Symplectic geometry of semisimple orbits

by Hassan Azad [a], Erik van den Ban [b] and Indranil Biswas [c]

[a] *Department of Mathematics and Statistics, King Fahd University, Saudi Arabia*

[b] *Mathematisch Instituut, Universiteit Utrecht, PO Box 80 010, 3508 TA Utrecht, The Netherlands*

[c] *School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay 400005, India*

ABSTRACT

Let G be a complex semisimple group, $T \subset G$ a maximal torus and B a Borel subgroup of G containing T. Let Ω be the Kostant–Kirillov holomorphic symplectic structure on the adjoint orbit $\mathcal{O} = \mathrm{Ad}(G)c \simeq G/Z(c)$, where $c \in \mathrm{Lie}(T)$, and $Z(c)$ is the centralizer of c in G. We prove that the real symplectic form $\mathrm{Re}\,\Omega$ (respectively, $\mathrm{Im}\,\Omega$) on $\mathcal{O}$ is exact if and only if all the eigenvalues $\mathrm{ad}(c)$ are real (respectively, purely imaginary). Furthermore, each of these real symplectic manifolds is symplectomorphic to the cotangent bundle of the partial flag manifold $G/Z(c)B$, equipped with the Liouville symplectic form. The latter result is generalized to hyperbolic adjoint orbits in a real semisimple Lie algebra.

1. INTRODUCTION

This work grew out of attempts to understand the following theorem of Arnold [1, p. 100, Theorem 1].

Theorem 1.1 [1]. *Let Ω be the standard complex symplectic structure on a regular coadjoint orbit of the group $\mathrm{SL}(n + 1, \mathbb{C})$. This orbit, equipped with the real symplectic structure $\mathrm{Im}(\Omega)$, is isomorphic to the total space of the cotangent bundle of the variety parameterizing the complete flags in $\mathbb{C}^{n+1}$, equipped with the standard Liouville symplectic structure on it, if and only if all the eigenvalues of some (and hence any) matrix in the orbit are pure imaginary.*

E-mails: hassanaz@kfupm.edu.sa (H. Azad), e.p.vandenban@uu.nl (E. van den Ban), indranil@math.tifr.res.in (I. Biswas).

A proof of this theorem is outlined in [1, pp. 100–101]. The assertion about the equivalence of the above mentioned symplectic structure $\mathrm{Im}(\Omega)$ with the one on total space of the cotangent bundle of the flag variety is made in lines 13–15 of [1, p. 101]. Apparently, the regular coadjoint orbit is identified with an adjoint orbit in $sl(n+1, \mathbb{C})$ through the non-degenerate bilinear form $(X, Y) \mapsto \mathrm{Tr}(XY)$, so that it makes sense to speak of eigenvalues of matrices in the orbit.

Arnold's result may be reformulated in terms of the theory of semisimple Lie groups. In the present paper we will state this reformulation and prove a generalization of it.

Let G be a connected complex semisimple Lie group. Its Lie algebra, which will be denoted by $\mathfrak{g}$, comes equipped with the Killing form B, which is an $\mathrm{Ad}(G)$-invariant symmetric non-degenerate bilinear form. Given an element $c \in \mathfrak{g}$, we denote by $B(c)$ the complex linear functional on $\mathfrak{g}$ defined by $X \mapsto B(c, X)$. Accordingly, the Killing form is viewed as a G-equivariant linear isomorphism

$$B : \mathfrak{g} \xrightarrow{\simeq} \mathfrak{g}^*.$$

Unless specified otherwise, we will use B to identify $\mathfrak{g}$ with $\mathfrak{g}^*$. In particular, by pull-back under B of the canonical Kostant–Liouville holomorphic symplectic form on any coadjoint orbit $\mathcal{O} \subset \mathfrak{g}^*$ may be viewed as a holomorphic symplectic form on the associated adjoint orbit $B^{-1}(\mathcal{O})$.

If $c \in \mathfrak{g}$, then by $\mathrm{ad}(c)$ we denote the endomorphism $Y \mapsto [c, Y]$ of $\mathfrak{g}$. The element c is called semisimple if and only if $\mathrm{ad}(c)$ diagonalizes. Equivalently, this means that c is contained in the Lie algebra of a maximal torus (or Cartan subgroup) T of G. The centralizer of c in G is denoted by $Z(c)$. If c is semisimple, then $Z(c)$ is known to be the Levi component of a parabolic subgroup P of G. In fact, one may take $P = Z(c)B$, where B is a Borel subgroup containing a maximal torus which contains c. We will prove the following generalization of Arnold's result.

Theorem 1.2. *Let G be a connected complex semisimple group, and let c be a semisimple element of its Lie algebra $\mathfrak{g}$. Let Ω be the Kostant–Kirillov holomorphic symplectic form on the orbit $\mathcal{O} = \mathrm{Ad}(G)c \simeq G/Z(c)$. Then the real and imaginary parts $\mathrm{Re}\,\Omega$ and $\mathrm{Im}\,\Omega$ are real symplectic forms on $\mathcal{O}$. Moreover, the following hold.*

(a) *The form $\mathrm{Re}\,\Omega$ (respectively, $\mathrm{Im}\,\Omega$) on $\mathcal{O}$ is exact if and only if all eigenvalues of $\mathrm{ad}(c)$ are real (respectively, purely imaginary).*

(b) *In either case, these symplectic manifolds with exact real symplectic forms are symplectomorphic to the total space of the cotangent bundle of G/P, equipped with the Liouville symplectic form, where P is any parabolic subgroup of G with Levi component $Z(c)$.*

In fact, we will prove a refinement of assertion (b) in the more general context of a real hyperbolic adjoint orbit of a real semisimple Lie group; see Theorems 2.11 and 6.1.

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Here are a few words about our interpretation of the above mentioned result of Arnold.

Set $G = \mathrm{SL}(n + 1, \mathbb{C})$, and let $T \subset G$ be the subgroup of diagonal matrices. For any $c \in \mathrm{Lie}(T)$ with distinct eigenvalues we have $Z(c) = T$, so that the adjoint orbit of c can be identified with G/T. Let c_i denote the ith diagonal entry of c. The eigenvalues of $\mathrm{ad}(c)$ are all the numbers of the form $c_i - c_j$, with $1 \leqslant i, j \leqslant n$. As $\sum_j c_j = 0$, it follows that the eigenvalues of c are all real if and only if those of $\mathrm{ad}(c)$ are.

The group G naturally acts on the manifold $\mathcal{F}$ of full flags in $\mathbb{C}^{n+1}$. The stabilizer of the standard flag $\mathbb{C} \subset \mathbb{C}^2 \subset \cdots \subset \mathbb{C}^n$ is the subgroup $B \subset G$ of upper triangular matrices. Consequently, $\mathcal{F} \simeq G/B$, as G-manifolds. Arnold's result asserts that $\mathrm{Ad}(G)c \simeq T^*\mathcal{F} \simeq T^*(G/B)$ as real symplectic manifolds. In the present set-up, our generalization concerns the analogue for an arbitrary diagonal matrix c and the associated partial flag manifold G/P.

As $G/T \simeq G \times_B (B/T)$, the natural projection

$$\psi : G/T \to G/B$$

makes G/T a fiber bundle over the full flag manifold G/B; its fibers are translates of B/T. Since G/B is a complete variety, and $G/T \simeq \mathcal{O}$ an affine variety, the bundle ψ does not admit any holomorphic sections.

On the other hand, let $K = \mathrm{SU}(n + 1)$. Then the natural map $j : K/K \cap T \to G/T$ determines a real analytic section of ψ. Indeed, since $G = KB$ and $K \cap B = K \cap T$, the map $K/K \cap T \to G/B$ is a real analytic diffeomorphism. Composing its inverse with j we obtain a section $s : G/B \to G/T$. Moreover,

$$G/T \simeq G \times_B (B/T) = KB \times_B (B/T)$$
$$\simeq K \times_{K \cap B} (B/T) \simeq K \times_{K \cap T} R_u(B),$$

where $R_u(B)$ denotes the unipotent radical of B. This unipotent radical has the structure of a complex linear space on which the adjoint action of T linearizes. Therefore, the last isomorphism realizes G/T as a real analytic vector bundle over $K/(K \cap T) \simeq G/B$ with s corresponding to the zero section. This real analytic vector bundle is in fact isomorphic to the cotangent bundle of $K/(K \cap T)$. It follows that the inclusion $K/(K \cap T) \to G/T$, and hence the real analytic section $s : G/B \to G/T$, induces an isomorphism on the cohomology algebras of these spaces. Hence, one can decide whether a given closed differential two-form on G/T is exact from its restriction to $K/(K \cap T)$. This is roughly a translation in group theoretic terms of [1, p. 100–101]. The generalization of this argument to our more general setting is worked out in the next section and leads to part (a) of Theorem 1.2.

Assertion (b) in Theorem 1.2 is based on the crucial observation that the fibration $\psi : G/Z(c) \to G/P$ has Lagrangian fibers and that $K/K \cap Z(c) \hookrightarrow G/T$ defines a Lagrangian section. This implies the existence of commuting vertical vector fields on the bundle ψ and is enough to establish the existence of a local symplectic isomorphism along the section $K/K \cap Z(c)$; see Section 3. This argument is

indicated in [1], but an argument for the existence of a globally defined symplectic isomorphism seems to be lacking.

We prove the existence of such a global symplectomorphism in Section 6 by showing that the mentioned vertical vector fields have complete flows which can be used to construct global coordinates along the fibers of ψ. Moreover, we give this argument of integration in the more general setting of real hyperbolic adjoint orbits for a real semisimple Lie group.

The above mentioned commuting vector fields are used to construct a K-equivariant diffeomorphism

$$\phi : K \times_{K \cap Z(c)} \left(\mathfrak{g} / \operatorname{Lie}(P) \right)^* \to G/Z(c).$$

The pull back of $\operatorname{Re}(\Omega)$ – in the notation of Theorem 1.2 – is the Liouville form on $K \times_{K \cap Z(c)} (\mathfrak{g}/ \operatorname{Lie}(P))^*$ identified with $T^*(G/P)$.

2. COMPLEX SEMISIMPLE ORBITS

We will recall some generalities concerning the Kostant–Kirillov symplectic form, after fixing the notation. At first we assume that G is a connected Lie group over the base field k, which is either $\mathbb{R}$ or $\mathbb{C}$. Let η be an element of $\mathfrak{g}^*$, the k-linear dual of $\mathfrak{g}$. Let $Z(\eta)$ denote the stabilizer of η in G, and let $\mathfrak{z}(\eta)$ be the Lie algebra of $Z(\eta)$. The map $x \mapsto \eta \circ \operatorname{Ad}(x)^{-1}$ induces a G-equivariant diffeomorphism from $G/Z(\eta)$ onto the coadjoint orbit $\mathcal{O} = \mathcal{O}_\eta \subset \mathfrak{g}^*$ through η.

