}(2,\mathbb{R})$, we have*

$$\mathrm{Im}(g \cdot z) = |cz + d|^{-2}\mathrm{Im}(z).$$

Proof. From $\mathrm{Im}\, z \neq 0$ it follows that $cz + d \neq 0$. Hence,

$$\begin{aligned} g \cdot z &= \frac{az+b}{cz+d} \\ &= \frac{(az+b)(c\bar{z}+d)}{|cz+d|^2} \\ &= \frac{(adz + bc\bar{z} + ac|z|^2 + bd)}{|cz+d|^2}. \end{aligned}$$

Taking imaginary parts, we find

$$\mathrm{Im}(g \cdot z) = \frac{(ad - bc)\mathrm{Im}(z)}{|cz+d|^2} = |cz + d|^{-2}\mathrm{Im}(z).$$

□

It follows from the above that the action of $\mathrm{SL}(2,\mathbb{R})$ is not transitive on $\widehat{\mathbb{C}}$. In fact, let H^+ denote the upper half plane in $\mathbb{C}$, consisting of $z \in \mathbb{C}$ such that $\mathrm{Im} z > 0$ and let H^- denote the lower half plane $-H^+$. Then it follows from the above lemma that both H^+ and H^- are invariant under $\mathrm{SL}(2,\mathbb{R})$.

Lemma 4.2 *The action of $\mathrm{SL}(2,\mathbb{R})$ on $\widehat{\mathbb{C}}$ has three orbits: the open orbits H^+ and H^- and the closed orbit $\widehat{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$ (a circle).*

Proof. We first observe that $g \cdot 0 \in \widehat{\mathbb{R}}$ for $g \in \mathrm{SL}(2,\mathbb{R})$. Since the rotation group R is a subgroup of $\mathrm{SL}(2,\mathbb{R})$ it follows that $\mathrm{SL}(2,\mathbb{R}) \cdot 0 \supset R \cdot 0 = \widehat{\mathbb{R}}$. We conclude that $\mathrm{SL}(2,\mathbb{R}) \cdot 0 = \widehat{\mathbb{R}}$. It follows that $H^+ \cup H^- = \widehat{\mathbb{C}} \setminus \widehat{\mathbb{R}}$ is invariant under the action of $\mathrm{SL}(2,\mathbb{R})$. We now observe that for $x \in \mathbb{R}$ the element

$$n_x = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

belongs to $\mathrm{SL}(2,\mathbb{R})$ and that $n_x \cdot w = w + x$ for all $w \in \mathbb{C}$ and $x \in \mathbb{R}$. Furthermore, for $t \in \mathbb{R}$ put

$$a_t := \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}.$$

Then $a_t \cdot w = e^{2t}w$ for all $w \in \mathbb{C}$ and $t \in \mathbb{R}$. Let N and A be the subgroups of $\mathrm{SL}(2,\mathbb{R})$ given by

$$N = \{n_x \mid x \in \mathbb{R}\}, \qquad A = \{a_t \mid t \in \mathbb{R}\}.$$

Then from the above we see that $A \cdot i = i\mathbb{R}^+$ and $NA \cdot i = H^+$. It follows that H^+ is contained in a single $\mathrm{SL}(2, \mathbb{R})$-orbit. On the other hand, we noticed already that H^+ is invariant under the action of $\mathrm{SL}(2, \mathbb{R})$. It thus follows that H^+ is a single $\mathrm{SL}(2, \mathbb{R})$-orbit. By applying complex conjugation, we see that H^- is a single $\mathrm{SL}(2, \mathbb{R})$-orbit as well. $\square$

Lemma 4.3 *The map* $\Psi : N \times A \times \mathrm{SO}(2) \to \mathrm{SL}(2, \mathbb{R})$, $(n, a, k) \mapsto nak$ *is a bijection.*