The Kostant–Kirillov form $\Omega = \Omega_\eta$ on $\mathcal{O}$ is defined as follows. The action of G on $\mathcal{O}$ gives rise to a Lie algebra homomorphism from $\mathfrak{g}$ to the space $\operatorname{Vect}(\mathcal{O})$ of vector fields on $\mathcal{O}$. Given $X \in \mathfrak{g}$, the associated vector field $\bar{X}$ on $\mathcal{O}$ is given by $\bar{X}_\xi =: -\xi \circ \operatorname{ad} X \in T_\xi \mathcal{O} \subset \mathfrak{g}^*$, where $\xi \in \mathcal{O}$. We agree to write X_ξ for $\bar{X}_\xi$ and note that the map $X \mapsto X_\xi$ descends to an isomorphism from $\mathfrak{g}/\mathfrak{z}(\xi)$ onto $T_\xi \mathcal{O}$. The two-form Ω on $\mathcal{O}$ is given by the formula

$$(2.1) \qquad \Omega_\xi(X_\xi, Y_\xi) = \xi([X, Y]),$$

where $\xi \in \mathcal{O}$ and $X, Y \in \mathfrak{g}$. Here we note that the expression on the right-hand side of (2.1) depends on X and Y through their images in $\mathfrak{g}/\mathfrak{z}(\xi)$, so that Ω is well defined. The form Ω is G-invariant. Moreover, it is readily seen to be closed and non-degenerate at the point ξ, hence it is a symplectic form. See [5, p. 6]. Note that if $k = \mathbb{C}$, then $\mathcal{O}$ is a complex submanifold of $\mathfrak{g}^*$, and Ω is a holomorphic symplectic form.

Via the natural diffeomorphism $G/Z(\eta) \to \mathcal{O}$ the form Ω may be pulled-back to a form on $G/Z(\eta)$. The resulting form, also denoted by Ω, is the unique G-invariant two-form which at the element $\bar{e} := eG(\eta)$ is given by the formula

$$(2.2) \qquad \Omega_{\bar{e}}(X_{\bar{e}}, Y_{\bar{e}}) = \eta([X, Y]), \quad X, Y \in \mathfrak{g}.$$

We now assume that G is a semisimple connected Lie group over k, so that the Killing form $B(X, Y) = \operatorname{Tr}(\operatorname{ad}(X) \operatorname{ad}(Y))$ is non-degenerate on $\mathfrak{g}$. The form B is

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G-invariant and symmetric. Hence it induces a G-equivariant linear isomorphism $\mathfrak{g} \to \mathfrak{g}^*$ that maps adjoint orbits diffeomorphically and G-equivariantly onto coadjoint orbits. Let $\xi \in \mathfrak{g}^*$ and let $c = c_\xi = B^{-1}(\xi)$. This means that

$$(2.3) \qquad \eta(Y) = B(c, Y), \quad Y \in \mathfrak{g}.$$

Then $Z(\eta)$ coincides with $Z(c)$, the centralizer of c in G. Via pull back under B, the form Ω may be realized as a form on the adjoint orbit $\mathrm{Ad}(G)c$.

In the rest of this section we assume that $k = \mathbb{C}$, so that G is a connected complex semisimple Lie group. We assume that $\eta \in \mathfrak{g}^*$ is such that $c = c_\eta$ is semisimple, i.e., the endomorphism $\mathrm{ad}(c) \in \mathrm{End}(\mathfrak{g})$ given by $X \mapsto [c, X]$ is diagonalizable. Equivalently, this means that c is contained in the Lie algebra of a maximal torus in G.

We fix a maximal torus T of G whose Lie algebra contains c, and in addition a maximal compact subgroup K of G for which $K \cap T$ is a maximal torus. Writing $\mathfrak{k}$ for the (real) Lie algebra of K, we have

$$(2.4) \qquad \mathfrak{g} = \mathfrak{k} \oplus \sqrt{-1} \cdot \mathfrak{k}$$

as a direct sum of real linear spaces. In particular, $\mathfrak{k}$ is a real form of $\mathfrak{g}$. The associated conjugation map $\theta : \mathfrak{g} \to \mathfrak{g}$ is called the Cartan-involution associated with K.

Lemma 2.1. *With notation as above, let Ω be the holomorphic Kostant–Kirillov symplectic form on $G/Z(c) = G/Z(\eta)$. Then both $\mathrm{Re}\,\Omega$ and $\mathrm{Im}\,\Omega$ are real symplectic forms on $G/Z(c)$.*

Proof. We will write $\mathfrak{g}_{\mathbb{R}}$ for $\mathfrak{g}$, viewed as a real Lie algebra. Accordingly, we put $\mathfrak{g}_{\mathbb{R}}^*$ for the real linear dual of $\mathfrak{g}_{\mathbb{R}}$. Then $\mathfrak{g}_{\mathbb{R}}^* = \mathrm{Hom}_{\mathbb{R}}(\mathfrak{g}, \mathbb{R})$. Both $\mathrm{Re}\,\eta$ and $\mathrm{Im}\,\eta$ belong to $\mathfrak{g}_{\mathbb{R}}^*$. Let $Z(\mathrm{Re}\,\eta)$ and $Z(\mathrm{Im}\,\eta)$ be the stabilizers of these elements for the coadjoint action for G, viewed as a real Lie group. We claim that

$$Z(\eta) = Z(\mathrm{Re}\,\eta) = Z(\mathrm{Im}\,\eta).$$

Indeed, this is seen as follows. Let J denote the linear automorphism of $\mathfrak{g}$ given by $X \mapsto \sqrt{-1} \cdot X$. Pull-back by J induces the real linear automorphism J^* of $\mathfrak{g}_{\mathbb{R}}^*$ given by $\xi \mapsto \xi \circ J$. As G is a complex Lie group, the adjoint action of G on $\mathfrak{g}$ commutes with J. Therefore, the coadjoint action of G on $\mathfrak{g}_{\mathbb{R}}^*$ commutes with J^*. It follows that $Z(J^*\xi) = Z(\xi)$ for all $\xi \in \mathfrak{g}_{\mathbb{R}}^*$. Now $J^* \mathrm{Re}\,\eta = \mathrm{Re}(i\eta) = -\mathrm{Im}\,\eta$, from which we see that $Z(\mathrm{Re}\,\eta) = Z(\mathrm{Im}\,\eta)$. Since $Z(\eta) = Z(\mathrm{Re}\,\eta) \cap Z(\mathrm{Im}\,\eta)$, the claim follows.

We now observe that $\mathrm{Re}\,\Omega$ is the unique G-invariant two-form on $G/Z(\eta) = G/Z(\mathrm{Re}\,\eta)$ given by $\mathrm{Re}\,\Omega_{\bar{e}}(X_{\bar{e}}, Y_{\bar{e}}) = [\mathrm{Re}\,\eta](X, Y)$. This implies that $\mathrm{Re}\,\Omega$ is just the Kostant–Kirillov form associated with the coadjoint orbit through $\mathrm{Re}\,\eta$ in $\mathfrak{g}_{\mathbb{R}}$, with G viewed as a real semisimple Lie group. Likewise, $\mathrm{Im}\,\Omega$ is the form associated with the coadjoint orbit through $\mathrm{Im}\,\eta$ in $\mathfrak{g}_{\mathbb{R}}^*$. $\quad\square$

In the rest of this section we will prove the following theorem.

Theorem 2.2. *Let* $c \in \mathrm{Lie}(T)$, *let* $\eta = B(c, \cdot) \in \mathfrak{g}^*$ *and let* Ω *be the holomorphic Kostant–Kirillov symplectic form on* $G/Z(c)$ *defined by* (2.2).

The real symplectic form $\mathrm{Re}\,\Omega$ *(respectively,* $\mathrm{Im}\,\Omega$*) on* $G/Z(c)$ *is exact if and only if all eigenvalues of* $\mathrm{ad}(c)$ *are real (respectively, purely imaginary).*

We will prove Theorem 2.2 through a number of lemmas.

Lemma 2.3. *The centralizer* $\mathfrak{z}(c)$ *is stable under* θ. *Equivalently,* $\mathfrak{k} \cap \mathfrak{z}(c)$ *is a real form of* $\mathfrak{z}(c)$.

Proof. Write $\mathfrak{t}$ for the Lie algebra of the maximal torus T. Since $T \cap K$ is a maximal torus of K, we have

$$\mathfrak{t} = \mathfrak{t} \cap \mathfrak{k} + \sqrt{-1}(\mathfrak{t} \cap \mathfrak{k}).$$

Accordingly, we write $c = a + \sqrt{-1}b$, where a and b belong to $\mathfrak{t} \cap \mathfrak{k}$. Fix a positive definite K-invariant Hermitian inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$. Then $\mathrm{ad}(a)$ is anti-Hermitian, hence diagonalizable with purely imaginary eigenvalues. Similarly, $\mathrm{ad}(\sqrt{-1}b)$ is diagonalizable with real eigenvalues. Since $\mathrm{ad}(a)$ and $\mathrm{ad}(\sqrt{-1}b)$ commute, they allow a simultaneous diagonalization. From this we see that $\ker(c)$ is the intersection of $\ker \mathrm{ad}(a)$ and $\ker \mathrm{ad}(b)$. Since both a and b are θ-stable, it follows that $\mathfrak{z}(c) = \ker \mathrm{ad}(c)$ is θ-stable. $\quad\square$

If $g \in G$ centralizes c, then g also centralizes the one–parameter subgroup $\{\exp(tc) \mid t \in \mathbb{C}\}$ of G. The closure of this one-parameter subgroup will be denoted by S. Clearly g centralizes S. In other words, we have $Z(c) = Z(S)$.

It is well known that the centralizers of tori are connected reductive. More precisely, $Z(c) = Z(S)$ is the Levi complement of a parabolic subgroup of G [4, p. 26, Proposition 1.22], [3].

Fix a simple system Δ_1 of roots of the reductive group $Z(c)$ relative to the maximal torus T and extend it to a simple system Δ of roots of G relative to the same maximal torus. Let B be the Borel subgroup of G defined by the simple system of roots Δ. Then $P = Z(c)B$ is a parabolic subgroup of G. Its Levi-complement is $Z(c)$, and its unipotent radical $R_u(P)$ is given by the roots in B whose supports are not contained in Δ_1. So $P = Z(c)R_u(P)$, and $G = KB = KP$. We agree to write $Z_K(c)$ for $K \cap Z(c)$, the centralizer of c in K.

Lemma 2.4. *The manifold* $G/Z(c)$ *is real analytically a vector bundle over* $K/Z_K(c)$.

Proof. This is a consequence of basic results of Mostow [7]. A direct argument is as follows.

The exponential map induces a holomorphic diffeomorphism from $\mathrm{Lie}(R_u(P))$ onto $R_u(P) \simeq P/Z(c)$. Accordingly, we equip $P/Z(c)$ with the structure of a complex vector space. As $k \exp X Z(c) = \exp \mathrm{Ad}(k) X Z(c)$ for $X \in R_u(P)$ and

$k \in K \cap P = K \cap Z(c)$, the action of $K \cap P$ on $P/Z(c)$ by left translation is linear for this structure. Accordingly,

$$K \times_{K \cap P} P/Z(c) \to K/K \cap P = K/Z_K(c)$$

has the structure of a real analytic vector bundle over $K/Z_K(c)$.

The multiplication map induces a surjective and submersive real analytic map $K \times P \to G$, which factors to a submersive real analytic map

$$K \times_{K \cap P} P \to G.$$

This map is clearly injective, hence a real analytic diffeomorphism. Therefore, the induced map

$$K \times_{K \cap P} P/Z(c) \to G/Z(c)$$

is a real analytic diffeomorphism as well. It realizes $G/Z(c)$ as a real analytic vector bundle over $K/Z_K(c)$. □

Lemma 2.5. *Let Ω be the holomorphic symplectic form on $G/Z(c)$ defined in (2.2). The K-orbit through $\bar{e} = eZ(c)$ is Lagrangian relative to $\operatorname{Im}\Omega$ (respectively, $\operatorname{Re}\Omega$) if and only if all the eigenvalues of $\operatorname{ad}(c)$ are purely imaginary (respectively, real).*

Proof. In Lemma 2.1 we established that $\operatorname{Re}\Omega$ and $\operatorname{Im}\Omega$ are real symplectic forms on $G/Z(c)$. It follows from Lemma 2.3 that $K\bar{e} \simeq K/Z_K(c)$ is a real form for $G/Z(c)$. In particular, $K\bar{e}$ has half the real dimension of $G/Z(c)$. Hence it suffices to establish the above assertion with the word Lagrangian replaced by isotropic.

It follows from (2.2) combined with (2.3) that the form $\operatorname{Im}\Omega$ is at $\bar{e} = eZ(c)$ given by

$$\operatorname{Im}\big(\Omega_{\bar{e}}(X_{\bar{e}}, Y_{\bar{e}})\big) = \operatorname{Im} B(c, [X, Y]), \quad X, Y \in \mathfrak{g}.$$

We write $c = a + \sqrt{-1}b$ with $a, b \in \mathfrak{t} \cap \mathfrak{k}$, as in the proof of Lemma 2.3. Since B is real-valued on $\mathfrak{k}$, it follows that

$$\operatorname{Im}\big(\Omega_{\bar{e}}(X_{\bar{e}}, Y_{\bar{e}})\big) = B(b, [X, Y])$$

for all $X, Y \in \mathfrak{k}$.

If $K\bar{e}$ is isotropic, then taking into account that $[\mathfrak{k}, \mathfrak{k}] = \mathfrak{k}$, we see that $B(Z, b) = 0$ for all $Z \in \mathfrak{k}$, and hence also for all $Z \in \mathfrak{g} = \mathfrak{k}^{\mathbb{C}}$. It follows that $b = 0$. Hence $c = a \in \mathfrak{t} \cap \mathfrak{k}$ and it follows that the eigenvalues of $\operatorname{ad}(c)$ are all purely imaginary.

Conversely, assume that all eigenvalues of $\operatorname{ad}(c)$ are purely imaginary. Then $c \in \mathfrak{t} \cap \mathfrak{k}$, so that $b = 0$. It follows that K is isotropic at the point $\bar{e}$. By invariance, K is isotropic everywhere. This completes proof of the result involving $\operatorname{Im}\Omega$. The proof for $\operatorname{Re}\Omega$ is similar. □

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Lemma 2.6. *The K-orbit of $\bar{e} = eZ(c)$ is Lagrangian with respect to $\operatorname{Im}\Omega$ if and only if the form $\operatorname{Im}\Omega$ is exact.*

Proof. As in the proof of the previous lemma, it suffices to prove the assertion with the word Lagrangian replaced by isotropic.

By Lemma 2.4, the G-orbit $G\bar{e} \simeq G/L$ can be retracted onto the K-orbit of $K\bar{e} \simeq K/K \cap L$. Hence, the inclusion $K\bar{e} \to G/Z(c)$ induces an isomorphism on de Rham cohomology. Therefore, the closed form $\omega = \operatorname{Im}\Omega$ is exact if and only if its restriction to $K\bar{e}$ is exact. Now the lemma is a consequence of the following more general result. $\square$

Lemma 2.7. *Let K be a compact Lie group and $H \subset K$ a compact subgroup containing a maximal torus of K. Let ω be a K-invariant closed two-form on K/H. Then ω is exact if and only if $\omega = 0$.*

Proof. We need to show that if ω is exact, then ω is identically zero.

Assume that $\omega = d\eta$. By integrating the left-translates $l_k^* \eta$ over $k \in K$ with respect to the Haar measure on K of total volume 1, we may assume that the form η is also K-invariant.

Let T_0 be a maximal torus of K contained in H and consider the natural fibration $\pi : K/T_0 \to K/H$. The pull back

$$\widetilde{\eta} := \pi^* \eta$$

is a K-invariant one-form. Let $\bar{e} = eT$. Then the evaluation $\widetilde{\eta}(\bar{e})$ is an $\operatorname{Ad}(T_0)$-invariant linear functional on the tangent space $T_{\bar{e}}(K/T_0)$. Its complex linear extension is therefore an $\operatorname{Ad}(T_0)$-invariant $\mathbb{C}$-linear functional on the complexification $T_{\bar{e}}(K/T_0) \otimes_{\mathbb{R}} \mathbb{C}$.

A basis for this complexification is given by the canonical images of root vectors $\{X_\alpha\}_{\alpha \in R}$, where R is a system of roots of $K^{\mathbb{C}}$ relative to $T_0^{\mathbb{C}}$. The $\operatorname{Ad}(T_0)$-invariance of $\widetilde{\eta}(\bar{e})$ implies that $\widetilde{\eta}(\bar{e}) = 0$ on each of these root vectors, hence on $T_{\bar{e}}(K/T_0)$. By K-invariance, it follows that $\widetilde{\eta} = 0$. Since π is a surjective submersion, this in turn implies that $\eta = 0$. $\square$

Lemma 2.5 and Lemma 2.6 together complete the proof of Theorem 2.2. In view of Lemma 2.1 this completes the proof of Theorem 1.2(a). For the remaining part of the proof of Theorem 1.2, the following observation will be of fundamental importance.

Lemma 2.8. *The fibers of the fibration $\psi : G/Z(c) \to G/P$ are isotropic for the holomorphic symplectic form Ω.*

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Proof. Put $\bar{e} := eZ(c)$. By G-invariance, it suffices to show that $\Omega_{\bar{e}}$ vanishes on the tangent space at $\bar{e}$ to the fiber $\psi^{-1}(eP) = PZ(c) = R_u(P)\bar{e}$. In view of (2.2) and (2.3) it suffices to show that

$$B(c, [X, Y]) = 0$$

for all $X, Y \in \mathrm{Lie}(R_u(P))$. By linearity it suffices to prove this identity for X, Y contained in root spaces of $R_u(P)$. If $[X, Y] = 0$, the identity is trivially valid, so we may assume $[X, Y] \neq 0$. Then $[X, Y]$ is contained in a root space for a root α of P. Let $t \in T$ be such that $t^\alpha \neq 1$. Then by G-invariance of B,

$$
\begin{aligned}
B(c, [X, Y]) &= B\big(\mathrm{Ad}(t^{-1})c, [X, Y]\big) \\
&= B\big(c, \mathrm{Ad}(t)[X, Y]\big) = t^\alpha B(c, [X, Y]).
\end{aligned}
$$

The lemma follows. $\quad\square$

The rest of the paper will be devoted to the proof of Theorem 1.2(b), or rather its generalization to the setting of real semisimple Lie algebras. We will proceed under the assumption that all eigenvalues of $\mathrm{ad}(c)$ are real. The case with all eigenvalues purely imaginary is treated similarly. Thus, $\mathrm{Re}\,\Omega$ is a real symplectic form on $G/Z(c)$ and $K/Z_K(c)$ is a Lagrangian submanifold for this form. Moreover, by Lemma 2.8 the fibers of the fibration $G/Z(c) \to G/P$ are Lagrangian for $\mathrm{Re}\,\Omega$.

In order to facilitate the comparison with the theory of real semisimple Lie algebras, we make a few more remarks about the real Lie algebra $\mathfrak{g}_\mathbb{R}$ (see the proof of Lemma 2.1). This algebra has a real Killing form which we denote by $B_\mathbb{R}$.

Lemma 2.9. *As maps $\mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$, the Killing forms B and $B_\mathbb{R}$ are related by $B_\mathbb{R} = 2\,\mathrm{Re}\,B$.*

Proof. Let $A : \mathfrak{g} \to \mathfrak{g}$ be a complex linear map. Its complex trace is denoted by $\mathrm{Tr}_\mathbb{C}\,A$. At the same time A defines a real linear endomorphism of $\mathfrak{g}_\mathbb{R}$. As such, its trace is denoted by $\mathrm{Tr}_\mathbb{R}\,A$. It is straightforward to check that $\mathrm{Tr}_\mathbb{R}\,A = 2\,\mathrm{Re}\,\mathrm{Tr}_\mathbb{C}\,A$. Hence, for $X, Y \in \mathfrak{g}$ we have $B_\mathbb{R}(X, Y) = \mathrm{Tr}_\mathbb{R}(\mathrm{ad}(X) \circ \mathrm{ad}(Y)) = 2\,\mathrm{Re}\,\mathrm{Tr}_\mathbb{C}(\mathrm{ad}(X) \circ \mathrm{ad}(Y)) = 2\,\mathrm{Re}\,B(X, Y)$. $\quad\square$

If $\lambda \in \mathfrak{g}_\mathbb{R}^*$ we denote by X_λ the dual of λ relative to $B_\mathbb{R}$, i.e., $\lambda(Y) = B_\mathbb{R}(X_\lambda, Y)$ for all $Y \in \mathfrak{g}_\mathbb{R}$.

Lemma 2.10. $\quad c_\eta = 2X_{\mathrm{Re}\,\eta}$.

Proof. For every $Y \in \mathfrak{g}$ we have

$$B_\mathbb{R}(2X_{\mathrm{Re}\,\eta}, Y) = 2\,\mathrm{Re}\,\eta(Y) = 2\,\mathrm{Re}\,B(c_\eta, Y) = B_\mathbb{R}(c_\eta, Y).$$

The result now follows from the non-degeneracy of $B_\mathbb{R}$. $\quad\square$

We assumed that all eigenvalues of $c = c_\eta$ are real. Because of Lemma 2.10 it follows that the element $X_{\mathrm{Re}\,\eta}$ is real hyperbolic in the real semisimple Lie algebra $\mathfrak{g}_\mathbb{R}$, in the sense of Section 6. Let B be a Borel subgroup of G containing T and such that the roots of $R_u(B)$ are non-negative on c. Then the parabolic subgroup $P = Z(c)B$ corresponds to the parabolic subgroup $P(\mathrm{Re}\,\eta)$ introduced in Section 6. Therefore, the results of that section apply to the present setting. In particular, the following result is a special case of Theorem 6.1.

Theorem 2.11. *Let $\eta \in \mathfrak{g}^*$ be such that $c = c_\eta$ belongs to $\mathrm{Lie}(T)$ and such that $\mathrm{ad}(c)$ has real eigenvalues. Then the projection*

$$G/Z(c) \to G/P$$

is a Lagrangian fibration with Lagrangian section $K/Z_K(\eta)$ relative to the symplectic form $\mathrm{Re}\,\Omega_\eta$. Moreover, there exists a unique symplectic isomorphism from this fibration onto the cotangent fibration $T^*(G/P) \to G/P$ equipped with the Liouville symplectic form, mapping $K/Z_K(\eta)$ to the zero section.

Theorem 1.2(b) follows from this result.

3. BACKGROUND IN SYMPLECTIC GEOMETRY

In this section we will discuss some background from symplectic geometry. Let M be a smooth manifold, and let $\pi : Z \to M$ be a fiber bundle whose total space Z is equipped with a symplectic form Ω. The bundle π is called *Lagrangian* if for each point $x \in M$ the fiber $\pi^{-1}(x)$ is a Lagrangian submanifold of Z. A section

$$s : M \to Z$$

is said to be *Lagrangian* if the image $s(M)$ is a Lagrangian submanifold of Z. If $\pi : Z \to M$ is Lagrangian, then by application of the Darboux theorem, it follows that for any point $z_0 \in Z$ there exists a Lagrangian section s of Z locally defined in a neighborhood of $m_0 = \pi(z_0)$ and with $s(m_0) = z_0$.