Proof. The maps $\mathbb{R} \to A$, $t \mapsto a_t$ and $\mathbb{R} \to N$, $x \mapsto n_x$ are bijective, so it suffices to show that the map

$$\psi : \mathbb{R} \times \mathbb{R} \times \mathrm{SO}(2) \to \mathrm{SL}(2, \mathbb{R}), \quad (x, t, k) \mapsto n_x a_t k$$

is a bijection. To see this, note that

$$\psi(x, t, k) \cdot i = e^{2t} i + x,$$

from which it readily follows that ψ is injective. On the other hand, if $g \in \mathrm{SL}(2, \mathbb{R})$, then $g \cdot i \in H^+$. Write $g \cdot i = x + iy \in H^+$, then there exists $t > 0$ such that $y = e^{2t}$. Therefore, $\psi(x, t, e) \cdot i = g \cdot i$ and it follows that $g^{-1}\psi(x, t)$ stabilizes i, from which $g^{-1}\psi(x, t) = k^{-1} \in \mathrm{SO}(2)$. This implies $\psi(t, x, e) = gk^{-1}$ hence $\psi(t, x, k) = g$ and we see that ψ is surjective. $\square$

Exercise 4.4 Show that the map Ψ of Lemma 4.3 is a homeomorphism. Show that the mentioned map is in fact a diffeomorphism, i.e., both Ψ and its inverse are C^∞ maps (between manifolds).

Remark 4.5 The above decomposition is known as the Iwasawa decomposition. Note that it follows from this decomposition that $\mathrm{SL}(2, \mathbb{R})$ is homeomorphic (even diffeomorphic) to $\mathbb{R}^2 \times S^1$.

For the sake of completeness, we mention another important decomposition for $\mathrm{SL}(2, \mathbb{R})$. Let $\mathfrak{s}$ denote the space of symmetric matrices in $\mathrm{M}(2, \mathbb{R})$ of trace zero. Then $\exp \mathfrak{s} = \{e^X \mid X \in \mathfrak{s}\}$ equals the set of positive definite symmetric matrices of determinant one. The following decomposition is known as the polar or Cartan decomposition.

Lemma 4.6 *The map* $\mathfrak{s} \times \mathrm{SO}(2) \to \mathrm{SL}(2, \mathbb{R})$, $(X, k) \mapsto e^X k$ *is a homeomorphism (even a diffeomorphism).*

Proof. We will first show that the mentioned map, f, is a bijection. Let $g \in \mathrm{SL}(2, \mathbb{R})$. Then $x := gg^{\mathrm{T}}$ belongs to $\mathrm{SL}(2, \mathbb{R})$, and is positive definite symmetric. It follows that $x = \exp(2X_s)$ for a symmetric matrix in $\mathrm{M}(2, \mathbb{R})$. As $\det x = 1$, it follows by an argument involving diagonalisation of X_s that X_s has trace zero.

Consider the element $k = \exp(-X_s)g$, This element belongs to $\mathrm{SL}(2, \mathbb{R})$ and

$$kk^{\mathrm{T}} = \exp(-X_s)gg^{\mathrm{T}} \exp(-X_s) = I$$

hence $k \in \mathrm{SO}(2)$ and we see that $k \in \mathrm{SO}(2)$. We conclude that $g = f(X_s, k)$ and have shown that f is surjective.

On the other hand, for injectivity, assume that $f(X_s, k) = g = f(X_s', k')$, then $\exp 2X_s = gg^{\mathrm{T}} = \exp 2X_s'$. By a straightforward argument involving eigenspaces, one sees that $X_s = X_s'$. It then readily follows that $k = k'$ and so f is injective.

Clearly the map f is continuous (in fact C^∞). We will show that f^{-1} is continuous as well.

We write $f^{-1}(g) = (X_s(g), k(g))$ and will show that both components depend continuously on $g \in \mathrm{SL}(2, \mathbb{R})$.