The following result is well known in basic symplectic geometry and can be found in [2], Sect. 4.2. See also [9], where the result is established in the context of Banach manifolds, with a useful review of the finite-dimensional case. A manifold M will be identified with a submanifold of its cotangent bundle T^*M through the zero section.

Theorem 3.1. *Let $\pi : Z \to M$ be a fiber bundle whose total space Z is equipped with a symplectic form Ω. Assume that:*

(1) π has Lagrangian fibers;
(2) π admits a Lagrangian section s.

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Let $p : T^*M \to M$ be the cotangent bundle of M equipped with the Liouville symplectic structure σ. Then there exists an open neighborhood U of M in T^*M and an open embedding $\varphi : U \to Z$ such that

(a) $\pi \circ \varphi = p$ on U;
(b) $\varphi = s$ on M;
(c) $\varphi^*(\Omega) = \sigma$.

If $\varphi' : U' \to T^*Z$ is a second such embedding, then $\varphi' = \varphi$ on an open neighborhood of M in $U \cap U'$.

Although this result is well known, we include a proof to prepare for our later arguments leading to the proof of Theorem 6.1, see also Theorem 2.11. The point is that there is a canonical way to define the map φ.

We agree to write n for the dimension of M. Then $s(M)$ is a submanifold of Z of dimension n. Since this submanifold is Lagrangian, the dimension of Z must be $2n$. The fibers of π have dimension n.

Let $x \in M$ and $\eta \in T_x^*M$. For each $z \in \pi^{-1}(x)$ we define a vector $H_\eta(z) \in T_z Z$ by the requirement that

$$(3.1) \qquad \Omega_z\big(X, H_\eta(z)\big) = \eta\big(d\pi(z)X\big), \quad \forall X \in T_z Z.$$

Since $d\pi(z) = 0$ on $T_z \pi^{-1}(x)$, which is a Lagrangian subspace of $T_z Z$, it follows that $H_\eta(z)$ belongs to this Lagrangian subspace. Hence $H_\eta(z)$ is tangent to the fiber $\pi^{-1}(x)$ at any of its points z. Accordingly, H_η will be viewed as an element of $\mathrm{Vect}(\pi^{-1}(x))$, the space of vector fields on $\pi^{-1}(x)$.

We will use the flows of these vector fields to define φ. The motivation for the above definition is the following relation to Hamilton vector fields of functions that are constant along the fibers of π.

Lemma 3.2. *Let $x \in M$, $\eta \in T_x^*M$ and let $\bar{f} : M \to \mathbb{R}$ be a smooth function such that $d\bar{f}(x) = \eta$. Let $f = \pi^*(\bar{f})$ and let H_f be the associated Hamilton vector field. Then*

$$H_f = H_\eta \quad \text{on } \pi^{-1}(x).$$

Proof. This is an immediate consequence of the definitions of H_η and H_f. $\square$

Corollary 3.3. *Let $x \in M$ and $\eta_1, \eta_2 \in T_x^*M$. Then H_{η_1} and H_{η_2} commute as vector fields on the fiber $\pi^{-1}(x)$.*

Proof. We select smooth functions $\bar{f}_j : M \to \mathbb{R}$ with $d\bar{f}_j(x) = \eta_j$ and define $f_j = \pi^*(\bar{f}_j)$. Then $H_{f_1} f_2 = 0$, hence $\{f_1, f_2\} = 0$ and it follows that H_{f_1} and H_{f_2} commute. These vector fields are tangent to the fiber $\pi^{-1}(x)$, hence their restrictions to the fiber commute. These restrictions equal H_{η_1} and H_{η_2} by the lemma above. $\square$

Proof of Theorem 3.1. If $\eta \in T_x^* M$ we denote by $t \mapsto e^{tH_\eta} s(x)$ the integral curve of H_η in $\pi^{-1}(x)$ with starting point $s(x)$. Its maximal interval of definition is denoted by I_η. There exists an open neighborhood U of M in $T^* M$ such that for each $x \in M$, the open set $T_x^* M \cap U$ is star shaped and for each $\eta \in T_x^* \cap U$ the interval I_η contains $(-2, 2)$. We define $\varphi : U \to Z$ by

$$(3.2) \qquad \varphi(\eta) = e^{H_\eta} s\big(p(\eta)\big), \quad \eta \in U.$$

Then φ is a local diffeomorphism at each point of M and coincides with an embedding on M. Shrinking U if necessary, we may arrange that in addition to the above, φ becomes a diffeomorphism from U onto an open neighborhood of $s(M)$ in Z. From the construction it is clear that (a) and (b) of Theorem 3.1 are satisfied.

We will now establish (c). As this is a local statement, we may assume that there exists a diffeomorphism $\bar{f} = (\bar{f}_1, \ldots, \bar{f}_n)$ from M onto an open subset of $\mathbb{R}^n$. Put $f = \pi^*(\bar{f}) = (f_1, \ldots, f_n)$. Then by Lemma 3.2 the Hamilton vector fields H_{f_i} are all tangent to the fibers of π, from which we deduce that $\Omega(H_{f_i}, H_{f_j}) = 0$, for all $1 \leqslant i, j \leqslant n$.

Define $g : \varphi(U) \to (\mathbb{R}^n)^*$ by

$$(3.3) \qquad g\big(\varphi(\eta)\big) = d\bar{f}\big(\pi(\eta)\big)^{-1*} \eta, \quad \eta \in U.$$

If $t = (t_1, \ldots, t_n) \in (\mathbb{R}^n)^*$ we agree to write $t\bar{f} = t \circ \bar{f} = t_1 \bar{f}_1 + \cdots + t_n \bar{f}_n$ and $\xi(x, t) = d(t\bar{f})(x) = t \circ d\bar{f}(x) = t_1 d\bar{f}_1(x) + \cdots + t_n d\bar{f}_n(x)$. Then

$$g_j\big(e^{t_1 H_{f_1}} \circ \cdots \circ e^{t_n H_{f_n}} s(x)\big) = g_j\big(e^{H_{\xi(x,t)}} s(x)\big) = \mathrm{pr}_j \, d\bar{f}(x)^{-1*} \xi(x, t) = t_j,$$

for (x, t) in a suitable neighborhood of the zero section in $M \times (\mathbb{R}^n)^*$. From this we see that $H_{f_i} g_j = \delta_{ij}$, so that $\Omega(H_{f_i}, H_{g_j}) = \delta_{ij}$ for all $1 \leqslant i, j \leqslant n$. The functions g_j are constant on the Lagrangian submanifold $s(M)$ of Z. Therefore, the vector fields H_{g_j} are tangent to $s(M)$, and it follows that $\{g_i, g_j\} = \Omega(H_{g_i}, H_{g_j}) = 0$ on $s(M)$. Now $H_{f_k} \{g_i, g_j\} = \{f_k, \{g_i, g_j\}\} = 0$ by application of the Jacobi identity. It follows that $\Omega(H_{g_i}, H_{g_j}) = 0$ on a suitable neighborhood of $s(M)$ in Z. We conclude that $\Omega = \sum_i df_i \wedge dg_j$ on this neighborhood, by evaluation on the vector fields H_{f_i}, H_{g_j}. Shrinking U if necessary, we may assume the identity to hold on $\varphi(U)$. Hence, $\Omega|_{\varphi(U)}$ is the pull-back under (f, g) of the standard symplectic form on $\bar{f}(M) \times (\mathbb{R}^n)^*$. Let $F : T^* M \to T^* \bar{f}(M) = \bar{f}(M) \times (\mathbb{R}^n)^*$ be the canonical symplectic isomorphism induced by $\bar{f}$. Then for (c) it suffices to prove that the following diagram commutes

$$
\begin{array}{ccc}
T^* M \supset U & \xrightarrow{\ \varphi\ } & \varphi(U) \subset Z \\
{\scriptstyle F} \searrow & & \swarrow {\scriptstyle (f,g)} \\
& \bar{f}(M) \times (\mathbb{R}^n)^* &
\end{array}
$$

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Let $\xi \in U$ and put $x = p(\xi)$. Then, by definition, $F(\xi) = (\bar{f}(x), d\bar{f}(x)^{-1*}\xi)$. On the other hand,

$$(f, g)(\varphi(\xi)) = (f(\pi(\varphi(\xi))), d\bar{f}(\pi(\xi))^{-1*}) = (\bar{f}(x), d\bar{f}(x)^{-1*}\xi),$$

by (3.3), and commutativity of the diagram follows.

It remains to establish uniqueness. Assume that φ satisfies the conditions of the theorem. We will show that it must be given by (3.2) in a neighborhood of the zero section. The cotangent bundle $p : T^*M \to M$ is Lagrangian, with M as a Lagrangian section. Hence, for $\eta \in T_x^*M$ and $\xi \in p^{-1}(x) = T^*M$, we may define $\widetilde{H}_\eta(\xi) \in T_\xi(p^{-1}(x))$ as H_η, but for the bundle p instead of π. Using that $\varphi^*\Omega = \sigma$, it is an easy matter to check from the definitions that

$$d\varphi(\xi)\widetilde{H}_\eta(\xi) = H_\eta(\varphi(\xi)),$$

for all ξ in a suitable neighborhood of M in T^*M. For the associated flows in the fibers $p^{-1}(x)$ and $\pi^{-1}(x)$ this implies that

$$\varphi \circ e^{t\widetilde{H}_\eta} = e^{tH_\eta} \circ \varphi.$$

A computation in local coordinates of M shows that $e^{t\widetilde{H}_\eta}\xi = \xi + t\eta$. On the other hand, $\varphi(0_x) = s(x) = s(\pi(\eta))$, and it follows that

$$\varphi(t\eta) = e^{tH_\eta}s(p(\eta)),$$

for all t in any interval containing zero on which both expressions are well defined. It follows that φ must be given by (3.2) on a suitable neighborhood of M in T^*M. $\quad\square$

4. REAL SEMISIMPLE GROUPS

In this section we recall some of the basic structure theory of real semisimple Lie groups and their Lie algebras. As a basic reference for this material we recommend [6].

Let G be a connected real semisimple group with finite center. The group G has a maximal compact subgroup K. All such are conjugate and connected. The Killing form B of $\mathfrak{g}$ is known to be negative definite on $\mathfrak{k}$ and positive definite on the orthocomplement $\mathfrak{p}$ of $\mathfrak{k}$. In particular,

(4.1) $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$

as a direct sum of linear spaces. It is known that $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$, $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$ and $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$. The decomposition (4.1) is called the Cartan decomposition of $\mathfrak{g}$ associated with the maximal compact subgroup K. It is readily seen that this decomposition is $\mathrm{Ad}(K)$-invariant.

The map $\theta : \mathfrak{g} \to \mathfrak{g}$ given by $\theta = I$ on $\mathfrak{k}$ and $\theta = -I$ on $\mathfrak{p}$ is called the associated Cartan involution. It commutes with the adjoint action of K. We define the bilinear

form $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$ by $\langle X, Y \rangle = -B(X, \theta Y)$. Then $\langle \cdot, \cdot \rangle$ is a K-invariant positive definite inner product on $\mathfrak{g}$; in other words, $\text{Ad}(K)$ acts by orthogonal transformations with respect to it. We note that $\text{ad}\,\mathfrak{p}$ consists of symmetric transformations.

It is known that the map $(k, X) \mapsto k \exp X$ is a real analytic diffeomorphism of $K \times \mathfrak{p}$ onto G. Define $\widetilde{\theta} : G \to G$ by $\widetilde{\theta}(k \exp X) = k \exp(-X)$, then it is readily verified that $\widetilde{\theta}$ is an involution of G with derivative equal to the Cartan involution θ of $\mathfrak{g}$. We agree to write θ for $\widetilde{\theta}$; this involution of G is also called the Cartan involution associated with K.

Remark 4.1. We note that if G is a complex semisimple group, then it may be viewed as a real semisimple Lie group with finite center. If K is a maximal compact subgroup, then $\mathfrak{p} = \sqrt{-1} \cdot \mathfrak{k}$, and θ is the involution associated with the real form $\mathfrak{k}$.

On the other hand, if G is linear, then G has a complexification $G^{\mathbb{C}}$ and

$$\widetilde{\mathfrak{k}} = \mathfrak{k} \oplus \sqrt{-1} \cdot \mathfrak{p}$$

is the Lie algebra of a maximal compact subgroup of $G^{\mathbb{C}}$.

Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$. It is known that all such are conjugate under K. For each linear functional $\lambda \in \mathfrak{a}^*$, we put

(4.2) $\qquad \mathfrak{g}_\lambda := \{ X \in \mathfrak{g} \mid [H, X] = \lambda(H) X, \ \forall H \in \mathfrak{a} \}.$

Since $\mathfrak{a}$ is abelian, and $\text{ad}(H)$ is symmetric for $\langle \cdot, \cdot \rangle$, for all $H \in \mathfrak{a}$, the adjoint representation of $\mathfrak{a}$ in $\mathfrak{g}$ has a simultaneous diagonalization. It follows that $\mathfrak{g}$ decomposes as a finite direct sum of joint eigenspaces of the form (4.2). Let Σ be the set of nonzero $\lambda \in \mathfrak{a}^*$ with $\mathfrak{g}(\lambda) \neq 0$. Then

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Sigma} \mathfrak{g}_\alpha.$$

It is known that $(\mathfrak{a}, \Sigma)$ is a root system, which is possibly non-reduced. A root $\alpha \in \Sigma$ is called reduced if $\frac{1}{2}\alpha$ is not a root. The set Σ_0 of all reduced roots forms a genuine root system in $\mathfrak{a}^*$. For each $\alpha \in \Sigma$ there exists a unique $\alpha_0 \in \Sigma_0$ such that $\alpha \in \{\alpha_0, 2\alpha_0\}$. We note that $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset [\mathfrak{g}_{\alpha+\beta}]$ for all $\alpha, \beta \in \Sigma$.

A positive system for Σ is a subset Π of Σ such that $\Sigma = \Pi \cup (-\Pi)$, and Π and $-\Pi$ are separated by a hyperplane in $\mathfrak{a}^*$, i.e., there exists a $H \in \mathfrak{a}$ such that $\Pi = \{\alpha \in \Sigma \mid \alpha(H) > 0\}$. It follows that $\Pi \mapsto \Pi_0 := \Pi \cap \Sigma_0$ defines a bijection from the set of positive systems of Σ onto the set of positive systems for Σ_0. Let $\mathfrak{a}^{\text{reg}}$ be the complement in $\mathfrak{a}$ of the union of all root hyperplanes $\ker \alpha$ for $\alpha \in \Sigma$. Then the connected components of $\mathfrak{a}^{\text{reg}}$ are called the open Weyl chambers of the root system Σ. There is an obvious bijection between the set of all such chambers and the set of positive systems for Σ.

Clearly, $\mathfrak{g}_0$ is the centralizer of $\mathfrak{a}$ in $\mathfrak{g}$. As $\theta = -I$ on $\mathfrak{a}$, it follows that $\mathfrak{g}_0$ is invariant under θ. The centralizer of $\mathfrak{a}$ in $\mathfrak{k}$ is denoted by $\mathfrak{m}$. Since $\mathfrak{a}$ is maximal abelian in $\mathfrak{p}$, the intersection $\mathfrak{g}_0 \cap \mathfrak{p}$ equals $\mathfrak{a}$. Therefore,

$$\mathfrak{g}_0 = \mathfrak{m} \oplus \mathfrak{a}.$$

Remark 4.2. In the notation of Remark 4.1, $t := \sqrt{-1} \cdot \mathfrak{a} \oplus \mathfrak{a}$ is a maximal torus of $\mathfrak{g}$ whose intersection with $\mathfrak{k}$ is a maximal torus of $\mathfrak{k}$. Moreover, $\mathfrak{a}$ is the real subspace of t consisting of all points on which the roots of t are real. Let R be the set of t-roots. Then restriction to $\mathfrak{a}$ induces an isomorphism $R \to \Sigma$. In particular, the root system Σ is reduced in this setting. Accordingly, the root spaces for t coincide with those for $\mathfrak{a}$. Finally, $\mathfrak{g}_0 = t$ and $\mathfrak{m} = t \cap \mathfrak{k} = \sqrt{-1} \cdot \mathfrak{a}$.

Fix a positive system Σ^+ for Σ. Let $\mathfrak{n}$ be the sum of all positive root spaces $\mathfrak{g}_\alpha$, for $\alpha \in \Sigma^+$. Since $\theta = -I$ on $\mathfrak{a}$, we have

$$\theta(\mathfrak{g}_\alpha) = \mathfrak{g}_{-\alpha},$$

for every $\alpha \in \Sigma$. Hence,

$$\mathfrak{g} = \theta(\mathfrak{n}) \oplus \mathfrak{g}_0 \oplus \mathfrak{n}.$$

As $\mathfrak{k}$ is the eigenspace of θ for the eigenvalue 1, we see that

$$(4.3) \qquad \mathfrak{k} = \mathfrak{m} \oplus \sum_{\alpha \in \Sigma^+, X \in \mathfrak{g}_\alpha} \left(X + \theta(X) \right).$$

It now follows that

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n},$$

as a direct sum of vector spaces. The exponential map $\exp : \mathfrak{g} \to G$ maps $\mathfrak{a}$ and $\mathfrak{n}$ diffeomorphically onto closed subgroups A and N of G, respectively. Moreover, one has the so-called Iwasawa decomposition

$$(4.4) \qquad G = KAN,$$

the multiplication map $(k, a, n) \mapsto kan$ being a diffeomorphism $K \times A \times N \to G$.

5. PARABOLIC SUBGROUPS

We recall that a Borel subalgebra of the complexified semisimple Lie algebra $\mathfrak{g}^{\mathbb{C}}$ is by definition a maximal solvable subalgebra. A subalgebra of $\mathfrak{g}^{\mathbb{C}}$ which contains a Borel subalgebra is said to be parabolic. It is well known that such a subalgebra equals its own normalizer in $\mathfrak{g}^{\mathbb{C}}$.

A parabolic subalgebra of $\mathfrak{g}$ is defined to be a subalgebra $\mathfrak{P}$ whose complexification $\mathfrak{P}^{\mathbb{C}}$ is parabolic in $\mathfrak{g}^{\mathbb{C}}$. Such an algebra $\mathfrak{P}$ equals its own normalizer in $\mathfrak{g}$.

A parabolic subgroup of G is defined to be a subgroup P which is the normalizer of a parabolic subalgebra $\mathfrak{P}$ of $\mathfrak{g}$. Being its own normalizer, $\mathfrak{P}$ is the Lie algebra of P. We proceed by describing the basic structure theory of parabolic subgroups of G. Details can be found in, e.g., [8], p. 279.

The algebra $\mathfrak{m} + \mathfrak{a} + \mathfrak{n} = \mathfrak{g}_0 + \mathfrak{n}$ is a parabolic subalgebra of $\mathfrak{g}$. It is known to be minimal in the sense that it does not contain any strictly smaller parabolic subalgebra. The associated minimal parabolic subgroup of G is given by

$$P_0 = MAN.$$

Note that this decomposition is compatible with the Iwasawa decomposition (4.4). In particular, the multiplication map $M \times A \times N \to P_0$ is a diffeomorphism. Moreover, from the Iwasawa decomposition (4.4) it follows that

(5.1) $G = KP_0.$

It is known that every parabolic subgroup of G is conjugate to one containing P_0. The parabolic subgroups containing P_0 are finite in number, and may be described as follows. Let

$$\bar{\mathfrak{a}}^+ := \{H \in \mathfrak{a} \mid \alpha(H) \geqslant 0, \forall \alpha \in \Pi\}$$

be the closed positive Weyl chamber in $\mathfrak{a}$. Given $c \in \bar{\mathfrak{a}}^+$ we define

$$\mathfrak{P}(c) = \bigoplus_{\alpha \in \Sigma, \alpha(c) \geqslant 0} \mathfrak{g}_\alpha.$$

Clearly, this is a subalgebra of $\mathfrak{g}$ containing $\mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$, hence parabolic. Moreover, it depends on c through the set $\Pi(c) := \{\alpha \in \Pi \mid \alpha(c) > 0\}$. It can be shown that every parabolic subalgebra of $\mathfrak{g}$ containing $\mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$ is of the form $\mathfrak{P}(c)$ for some $c \in \bar{\mathfrak{a}}^+$. In particular, we see that these parabolic subalgebras are finite in number.

Let $\mathfrak{n}(c)$ be the sum of the root spaces $\mathfrak{g}_\alpha$ for $\alpha \in \Pi(c)$. Then

$$\mathfrak{P}(c) = \mathfrak{z}(c) \oplus \mathfrak{n}(c).$$

As $\mathfrak{z}(c)$ is reductive and normalizes the nilpotent subalgebra $\mathfrak{n}(c)$, this is a Levi decomposition of $\mathfrak{P}(c)$. In particular, $\mathfrak{n}(c)$ is the nilpotent radical of $\mathfrak{P}(c)$. Since $\mathfrak{n}(c) \subset \mathfrak{n}$, the exponential map maps $\mathfrak{n}(c)$ diffeomorphically onto a closed subgroup $N(c)$ of G. Let $P(c)$ be the normalizer of $\mathfrak{P}(c)$ in G, then we have the semi direct product decomposition

$$P(c) = Z(c) \ltimes N(c).$$

In particular, $N(c)$ is the unipotent radical of P.

Finally, we note that $\mathfrak{P}(c)$ is the sum of the eigenspaces for the nonnegative eigenvalues of $\text{ad}(c)$. Now this definition can be given for any element $c \in \mathfrak{g}$ which is *real hyperbolic*, i.e., for which $\text{ad}(c)$ diagonalizes with real eigenvalues. Moreover, $\text{Ad}(x)\mathfrak{P}(c) = \mathfrak{P}(\text{Ad}(x)c)$, for each $x \in G$. It is known that every real hyperbolic element is conjugate to an element of $\bar{\mathfrak{a}}^+$. From what we just said, it follows that every algebra of the form $\mathfrak{P}(c)$, with c real hyperbolic, is a parabolic subalgebra of $\mathfrak{g}$. Moreover, since all minimal parabolic subalgebras are conjugate, it follows that every parabolic subalgebra arises in this way.

We retain the assumption that $c \in \bar{\mathfrak{a}}^+$. Let $P = P(c)$ be the associated parabolic subgroup of G. Since $G/Z(c) \simeq G \times_P P/Z(c)$, the natural projection

$$\pi : G/Z(c) \to G/P$$

gives the quotient manifold $G/Z(c)$ the structure of a real analytic fiber bundle over the real flag manifold G/P, with fiber $P/Z(c) \simeq N(c)$.