It is sufficient to prove the claim that $X_s(g)$ depends continuously on g, for then obviously $k(g) = \exp(-X_s)(g)g$ depends continuously on g.

To see that the claim is valid, we note that $x(g) := gg^{\mathrm{T}}$ depends continuously on g and that $x(I) = I$. The matrix $x(g)$ has determinant one, and is positive definite symmetric, hence it has two eigenvalues $\lambda(g) \geq 1$ and $\mu(g) = \lambda(g)^{-1} \leq 1$. If $x(g) \neq I$ it follows that the eigenvalues of $x(g)$ as well as the corresponding eigenspaces are distinct and depend continously (even C^∞) on g. Hence also $X_s(g)$ depends continuously (in fact C^∞) on g

Let $g_0 \in \mathrm{SL}(2, \mathbb{R})$ be such that $x(g_0) = I$, or, equivalently, $g_0 \in \mathrm{SO}(2)$. If $g \to g_0$, then it follows that $x(g) \to I$, from which it readily follows that $X_s(g) \to 0$. Thus, the map $g \mapsto X_s(g)$ is continuous on all of $\mathrm{SL}(2, \mathbb{R})$ and it follows that f^{-1} is continuous. Hence, f is a homeomorphism.

With a bit more work we can show that f^{-1} is C^∞. From the above argument it should be clear that this is true at elements $g_0 \in \mathrm{SL}(2, \mathbb{R}) \setminus \mathrm{SO}(2)$.

Let $\mathfrak{p}$ be the set of all symmetric matrices in $\mathrm{M}(2, \mathbb{R})$ and let P be the set of matrices in $\mathfrak{p}$ which are positive definite. Then $\exp : \mathfrak{p} \to P$ is a smooth map. Its total derivative at 0 is readily seen to be the identity map $\mathfrak{p} \to \mathfrak{p}$. By the inverse function theorem it follows that there exists an open neighborhood $U \ni 0$ in $\mathfrak{p}$ such that $\epsilon := \exp|_U$ is a diffeomorphism onto an open neighborhood V of I in P. Returning to the above setting, let W be the set of elements $g \in \mathrm{SL}(2, \mathbb{R})$ such that $gg^{\mathrm{T}} \in V$. Then W is an open neighborhood of $\mathrm{SO}(2)$ in $\mathrm{SL}(2, \mathbb{R})$. Moreover, $\epsilon^{-1}(gg^{\mathrm{T}})$ depends C^∞ on $g \in W$. On the other hand, $\epsilon^{-1}(gg^{\mathrm{T}}) = X_s(g)$ and the smoothness of X_s on W follows. □

5 Hyperbolic geometry

We will now use the bijection $\mathrm{SL}(2, \mathbb{C})/\mathrm{SO}(2) \simeq H^+$ to equip H^+ with the structure of a smooth $\mathrm{SL}(2, \mathbb{C})$-invariant Riemannian metric.

A smooth Riemannian metric on H^+ is defined to be a smooth map $H^+ \to (\mathbb{R}^2 \otimes \mathbb{R}^2)^*$, $\beta : z \mapsto \beta_z$, with values in the set of positive definite inner products. By an isometry of (H^+, β) we mean a diffeomorphism $\varphi : H^+ \to H^+$ such that $D\varphi(z) : \mathbb{R}^2 \to \mathbb{R}^2$ is isometric relative to the metrics β_z and $\beta_{\varphi(z)}$, for every $z \in H^+$. The metric β is said to be invariant under the action of $\mathrm{SL}(2, \mathbb{R})$ on H^+ if T_g is an isometry, for every $g \in \mathrm{SL}(2, \mathbb{R})$.

Remark 5.1 In general, a Riemannian metric on a manifold M is a family of positive definite inner products β_m on $T_m M$, for $m \in M$, which depends smoothly on $m \in M$. An isometry of M is then a diffeomorphism $\varphi : M \to M$ such that the derivative or tangent map $T_m\varphi : T_m M \to T_{\varphi(m)} M$ is isometric with respect to the given inner products β_m and $\beta_{\varphi(m)}$, for all $m \in M$.