We note that P contains P_0, so that $G = KP$, by (5.1). It follows that the inclusion map $K \to G$ induces a diffeomorphism $K/K \cap P \simeq G/P$. Now $K \cap P = K \cap P \cap \theta P = K \cap Z(c)$. Put $Z_K(c) = K \cap Z(c)$. Then

$$(5.2) \qquad G/Z(c) \simeq KP/Z(c) \simeq K \times_{Z_K(c)} P/Z(c) \simeq K \times_{Z_K(c)} N(c),$$

exhibiting $G/Z(c)$ as a K-equivariant real analytic vector bundle over $K/Z_K(c) \simeq G/P$.

From (4.3) it follows that the map $X \mapsto X + \theta X$ induces a linear isomorphism from $\mathfrak{n}(c)$ onto $\mathfrak{k}/\mathfrak{k} \cap \mathfrak{z}(c)$. This implies that $K/Z_K(c)$ is a submanifold of $G/Z(c)$ of half the dimension.

6. REAL FLAG MANIFOLDS

Let G be a connected real semisimple Lie group with finite center, and let $\lambda \in \mathfrak{g}^*$ be a real linear functional. We write $X_\lambda = B^{-1}(\lambda)$, i.e.,

$$\lambda(Y) = B(X_\lambda, Y), \quad Y \in \mathfrak{g}.$$

The element λ is called *real hyperbolic* if $\mathrm{ad}(X_\lambda) \in \mathrm{End}(\mathfrak{g})$ is diagonalizable with real eigenvalues. From now on we assume λ to be real hyperbolic.

From the discussion in the previous section we know that the element X_λ is conjugate to an element of the positive chamber in $\mathfrak{a}$. Thus, for the purpose of studying the symplectic geometry of the coadjoint orbit through λ, we may – and will – assume that X_λ is contained in the positive chamber in $\mathfrak{a}$ from the start.

Let $\Omega = \Omega_\lambda$ be the Kostant–Kirillov symplectic form on the coadjoint orbit $G \cdot \lambda$. The centralizer $Z(\lambda)$ of λ in G equals $Z(X_\lambda)$, by invariance of the Killing form. Via the natural G-equivariant diffeomorphism

$$G/Z(\lambda) \to G \cdot \lambda,$$

the form Ω may be pulled back to a symplectic form on $G/G(\lambda)$. For convenience, the latter form will also be denoted by Ω.

With notation as in the previous section, we write $\mathfrak{n}(\lambda) = \mathfrak{n}(X_\lambda)$ and $\mathfrak{P}(\lambda) = \mathfrak{P}(X_\lambda)$, and likewise $N(\lambda) = N(X_\lambda)$ and $P(\lambda) = P(X_\lambda)$. Then $P(\lambda)$ is a parabolic subgroup of G with Levi decomposition $P(\lambda) = Z(\lambda)N(\lambda)$.

The projection

$$(6.1) \qquad \pi : G/Z(\lambda) \to G/P(\lambda)$$

is a G-equivariant fibration with fibers equal to the G-translates of $N(\lambda) \hookrightarrow G/Z(\lambda)$. On the other hand, the cotangent bundle $T^*(G/P(\lambda))$ comes equipped with the natural Liouville symplectic form σ.

In this section we will prove the following result.

Theorem 6.1. *There exists a unique diffeomorphism*

$$\varphi_\lambda : T^*\big(G/P(\lambda)\big) \to G/Z(\lambda)$$

satisfying the following conditions:

(a) $\pi \circ \varphi_\lambda$ *equals the projection of* $T^*(G/P(\lambda))$;
(b) φ_λ *maps the zero-section of* $T^*(G/P(\lambda))$ *to* $K/Z_K(\lambda)$;
(c) $\varphi_\lambda^*(\Omega_\lambda) = (\sigma)$.

The proof of this result is based on the ideas of the proof of Theorem 3.1 and will be given through a number of lemmas. We start by observing that the symplectic form Ω on $G/Z(\lambda)$ is G-invariant. At the origin $\bar{e} = eZ(\lambda)$ it is given by

$$(6.2) \qquad \Omega_{\bar{e}}(X_{\bar{e}}, Y_{\bar{e}}) = \lambda([X, Y]) = B(X_\lambda, [X, Y]) = -B(X, [X_\lambda, Y]),$$

for $X, Y \in \mathfrak{g}$.

The following lemma expresses that we are in the set-up of Theorem 3.1.

Lemma 6.2.

(a) *The fibers of* π, *defined in (6.1), are Lagrangian for* Ω.
(b) *The submanifold* $K/Z_K(\lambda) \hookrightarrow G/Z(\lambda)$ *is Lagrangian for* Ω.

Proof. From (6.2) one sees that $\Omega_{\bar{e}}$ vanishes on $\mathfrak{n} \times \mathfrak{n}$, so that $T_{\bar{e}}(N(\lambda)\bar{e})$ is isotropic in $T_{\bar{e}}(G/Z(\lambda))$. We agree to write $\bar{\mathfrak{n}} = \theta\mathfrak{n}$ and $\bar{\mathfrak{n}}(\lambda) = \theta\mathfrak{n}(\lambda)$. From the decomposition

$$\mathfrak{g} = \bar{\mathfrak{n}}(\lambda) \oplus \mathfrak{z}(\lambda) \oplus \mathfrak{n}(\lambda)$$

one sees that $\dim \mathfrak{g}/\mathfrak{z}(\lambda) = 2 \cdot \dim \mathfrak{n}$. Hence, the orbit $N(\lambda)\bar{e}$ is Lagrangian in $G/Z(\lambda)$. By equivariance, the fibers $gN(\lambda)\bar{e}$ are Lagrangian as well. This establishes assertion (a).

For (b) we observe that for $X, Y \in \mathfrak{k}$ we have

$$\Omega_{\bar{e}}(X, Y) = B(X_\lambda, [X, Y]) = B(\theta X_\lambda, [\theta X, \theta Y])$$
$$= -B(X_\lambda, [X, Y]) = -\Omega_{\bar{e}}(X, Y).$$

This implies that $T_{\bar{e}}(K/Z_K(\lambda))$ is isotropic in $T_{\bar{e}}(G/Z(\lambda))$. By K-invariance, $K/Z_K(\lambda)$ is isotropic in $G/Z(\lambda)$. In the text below equation (5.2), we observed that $K/Z_K(\lambda)$ has dimension equal to half the dimension of $G/Z(\lambda)$. $\square$

We proceed by following the ideas of the proof of Theorem 3.1. Our first goal is to define suitable vector fields along the fibers of the fibration (6.1). The natural projection $G \to G/P(\lambda)$ induces an isomorphism from $\mathfrak{g}/\mathfrak{P}(\lambda)$ onto the tangent space $T_{eP(\lambda)}(G/P(\lambda))$ which we use for identification of the two spaces.

Let $\eta \in (\mathfrak{g}/\mathfrak{P}(\lambda))^*$ and $n \in N(\lambda)$. Since $\Omega_{n\bar{e}}$ is non-degenerate on $T_{n\bar{e}}G/G(\lambda)$ we may define a tangent vector $H_{\lambda,\eta}(n\bar{e}) = H_\eta(n\bar{e}) \in T_{n\bar{e}}G/Z(\lambda)$ by

$$(6.3) \qquad \Omega_{n\bar{e}}\big(Z, H_\eta(n\bar{e})\big) = \eta\big(d\pi(n\bar{e})Z\big),$$

for all $Z \in T_{n\bar{e}}G/Z(\lambda)$. Viewing η as an element of $T^*_{eP(\lambda)}(G/P(\lambda))$, we see that this tangent vector coincides with the vector $H_\eta(n\bar{e})$ defined in (3.1). In particular, $H_\eta(n\bar{e})$ is tangent to the fiber $N(\lambda)\bar{e}$.

If X is a homogeneous space for G, and $g \in G$, we denote by l_g the left multiplication $x \mapsto gx$ on X. For the space $G/P(\lambda)$ we note that $dl_g(eP(\lambda))$ is a linear isomorphism from $\mathfrak{g}/\mathfrak{P}(\lambda)$ onto $T_{gP(\lambda)}(G/P(\lambda))$. Given $\eta \in (\mathfrak{g}/\mathfrak{P}(\lambda))^*$ and $g \in G$, we put

$$g \cdot \eta := dl_g\big(eP(\lambda)\big)^{-1*}\eta = \eta \circ dl_g\big(eP(\lambda)\big)^{-1}.$$

Lemma 6.3. *Let $g \in G$ and $\eta \in (\mathfrak{g}/\mathfrak{P}(\lambda))^*$. Let $H_{g\cdot\eta} \in \mathrm{Vect}(gN(\lambda)\bar{e})$ be defined as in (3.1), for the bundle $\pi : G/Z(\lambda) \to G/P(\lambda)$. Then*

$$H_{g\cdot\eta}(gn\bar{e}) = dl_g(\bar{e})H_\eta(n\bar{e}), \quad n \in N(\lambda).$$

Proof. This is a straightforward consequence of the G-invariance of Ω and the G-equivariance of the projection map $\pi : G/Z(\lambda) \to G/P(\lambda)$. $\square$

At a later stage the situation that $g = m \in P(\lambda)$ will be of particular interest to us. As left translation by $P(\lambda)$ fixes the element $eP(\lambda)$ of $G/P(\lambda)$, we see that $(m, \eta) \mapsto m \cdot \eta$ defines an action of $P(\lambda)$ on $(\mathfrak{g}/\mathfrak{P}(\lambda))^* \simeq T_{eP(\lambda)}(G/P(\lambda))$.

Lemma 6.4. *The action $(m, \eta) \mapsto m \cdot \eta$ of $P(\lambda)$ on $(\mathfrak{g}/\mathfrak{P}(\lambda))^*$ is induced by the adjoint action of $P(\lambda)$ on $\mathfrak{g}$.*

Proof. Let $\mathrm{pr}: G \to G/P(\lambda)$ be the natural projection and for $m \in P(\lambda)$, let $C_m : G \to G$ denote the conjugation map $x \mapsto mxm^{-1}$. Then $\mathrm{pr} \circ C_m = l_m \circ \mathrm{pr}$. Differentiating this expression at the identity element, we see that $dl_m(eP(\lambda) \in \mathrm{End}(\mathfrak{g}/\mathfrak{P}(\lambda))$ is induced by $dC_m(e) = \mathrm{Ad}(m)$. The result follows. $\square$

We will now derive a formula for the vector field H_η along $N(\lambda)\bar{e}$ that will allow us to understand the global behavior of its flow.

Lemma 6.5. *Let $n \in N(\lambda)$. The map $\eta \mapsto H_\eta(n\bar{e})$ is a linear isomorphism from $(\mathfrak{g}/\mathfrak{P}(\lambda))^*$ onto $T_{n\bar{e}}(N(\lambda)\bar{e})$.*

Proof. Since $\Omega_{n\bar{e}}$ is non-degenerate, the map is injective. The expression on the right-hand side of (6.3) vanishes for all $Z \in T_{n\bar{e}}(N(\lambda)\bar{e})$. Since $T_{n\bar{e}}(N(\lambda)\bar{e})$ is Lagrangian for the form $\Omega_{n\bar{e}}$ it follows that $H_\eta(n\bar{e})$ belongs to $T_{n\bar{e}}(N(\lambda)\bar{e})$. The result now follows for dimensional reasons. $\square$

For each point $n \in N(\lambda)$ the natural map $n \mapsto n\bar{e}$ is an embedding of $N(\lambda)$ onto the closed submanifold $N(\lambda)\bar{e} = \pi^{-1}(eP(\lambda))$ of $G/Z(\lambda)$. The derivative of this embedding is a linear isomorphism from $T_n N(\lambda)$ onto $T_{n\bar{e}} N(\lambda)\bar{e}$ through which we shall identify these spaces.