Lemma 5.2 *The space H^+ has a unique* $\mathrm{SL}(2,\mathbb{R})$*-invariant Riemannian metric β such that the associated inner product β_i at i equals the standard inner product on $\mathbb{R}^2$. The metric is given by*

$$\beta_z = y^{-2}\langle\ \cdot\ ,\ \cdot\ \rangle_{\mathrm{st}} = y^{-2}(dx^2 + dy^2). \tag{4}$$

for $z = x + iy \in H^+$. The subscript st *indicates that the standard inner product on $\mathbb{R}^2$ is taken.*

Proof. Let $g = g_{a,b,c,d} \in \mathrm{SL}(2,\mathbb{R})$. Then $T_g : H^+ \to H^+$ is holomorphic. By a straightforward calculation it is seen that its complex derivative at $w \in H^+$ is given by

$$(T_g)'(w) = \frac{a(cw+d) - (aw+b)c}{(cw+d)^2} = \frac{1}{(cw+d)^2}.$$

It follows from this that the total derivative $D(T_g)(w) : \mathbb{R}^2 \to \mathbb{R}^2$ corresponds to the map $\mathbb{C} \to \mathbb{C}$, $\zeta \mapsto (cw+d)^{-2}\zeta$.

This multiplication map decomposes as a rotation (over the argument of $(cw+d)^{-2}$) and the real scalar multiplication by $|cw+d|^{-2}$.

Let β_{st} denote the standard inner product on $\mathbb{R}^2$. Then it follows that the pull-back

$$D(T_g)(w)^*\beta_{st} := \beta_{\mathrm{st}} \circ (D(T_g)(w) \times D(T_g)(w))$$

is given by

$$D(T_g)(w)^*\beta_{\mathrm{st}} = |cw+d|^{-4}\beta_{\mathrm{st}}. \tag{5}$$

We will now establish uniqueness. Let β be as asserted. Then by using the above formula to compare β_z with $\beta_i = \beta_{\mathrm{st}}$, we see that

$$\beta_z = C(z)\beta_{\mathrm{st}},$$

for a uniquely determined function $C : H^+ \to (0,\infty)$. There exists $g = g_{a,b,c,d}$ such that $g \cdot i = z$. From Lemma 4.1 we see that

$$y = \mathrm{Im}(g \cdot i) = |ci + d|^{-2}.$$

Now by invariance and using (5) with $w = i$ we find

$$\beta_{\mathrm{st}} = D(T_g)(i)^*\beta_z = C(z)D(T_g)(i)^*\beta_{\mathrm{st}} = C(z)y^2\beta_{\mathrm{st}},$$

so that $C(z) = y^{-2}$. This establishes uniqueness and the necessity of formula (4). We will now establish existence. For this we note that it suffices to show that the metric β defined by (4) is $\mathrm{SL}(2,\mathbb{R})$-invariant.

We write $C(z) = \mathrm{Im}(z)^{-2}$, for $z \in H^+$. Then $\beta_z = C(z)\beta_{\mathrm{st}}$. It suffices to show that for $g \in \mathrm{SL}(2,\mathbb{R})$ and $z \in H^+$ we have $D(T_g)(z)^*\beta_{g\cdot z} = \beta_z$. This is equivalent to

$$C(g \cdot z)D(T_g)(z)^*\beta_{\mathrm{st}} = C(z)\beta_{\mathrm{st}}.$$

In view of (5) with $w = z$ the latter equation is equivalent to

$$C(g \cdot z) = |cz + d|^4 C(z),$$

which in turn is a consequence of Lemma 4.1. □

In the following we agree to denote by $|\cdot|_z$ the norm on $\mathbb{R}^2$ determined by the inner product β_z, for $z \in H^+$. Then

$$\|\zeta\|_z = \sqrt{y^{-2}\langle\,\zeta,\zeta\,\rangle_{\rm st}} = y^{-1}\|\zeta\|_{\rm st}.$$

For a piecewise C^1-curve $\gamma : [p,q] \to H^+$ we define the length by

$$L(\gamma) = \int_p^q \|\gamma'(t)\|_{\gamma(t)}\, dt.$$

It is readily seen that the length of a curve is invariant under C^1 reparametrization, so that we may reduce to the situation $p = 0$ and $q = 1$.

Given two points $z, w \in H^+$ we define the Riemannian distance $d(z_1, d_2)$ to be the infimum of $L(\gamma)$ where γ ranges over the piecewise C^1-curves $[0,1] \to H^+$ with $\gamma(0) = z$ and $\gamma(1) = w$.

Exercise 5.3 Show that d is a distance function in the sense of metric spaces.

Exercise 5.4 Show that if $\varphi : H^+ \to H^+$ is an isometry, then $d(\varphi(z), \varphi(w)) = d(z,w)$, for all $z, w \in \mathbb{C}$.

Lemma 5.5 *Let $s, t \in \mathbb{R}$, $s \leq t$. Then the distance between $e^s i$ and $e^t i$ equals $|t-s|$.*

Proof. We consider the curve $\gamma : [0,1] \to H^+$ given by

$$\gamma(\tau) = e^{s+\tau(t-s)}\, i.$$

Then $\gamma'(\tau) = (t-s)\gamma(\tau)$, so that

$$\|\gamma'(\tau)\|_{\gamma(\tau)} = (t-s), \qquad (0 \leq \tau \leq 1).$$

It follows that $L(\gamma) = t - s$. Hence, $d(e^s i, e^t i) \leq t - s$. It remains to establish the converse inequality.

By the exercise below, for any piecewise C^1-curve $\gamma : [0,1] \to H^+$ with initial point $e^s i$ and final point $e^t i$ we have $L(\gamma) \geq t - s$. By definition of the distance function, this implies $d(e^s i, e^t i) \geq t - s$. ☐

Exercise 5.6 Show that for any piecewise C^1-curve $\gamma : [0,1] \to H^+$ with initial point z and final point w we have

$$L(\gamma) \geq |\log \mathrm{Im}(w) - \log \mathrm{Im}(z)|.$$

Hint: first do this in case γ is C^1.

Definition 5.7 A geodesic in H^+ is defined to be a C^1-curve $\gamma : I \to H^+$ with $I \subset \mathbb{R}$ an interval, such that

(a) for every subinterval $[p,q] \subset I$ the curve $\gamma|_{[p,q]}$ has length $d(\gamma(p), \gamma(q))$ (length minimalizing property);

(b) the function $t \mapsto \|\gamma'(t)\|_{\gamma(t)}$ is constant on $[p,q]$ (constant velocity property).

A complete geodesic is a geodesic as above with domain $I = \mathbb{R}$.

Exercise 5.8 Show that for every $\xi \in \mathbb{R}$ the curve

$$\mathbb{R} \to H^+, t \mapsto e^{t\xi} i$$

is a (complete) geodesic in H^+.

6 Caley transform and Poincaré disk

We may use the action of $\mathrm{SL}(2,\mathbb{C})$ to find another representation of the hyperbolic Riemannian structure on H^+ on the open unit disk $D = \{z \in \mathbb{C} \mid |z| < 1\}$. The disk, equipped with this other representation of the metric is called the Poincaré disk.