Since $\mathfrak{n}(\lambda)$ and $\mathfrak{P}(\lambda)$ are perpendicular for the Killing form B, the map $X \mapsto -B(X, \cdot)$ induces a linear map

$$(6.4) \qquad V \mapsto \eta_V, \qquad \mathfrak{n}(\lambda) \to \big(\mathfrak{g}/\mathfrak{P}(\lambda)\big)^*.$$

As B is non-degenerate, the map (6.4) is a linear isomorphism onto. Given $V \in \mathfrak{n}(\lambda)$ we agree to write

$$H_V(n) = H_{\eta_V}(n\bar{e}), \quad n \in N(\lambda),$$

viewed as an element of $T_n N(\lambda)$. Accordingly, H_V becomes a vector field on $N(\lambda)$. The following lemma gives an explicit formula for this vector field. It involves the endomorphism

$$(6.5) \qquad T_\lambda := \mathrm{ad}(X_\lambda)|_{\mathfrak{n}(\lambda)} \in \mathrm{End}\big(\mathfrak{n}(\lambda)\big).$$

As the roots of $\mathfrak{n}(\lambda)$ are positive on X_λ, this endomorphism is invertible. From (6.2) we see that

$$(6.6) \qquad \Omega_{\bar{e}}\big(X + \mathfrak{z}(\lambda), Y + \mathfrak{z}(\lambda)\big) = -B(X, T_\lambda Y),$$

for all $X, Y \in \mathfrak{g}$.

Lemma 6.6. *Let $V \in \mathfrak{n}(\lambda)$. Then for each $n \in N(\lambda)$,*

$$(6.7) \qquad H_V(n) = dl_n(e) \circ T_\lambda^{-1} \circ \mathrm{Ad}(n)^{-1}(V).$$

Proof. Let $\eta = \eta_V$. From (6.3) it follows that for every $X \in \mathfrak{g}$ we have

$$\Omega_{\bar{e}}\big(X + \mathfrak{P}(\lambda), H_\eta(\bar{e})\big) = \eta\big(X + \mathfrak{P}(\lambda)\big)$$
$$= -B(V, X).$$

On the other hand, since $H_\eta(\bar{e}) = H_V(e) + \mathfrak{P}(\lambda)$ it follows from (6.6) that

$$\Omega_{\bar{e}}\big(X + \mathfrak{P}(\lambda), H_\eta(\bar{e})\big) = -B\big(X, T_\lambda H_V(e)\big).$$

Comparing the two equalities, and using that the Killing form is non-degenerate, we find that

$$(6.8) \qquad H_V(e) = T_\lambda^{-1} V.$$

This establishes (6.7) for $n = e$.

To establish the formula in general, we observe that from Lemma 6.4 it follows that for every $n \in N(\lambda)$ we have

$$n \cdot [\eta_V] = \eta_V \circ \mathrm{Ad}(n)^{-1} = \eta_{\mathrm{Ad}(n)V}.$$

From Lemma 6.3 with $g = n^{-1}$ we now infer that

$$H_{\eta_V}(n\bar{e}) = H_{\eta_{\mathrm{Ad}(n)^{-1}V}}(\bar{e}).$$

Therefore,

$$H_V(n) = dl_n(e)H_{\mathrm{Ad}(n)^{-1}V}(e) = dl_n(e) \circ T_\lambda^{-1} \circ \mathrm{Ad}(n)^{-1}(V). \qquad \square$$

Proposition 6.7. *For every pair $V_1, V_2 \in \mathfrak{n}(\lambda)$, the associated vector fields H_{V_1} and H_{V_2} in $\mathrm{Vect}(N(\lambda))$ commute. Moreover, the flows of these vector fields are well defined as maps $\mathbb{R} \times N(\lambda) \to N(\lambda)$. The associated map*

$$V \mapsto \exp(H_V)e_N$$

induces a diffeomorphism from $\mathfrak{n}(\lambda)$ onto $N(\lambda)$.

Proof. The first assertion follows from Corollary 3.3 applied to the bundle $\pi : G/Z(\lambda) \to G/P(\lambda)$.

Let now $V \in \mathfrak{n}(\lambda)$. In our study of the flow of the vector field $H_V \in \mathrm{Vect}(N(\lambda))$ formula (6.7) will play a crucial role. Let h_V denote the pull back of the vector field H_V under the exponential map $\exp : \mathfrak{n}(\lambda) \to N(\lambda)$. Thus,

$$(6.9) \qquad h_V(U) = d\exp(U)^{-1}H_V(\exp U).$$

Now

$$(6.10) \qquad d\exp(U) = dl_{\exp U}(e) \circ [I + R(\mathrm{ad}\,U)],$$

where R is the analytic function $\mathbb{R} \to \mathbb{R}$ given by the convergent power series

$$R(t) = \frac{1 - e^{-t}}{t} - 1 = \sum_{n \geqslant 1} \frac{(-t)^n}{(n+1)!}.$$

Since $\mathfrak{n}(\lambda)$ is nilpotent, there exists a smallest positive integer N_0 such that $(\mathrm{ad}\,U)^{N_0+1} = 0$ for all $U \in \mathfrak{n}(\lambda)$. It follows that formula (6.10) is also valid with the polynomial

$$R(t) = \sum_{n=1}^{N_0} \frac{(-t)^n}{(n+1)!}.$$

Combining (6.7), (6.9) and (6.10), we see that the vector field h_V on $\mathfrak{n}(\lambda)$ is given

by

$$h_V(U) = [I + R(\operatorname{ad} U)]^{-1} \circ T_\lambda^{-1} \circ e^{-\operatorname{ad} U}(V)$$
$$= V + \rho_\lambda(\operatorname{ad} U)(V),$$

with $\rho_\lambda = \rho \in \mathbb{R}[t]$ a polynomial divisible by t; see (6.5) for the definition of T_λ.

We recall that, by assumption, the element $\operatorname{ad} X_\lambda$ diagonalizes with real eigenvalues. Let

$$\nu_1 < \nu_2 < \cdots < \nu_p$$

be the positive eigenvalues, and let $\mathfrak{n}(\nu_j)$ be the eigenspace associated with the eigenvalue ν_j. Let d_j be the dimension of this eigenspace. The sum of the eigenspaces $\mathfrak{n}(\nu_j)$ equals $\mathfrak{n}(\lambda)$. Accordingly, we fix a basis of eigenvectors $V_1, \ldots, V_n$ for $\operatorname{ad} X_\lambda$ in $\mathfrak{n}(\lambda)$, and put $\mathfrak{n}_j = \mathbb{R} V_j$. By choosing a suitable numbering we may arrange that for each k,

$$\bigoplus_{i \leqslant k} \mathfrak{n}(\nu_i) = \bigoplus_{j \leqslant d_1 + \cdots + d_k} \mathfrak{n}_j.$$

Given $1 \leqslant k \leqslant n$, we put

$$\mathfrak{n}_{\geqslant k} = \bigoplus_{j \geqslant k} \mathfrak{n}_j.$$

In addition, we put $\mathfrak{n}_{\geqslant k} = 0$ for $k > n$. The subspaces $\mathfrak{n}_{>k}, \mathfrak{n}_{\leqslant k}$ and $\mathfrak{n}_{<k}$ of $\mathfrak{n}(\lambda)$ are defined in a similar fashion.

For $1 \leqslant j \leqslant n$, we denote by pr_j the projection map $\mathfrak{n}(\lambda) \to \mathfrak{n}_j$ along the remaining summands $\mathfrak{n}_i, i \neq j$. Also, we define $\operatorname{pr}_{\geqslant k} = \sum_{j \geqslant k} \operatorname{pr}_j$. The projections $\operatorname{pr}_{>k}, \operatorname{pr}_{\leqslant k}$ and $\operatorname{pr}_{<k}$ are defined in a similar fashion.

By the Jacobi identity, $\operatorname{ad} X_\lambda$ acts on $[\mathfrak{n}(\nu_i), \mathfrak{n}(\nu_j)]$ as the scalar multiplication through $\nu_i + \nu_j$. Hence, for each $k \geqslant 1$,

$$[\mathfrak{n}(\lambda), \mathfrak{n}_k] \subset \mathfrak{n}_{>k}.$$

Let $V \in \mathfrak{n}(\lambda)$. Then the integral curve $t \mapsto U(t)$ of the vector field h_V with initial point $U(0) = U_0$ is determined by the initial value problem

(6.11) $U'(t) = V + \rho\big(\operatorname{ad}[U(t)]\big)(V), \quad U(0) = U_0.$

For the component $U_1 := \operatorname{pr}_1 \circ U$ in $\mathfrak{n}_1$ the equation becomes

$$U_1'(t) = \operatorname{pr}_1 V, \quad U_1(0) = \operatorname{pr}_1(U_0).$$

Indeed, $\rho(\operatorname{ad} U)V$ has its values in $\mathfrak{n}_{>1}$. The equation for U_1 has the solution

$$U_1(t) = t \operatorname{pr}_1 V + \operatorname{pr}_1(U_0),$$

which is linear in t. The remaining components may now be obtained by a recurrence procedure and integration. More precisely, let $k \geqslant 2$, assume that $U_j := \mathrm{pr}_j \circ U$ has been solved for each $1 \leqslant j \leqslant k-1$, and put

$$U_{<k} = \sum_{j<k} U_j.$$

Then U_k is determined by the initial value problem

$$\begin{cases} U_k'(t) = \mathrm{pr}_k V + \mathrm{pr}_k \circ \rho\big(\mathrm{ad}[U(t)]\big)(V) = \mathrm{pr}_k V + \mathrm{pr}_k \circ \rho\big(\mathrm{ad}[U_{<k}(t)]\big)(V), \\ U_k(0) = \mathrm{pr}_k(U_0). \end{cases}$$

This equation may be solved directly by integration. By induction one sees that the integral curve is defined for all $t \in \mathbb{R}$ and is in fact a polynomial function of t.

For the final part of the proof it is important to make the following observation. If $V \in \mathfrak{n}_{\geqslant k}$, then the integral curve $U(t)$ satisfies

(6.12) $U(t) - U_0 - tV \in \mathfrak{n}_{>k}$

for all $t \in \mathbb{R}$. Indeed, this follows by applying the projection $\mathrm{pr}_{\leqslant k}$ to the constituents of the equation (6.11) and solving the resulting equation.