Lemma 6.1 *There exists a unique $\kappa \in \mathrm{SL}(2,\mathbb{C})/\{+I,-I\}$ such that $\kappa \cdot i = 0, \kappa \cdot \infty = 1, \kappa \cdot 0 = -1$. This element κ sends the upper half plane H^+ biholomorphically onto the open unit disk D. It is given by*

$$\kappa = \begin{pmatrix} a & -ai \\ a & ai \end{pmatrix}, \qquad a = \pm\frac{1+i}{2}.$$

Proof. We will first establish uniqueness. Write $\kappa = g_{a,b,c,d} \in \mathrm{SL}(2,\mathbb{C})$. Then $\kappa \cdot 0 = 1$ implies $b = d$ and $\kappa \cdot \infty = -1$ implies $a = -c$. Finally, $g \cdot i = 0$ implies $ai + b = 0$, so that $b = -ai = ci = d$. Conversely, the latter condition implies $\kappa \cdot i = 0$, $\kappa \cdot 0 = 1$ and $\kappa \cdot \infty = i$. The condition $\det g = 1$ is now equivalent to $1 = ad - bc = -2a^2 i = 1$ so that $a^2 = \frac{i}{2}$ and existence and uniquess of κ follows, as well as the final assertion.

The element κ is readily seen to send $\mathbb{R}$ into the unit circle. Since it sends $\widehat{\mathbb{R}}$ onto a circle of $\widehat{\mathbb{C}}$, we see that κ must send $\widehat{\mathbb{R}}$ diffeomorphically onto the unit circle $\mathbb{T} := \partial D$. Now κ sends $H^+ \cup H^-$ homeomorphically onto $\widehat{\mathbb{C}} \setminus \mathbb{T}$ and since $\kappa \cdot i = 0$, we see that κ maps H^+ bi-holomorphically onto D and H^- bi-holomorphically onto $\widehat{\mathbb{C}} \setminus \bar{D}$. □

The associated transform $T_\kappa : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is given by

$$T_\kappa(z) = \frac{z-i}{z+i}$$

and known as the Caley-transform. The sets $D, \partial D$ and $\widehat{\mathbb{C}} \setminus \bar{D}$ are the images of H^+, $\widehat{\mathbb{R}}$ and H^- under T_κ respectively, and therefore equal to the orbits of the conjugate group $G' = \kappa \mathrm{SL}(2,\mathbb{R})\kappa^{-1}$.

Let $\langle \,\cdot\,,\,\cdot\,\rangle$ be the standard Hermitian inner product on $\mathbb{C}^2$ and let J be the 2×2 diagonal matrix, with $J_{11} = 1$ and $J_{22} = -1$. Then $\mathrm{SU}(1,1)$ is defined to be the stabilizer in $\mathrm{SL}(2,\mathbb{C})$ of the sesquilinear form $(z, z') \mapsto \langle z, Jz' \rangle$ on $\mathbb{C}^2$. That is, an element $g \in \mathrm{SL}(2,\mathbb{C})$ belongs to $\mathrm{SU}(1,1)$ if and only if

$$\langle gz, Jgz' \rangle = \langle z, Jz' \rangle \qquad (\forall z, z' \in \mathbb{C}^2).$$

The above is equivalent to $g^* Jg = J$, hence to

$$g^{-1} = Jg^* J.$$

Lemma 6.2 *The conjugate group $\kappa \mathrm{SL}(2,\mathbb{R})\kappa^{-1}$ equals* $\mathrm{SU}(1,1)$.

Proof. Let $g \in \mathrm{SL}(2,\mathbb{C})$. Then $\kappa g \kappa^{-1}$ belongs to $\mathrm{SU}(1,1)$ if and only if

$$\kappa g^{-1} \kappa^{-1} = J\kappa^{-1*} g^* \kappa^* J,$$

which in turn is equivalent to

$$g^{-1} = L^{-1} g^* L,$$

where

$$L = \kappa^* J \kappa = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

A simple calculation now leads to

$$L^{-1} g^*_{a,b,c,d} L = \begin{pmatrix} \bar{d} & -\bar{b} \\ -\bar{c} & \bar{a} \end{pmatrix}.$$

We thus see that $\kappa g_{a,b,c,d} \kappa^{-1}$ belongs to $\mathrm{SU}(1,1)$ if and only if $a, b, c, d \in \mathbb{R}$, or equivalently, $g_{a,b,c,d} \in \mathrm{SL}(2, \mathbb{R})$. $\square$

We will now determine the Riemannian structure β^D on D for which the inverse Caley transform $T_\kappa^{-1} : D \to H^+$ (and hence also the Cayley transform $T_\kappa : H^+ \to D$) becomes an isometry. This means that

$$\beta^D_z = D(T_{\kappa^{-1}})(z)^* \beta_{T_\kappa^{-1}(z)}, \qquad (z \in D).$$

We observe that

$$T_\kappa^{-1}(z) = \frac{z+1}{iz - i}.$$

The expression on the right may be rewritten as

$$\frac{z+1}{iz-i} = \frac{(z+1)(\bar{z}-1)}{i|z-1|^2} = \frac{\bar{z} - z + |z|^2 - 1}{i|z-1|^2},$$

from which we see that

$$\mathrm{Im}(T_\kappa^{-1}(z)) = \frac{1 - |z|^2}{|z-1|^2}.$$

It follows that

$$\beta^D_z = \frac{|z-1|^4}{(1-|z|^2)^2} |T'_{\kappa^{-1}}(z)|^2 \beta_{\mathrm{st}}.$$

Since the derivative of the inverse Cayley transform is given by

$$-i \frac{d}{dz} \frac{z+1}{z-1} = \frac{2i}{(z-1)^2},$$

it follows that

$$\beta^D_z = 4(1 - |z|^2)^{-2} \beta_{\mathrm{st}}.$$

The Poincaré disk is defined to be the unit disk D equipped with this metric.

As before, the Riemannian metric β^D_z induces a distance function on D which we denote by d^D.

Exercise 6.3 Show that for every isometry $\varphi : H^+ \to D$ we have $d^D(\varphi(z), \varphi(w)) = d(z, w)$, for all $z, w \in H^+$.

The notion of geodesic in D may be defined in a fashion analogous to Definition 5.7.

Exercise 6.4 Let $\varphi : H^+ \to D$ be an isometry. Let $I \subset \mathbb{R}$ be an interval and $\gamma : I \to H^+$ a C^1-curve. Show that γ is a geodesic in H^+ (for the metric β) if and only $T_\kappa \circ \gamma : I \to D$ is a geodesic in D (for the metric β^D).

Exercise 6.5 Let $s, t \in \mathbb{R}, s \leq t.$

(a) Show that the Caley-transform maps the line segment $[e^s, e^t]i$ onto $[\tanh \frac{s}{2}, \tanh \frac{t}{2}]$, which is a line segment contained in D.

(b) Show that the curve $c : \tau \mapsto \tanh(s + \tau(t - s))$ has length $e^t - e^s$ relative to the hyperbolic metric β^D.

(c) For $\varphi \in \mathbb{R}$ we define the diagonal matrix

$$d_\varphi := \begin{pmatrix} e^{-i\varphi} & 0 \\ 0 & e^{i\varphi} \end{pmatrix}.$$

(d) Show that

$$d_\varphi \cdot w = e^{-2i\varphi} w, \qquad (\varphi \in \mathbb{R}, w \in D).$$

Argue that for every $\varphi \in \mathbb{R}$ the curve

$$t \mapsto e^{i\varphi} \tanh t$$

is a geodesic in D.

(e) Show that the (images of the) complete geodesics in D are all intersections of D with circles in $\widehat{\mathbb{C}}$ that intersect ∂D perpendicularly. The images of these geodesics are also called: the straight lines of the Poincaré disk.

Exercise 6.6 The Poincaré disk is a model for hyperbolic geometry. Argue that the following assertions of hyperbolic geometry are valid.