The vector fields $h_j := h_{V_j}$, where $1 \leqslant j \leqslant n$, form a collection of commuting vector fields for which the flows are defined globally. Consequently, the associated flow maps $(t, c) \mapsto e^{th_j} c$ are smooth maps $\mathbb{R} \times \mathfrak{n}(\lambda) \to \mathfrak{n}(\lambda)$. Let $1 \leqslant k \leqslant n$. It follows from (6.12) that

(6.13) $\mathrm{pr}_{\leqslant k} e^{th_k} x = \mathrm{pr}_{\leqslant k} x + t V_k,$

for all $x \in \mathfrak{n}(\lambda)$ and $t \in \mathbb{R}$. Since the vector fields h_j commute, the map $\varphi : \mathbb{R}^n \times \mathfrak{n}(\lambda) \to \mathfrak{n}(\lambda)$ given by

$$\varphi(t, x) = e^{t_1 h_1} \circ \cdots \circ e^{t_n h_n}(x),$$

defines a smooth action of $(\mathbb{R}^n, +, 0)$ on $\mathfrak{n}(\lambda)$. It follows by repeated application of (6.13) that

$$\mathrm{pr}_k \varphi(0, t_k, \ldots, t_n, x) = \mathrm{pr}_k x + t_k V_k,$$

for all $t_k, \ldots, t_n \in \mathbb{R}$ and $x \in \mathfrak{n}(\lambda)$. We will use this observation to show that the action φ is proper.

For the proof of this, it is convenient to have the following notation. For a compact subset $C \subset \mathfrak{n}(\lambda)$ we write

$$T(C) := \mathrm{pr}_{\mathbb{R}^n}\big(\varphi^{-1}(C) \cap \mathbb{R}^n \times C\big) = \big\{t \in \mathbb{R}^n \mid \exists c \in C \colon \varphi(t, c) \in C\big\}.$$

For proving properness of the action, it suffices to show that for every compact subset $C \subset \mathfrak{n}(\lambda)$, the set $T(C)$ defined above is bounded. Indeed, this implies that $\varphi^{-1}(C) \cap \mathbb{R}^n \times C$ is compact. For $1 \leqslant k \leqslant n$, let $\pi_{\leqslant k} : \mathbb{R}^n \to \mathbb{R}^k$ be projection onto

the first coordinates. By induction on k we will show that for every compact set $C \subset \mathfrak{n}(\lambda)$ the set $\pi_{\leqslant k}(T(C))$ is bounded.

First, let $k = 1$, and let $t \in T(C)$. Then there exists some $c \in C$ such that $\varphi(t, c) \in C$. Since

$$\mathrm{pr}_1 \varphi(t, c) = \mathrm{pr}_1 c + t_1 V_1,$$

it follows that $t_1 V_1$ belongs to the vectorial sum $-\mathrm{pr}_1(C) + \mathrm{pr}_1(C)$, which shows that $\pi_{\leqslant 1}(T(C))$ is bounded.

Next, let $1 \leqslant k < n$, and assume that $\mathrm{pr}_{\leqslant k}(T(C))$ is bounded for every compact subset $C \subset \mathfrak{n}(\lambda)$. Let $t \in T(C)$. Then there exists a $c \in C$ such that $\varphi(t, c) \in C$. The element $(t_1, \ldots, t_k)$ lies in the subset

$$S := \mathrm{cl}\, \mathrm{pr}_{\leqslant k}\big(T(C)\big)$$

of $\mathbb{R}^k$, which is compact by the inductive hypothesis. It follows that

$$\varphi(0, t_{k+1}, \ldots, t_n, c) = e^{-t_1 h_1} \cdots e^{-t_k h_k} \varphi(t, c)$$

lies in the compact subset

$$C' = \big\{ e^{-t_1 h_1} \cdots e^{-t_k h_k} c \mid (t, c) \in S \times C \big\} \subset \mathfrak{n}(\lambda).$$

Now

$$\mathrm{pr}_{\leqslant k+1} \varphi(0, t_{k+1}, \ldots, t_n, c) = \mathrm{pr}_{\leqslant k+1} c + t_{k+1} V_{k+1},$$

from which we see that $t_{k+1} V_{k+1}$ belongs to the vectorial sum $-\mathrm{pr}_{\leqslant k+1}(C) + \mathrm{pr}_{\leqslant k+1}(C')$. From this we conclude that $\pi_{\leqslant k+1} T(C)$ is bounded.

We now come to the final assertion. It follows from the above that the map

$$\psi : \mathbb{R}^n \to \mathfrak{n}(\lambda)$$

defined by $t \mapsto \varphi(t, 0)$ is proper. Moreover, since at every point the vector fields $h_1, \ldots, h_n$ are linearly independent, it follows that ψ is a local diffeomorphism. Therefore, ψ has open and closed image, hence it is surjective onto $\mathfrak{n}(\lambda)$. Moreover, the fibers of ψ are finite and discrete. Hence, ψ is a covering map. Since $\mathfrak{n}(\lambda)$ is simply connected, it follows that ψ is a diffeomorphism from $\mathbb{R}^n$ onto $\mathfrak{n}(\lambda)$. We consider the linear bijection

$$\tau : \mathfrak{n}(\lambda) \to \mathbb{R}^n$$

given by $V = \sum_j \tau(V)_j V_j$. By linearity of the map $V \mapsto h_V$ it follows that for all $V \in \mathfrak{n}(\lambda)$,

$$\exp(h_V)(0) = \exp\big(\tau(V_1)h_1 + \cdots + \tau(V_n)h_n\big)(0) = \varphi\big(\tau(V), 0\big).$$

This implies that the map defined by $V \mapsto \exp(h_V)(0)$ is a diffeomorphism from $\mathfrak{n}(\lambda)$ onto $\mathfrak{n}(\lambda)$. Since h_V is the pull back of H_V by the diffeomorphism $\exp: \mathfrak{n}(\lambda) \to N(\lambda)$, it follows that

$$e^{H_V} = \exp \circ e^{h_V} \circ \exp^{-1}.$$

Hence, the map defined by $V \mapsto e^{H_V} e_N$ is a diffeomorphism from $\mathfrak{n}(\lambda)$ onto $N(\lambda)$. This completes the proof of the proposition. $\square$

Corollary 6.8. *The map* $\widetilde{\Phi}: K \times (\mathfrak{g}/\mathfrak{P}(\lambda))^* \to G/Z(\lambda)$ *given by*

$$\widetilde{\Phi}(k, \eta) = k \exp(H_\eta) \bar{e}$$

induces a K-equivariant diffeomorphism of fiber bundles

$$\Phi_\lambda: K \times_{Z_K(\lambda)} (\mathfrak{g}/\mathfrak{P}(\lambda))^* \to G/Z(\lambda).$$

Here the action of $Z_K(\lambda)$ *on* $(\mathfrak{g}/\mathfrak{P}(\lambda))^*$ *is induced by the adjoint action of* $Z_K(\lambda)$ *on* $\mathfrak{g}$.

Proof. The map $\widetilde{\Phi}$ is the composition of the diffeomorphism

$$\bar{\Phi}: K \times (\mathfrak{g}/\mathfrak{P}(\lambda))^* \to K \times N(\lambda)\bar{e}$$

defined by $(k, \eta) \mapsto (k, \exp(H_\eta)\bar{e})$ and the submersion

$$j: K \times N(\lambda)\bar{e} \to G/Z(\lambda)$$

defined by $(k, n\bar{e}) \mapsto kn\bar{e}$. The latter map factors to a diffeomorphism $K \times_{Z_K(\lambda)} N(\lambda)\bar{e} \to G/Z(\lambda)$. The quotient $K \times_{Z_K(\lambda)} N(\lambda)$ is defined by using the left action of $Z_K(\lambda)$ on $N(\lambda)\bar{e}$. It follows from Lemma 6.3 that for all $m \in Z_K(\lambda)$ we have

$$l_m \circ \exp(H_\eta) = \exp(H_{m \cdot \eta}) \circ l_m$$

on $N(\lambda)\bar{e}$. This implies that

$$\bar{\Phi}(km, \eta) = \bar{\Phi}(k, m \cdot \eta),$$

so that $\bar{\Phi}$ induces a diffeomorphism $K \times_{Z_K(\lambda)} (\mathfrak{g}/\mathfrak{P}(\lambda))^* \to K \times_{Z_K(\lambda)} N(\lambda)\bar{e}$. In view of Lemma 6.4 this completes the proof. $\square$

The action of K on $G/P(\lambda)$ naturally induces an action of K on the total space of the cotangent bundle $T^*(G/P(\lambda))$ through symplectomorphisms for the Liouville symplectic form σ on $T^*(G/P(\lambda))$. In particular, the stabilizer $Z_K(\lambda) = K \cap P(\lambda)$ acts linearly on the cotangent space of $T_{eP(\lambda)}G/P(\lambda)$ at $eP(\lambda)$. The latter is naturally identified with $(\mathfrak{g}/\mathfrak{P}(\lambda))^*$. By Lemma 6.4 the resulting action of $Z_K(\lambda)$ on $(\mathfrak{g}/\mathfrak{P}(\lambda))^*$ coincides with the one induced by the adjoint action of $Z_K(\lambda)$ on $\mathfrak{g}$.

The map defining the action of K on $T^*(G/P(\lambda))$ induces a submersion

$$K \times \left(\mathfrak{g}/\mathfrak{P}(\lambda)\right)^* \to T^*(G/P(\lambda))$$

which factors to an isomorphism of vector bundles

$$\Psi_\lambda : K \times_{Z_K(\lambda)} \left(\mathfrak{g}/\mathfrak{P}(\lambda)\right)^* \xrightarrow{\sim} T^*(G/P(\lambda)).$$

We will complete the proof of Theorem 6.1 by showing that the bundle isomorphism

$$\varphi_\lambda := \Phi_\lambda \circ \Psi_\lambda^{-1} : T^*(G/P(\lambda)) \to G/Z(\lambda)$$

satisfies the properties of the theorem.

Completion of the proof of Theorem 6.1. Since $K \cap P(\lambda) = Z_K(\lambda)$ and $G = KP(\lambda)$, the inclusion map $K \to G$ induces a diffeomorphism $K/Z_K(\lambda) \to G/P(\lambda)$, whose inverse will be denoted by $\bar{s}$. Let $j : K/Z_K(\lambda) \to G/Z(\lambda)$ be the embedding induced by the inclusion map; then $s = j \circ \bar{s}$ is a section of the bundle $\pi : G/Z(\lambda) \to G/P(\lambda)$.

Let $x \in G/P(\lambda)$ and $\xi \in T_x^*(G/P(\lambda))$. Fix $k \in K$ such that $kZ_K(\lambda) = s(x)$ and define $\eta := dl_k(eP(\lambda))^*\xi$. Then

$$\Psi_\lambda([k, \eta]) = k \cdot \eta = \xi,$$

so that

$$
\begin{aligned}
\varphi_\lambda(\xi) &= \Phi_\lambda([k, \eta]) \\
&= k \exp(H_\eta)\bar{e} \\
&= \exp(H_{k \cdot \eta})k\bar{e} \\
&= \exp(H_\xi)s(x).
\end{aligned}
$$

It follows that $\varphi = \varphi_\lambda$ equals the map defined by (3.2), for the bundle $\pi : G/Z(\lambda) \to G/P$. Moreover, in view of Lemma 6.3, Proposition 6.7 and Corollary 6.8, the proof of Theorem 3.1 works with $U = T^*(G/P(\lambda))$; in particular, all appearing flows of vector fields are defined without any restriction on their domains. This establishes conditions (a), (b) and (c) of Theorem 3.1 with $U = T^*M$ and $\varphi(U) = G/Z(\lambda)$. From these, conditions (a), (b) and (c) of Theorem 6.1 follow. Uniqueness of φ_λ follow by the arguments of the proof of Theorem 3.1 that are valid without any restrictions on domains. $\square$

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