(a) Given a hyperbolic line l in D and a point $a \in D \setminus l$ show that there is an infinite collection of lines $m \ni a$ with $m \cup l = \emptyset$ (such m is called parallel to l).

(b) Show that this collection of lines can be characterized by two extreme 'parallel' lines through a.

(c) Show that the hyperbolic metric determines a notion of angle between lines. Show that in present setting this notion coincides with the Euclidean notion of angle.

(d) Convince yourself that the sum of the angles in a geodesic triangle in D is strictly smaller than π.

Exercise 6.7 Let r_φ be the matrix of the rotation around 0 in $\mathbb{R}^2$ by angle φ.

(a) Show that for every $\varphi \in \mathbb{R}$ we have

$$\kappa r_\varphi \kappa^{-1} = d_\varphi$$

where d_φ is the diagonal matrix defined in Exercise 6.5 (c). In particular, this means that $\kappa \mathrm{SO}(2)\kappa^{-1}$ equals the group $\mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(1))$ of diagonal unitary matrices in $\mathrm{SL}(2, \mathbb{C})$.

(b) By using the Caley transform, conclude that the orbit $\mathrm{SO}(2) \cdot (e^t i)$ is a Euclidean circle C contained in H^+.

(e) Show that the circle C has center $(\cosh t)i$ (in the sense of Euclidean geometry). Hint: show that C is symmetric with respect to the imaginary axis, and determine the intersection $C \cap i\mathbb{R}$.

We are now in the position to prove that every two distinct points in the Riemannian manifolds D and H^+ can be connected by a unique geodesic.

Theorem 6.8 *Let $z, w \in D$ be two distinct points. Then there is a unique geodesic $\gamma : [0,1] \to D$ with initial point z and end point w.*

Proof. If $z, w \in (-1, 1) \subset D$ this result follows from Exercise 6.5. For arbitrary $z, w \in D$ we note that by transitivity of the action of $\mathrm{SU}(1,1)$ on D there exists $g_0 \in \mathrm{SU}(1,1)$ such that $g_0 \cdot z = 0$. Now the group $\mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(1))$ fixes the point 0, and acts on D by rotations about 0, see Exercise 6.7 (b). It follows that there exists a $\varphi \in \mathbb{R}$ such that $d_\varphi g_0 \cdot w \in D = [0, 1)$. We note that $d_\varphi g_0 \cdot z = 0$.

Put $g := d_\varphi g_0$, then $g \in \mathrm{SU}(1,1)$ so $T := T_g : D \to D$ is an isometry such that $T(z) = 0$ and $T(w) \in [0, 1)$. Let $c : [0,1] \to D$ be a C^1-curve connecting $T(z)$ and $T(w)$. Then $\gamma = T^{-1} \circ c$ is a C^1-curve connecting z and w. Since T is isometric, γ is a geodesic if and only if c is a geodesic. The result now follows from the special case mentioned at the beginning of the proof.

Exercise 6.9 Show that the geodesic connecting two elements z and w of D has as image the arc with boundary points z and w of a circle in $\widehat{\mathbb{C}}$ which intersects ∂D perpendicularly.

Exercise 6.10 Show that for any two points $z, w \in H^+$ there exists a unique geodesic $\gamma : [0, 1] \to H^+$ such that $\gamma(0) = z$ and $\gamma(1) = w$. Show that $\gamma([0, 1])$ is the arc of a circle in $\widehat{C}$ which intersects $\widehat{\mathbb{R}}$ perpendicularly.

Exercise 6.11 Let C be any circle in $\mathbb{C}$ which intersects $\mathbb{R}$ perpendicularly in a point z. Show that C intersects $\mathbb{R}$ perpendicularly in a second point $z' \in \mathbb{R}$. Show that there exists an element $g \in \mathrm{SL}(2, \mathbb{R})$ such that $C = T_g(i\mathbb{R})$.