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The Plancherel decomposition for a reductive symmetric space II. Representation theory

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Abstract. We obtain the Plancherel decomposition for a reductive symmetric space in the sense of representation theory. Our starting point is the Plancherel formula for spherical Schwartz functions, obtained in part I. The formula for Schwartz functions involves Eisenstein integrals obtained by a residual calculus. In the present paper we identify these integrals as matrix coefficients of the generalized principal series.

Contents

1 Introduction

In this paper we establish the Plancherel decomposition for a reductive symmetric space $\mathrm{X} = G/H$, in the sense of representation theory. Here G is a real reductive group of Harish-Chandra's class and H is an open subgroup

of the group G^σ of fixed points for an involution σ of G. This paper is a continuation of the paper [12] in the sense that we derive the Plancherel decomposition from its main result [12], Thm. 23.1, the Plancherel formula for the space $\mathcal{C}(X:\tau)$ of τ-spherical Schwartz functions on X. Here (τ, V_τ) is a finite dimensional unitary representation of K, a σ-stable maximal compact subgroup of G. At the end of the paper, we make a detailed comparison of our results with those of P. Delorme [21].

The results of this paper were found and announced in the fall of 1995, when both authors were visitors of the Mittag–Leffler Institute in Djursholm, Sweden. At the same time Delorme announced a proof of the Plancherel theorem. For more historical comments, we refer the reader to the introduction of [12].

Before giving a detailed outline of the results of this paper, we shall first give some background and describe the main result of [12], which serves as the basis for this paper. The space X carries an invariant measure dx; accordingly, the regular representation L of G in $L^2(X)$ is unitary. The Plancherel decomposition amounts to an explicit decomposition of L as a direct integral of irreducible unitary representations of G. These representations will turn out to be discrete series representations of X and generalized principal series representations of the form

$$\pi_{Q,\xi,\nu} = \mathrm{Ind}_Q^G(\xi \otimes \nu \otimes 1), \tag{1.1}$$

with $Q = M_Q A_Q N_Q$ a $\sigma\theta$-stable parabolic subgroup of G with the indicated Langlands decomposition, ξ a discrete series representation of the symmetric space $X_Q := M_Q/M_Q \cap H$, and ν a unitary character of $A_Q/A_Q \cap H$. To keep the exposition simple, we assume here, and in the rest of the introduction, that the number of open H-orbits on $Q\backslash G$ is one. In general, there are finitely many open orbits, parametrized by a set ${}^Q\mathcal{W}$ of representatives, and then ξ should be taken from the discrete series of the spaces $X_{Q,v} := M_Q/M_Q \cap vHv^{-1}$, for $v \in {}^Q\mathcal{W}$.

Let θ be the Cartan involution associated with K; it commutes with σ. Let $\mathfrak{a}_\mathfrak{q}$ be a maximal abelian subspace of the intersection of the -1 eigenspaces for θ and σ in $\mathfrak{g}$, the Lie algebra of G. We denote by $\mathcal{P}_\sigma$ the collection of $\theta\sigma$-stable parabolic subgroups of G containing $A_\mathfrak{q} := \exp \mathfrak{a}_\mathfrak{q}$. For $Q \in \mathcal{P}_\sigma$ we put $\mathfrak{a}_{Q\mathfrak{q}} := \mathfrak{a}_Q \cap \mathfrak{a}_\mathfrak{q}$. In [12] we defined a spherical Fourier transform $\mathcal{F}_Q$ in terms of a so called normalized Eisenstein integral

$$E^\circ(Q:\nu) = E_\tau^\circ(Q:\nu).$$

The Eisenstein integral is a $\mathbb{D}(X)$-finite and $1 \otimes \tau$-spherical function in $C^\infty(X) \otimes \mathrm{Hom}(\mathcal{A}_{2,Q}, V_\tau)$, depending meromorphically on the parameter $\nu \in \mathfrak{a}_{Q\mathfrak{q}\mathbb{C}}^*$. Here $\mathcal{A}_{2,Q} = \mathcal{A}_{2,Q}(\tau)$ is defined as the space of Schwartz functions $X_Q \to V_\tau$ that are $\tau_Q := \tau|_{K\cap M_Q}$-spherical and behave finitely under the algebra $\mathbb{D}(X_Q)$ of invariant differential operators on X_Q. The space $\mathcal{A}_{2,Q}$ is finite dimensional, and inherits the Hilbert structure from the bigger space $L^2(X_Q:\tau_Q)$. Without the simplifying assumption, $\mathcal{A}_{2,Q}$ is defined as a finite direct sum of similar function spaces for $X_{Q,v}$, as $v \in {}^Q\mathcal{W}$.

Let P_0 be a fixed minimal element of $\mathcal{P}_\sigma$. Then the Eisenstein integral $E^\circ(P_0 : \lambda)$ is essentially obtained as a (sum of) matrix coefficient(s) of a K-finite vector with an H-fixed distribution vector of a σ-minimal principal series representation of the form (1.1) with $Q = P_0$, see [4] and [7].

In contrast, for non-minimal $Q \in \mathcal{P}_\sigma$ the Eisenstein integral $E^\circ(Q : \nu)$ is obtained from $E^\circ(P_0 : \lambda)$ by means of a residual calculus in the variable $\lambda \in \mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$, see [12], Eqn. (8.7) and Lemmas 13.18 and 13.12. In particular, for such Q it is a priori not clear that the normalized Eisenstein integral $E^\circ(Q : \nu)$ is a matrix coefficient of the generalized principal series representation (1.1).

In terms of the Eisenstein integral, the spherical Fourier transform is defined by the formula

$$\mathcal{F}_Q f(\nu) = \int_{\mathrm{X}} E^\circ(Q : -\bar{\nu} : x)^* f(x)\, dx \in \mathcal{A}_{2,Q},$$

for $f \in \mathcal{C}(\mathrm{X} : \tau)$ and $\nu \in i\mathfrak{a}_{Q\mathrm{q}}^*$; see [12], § 19. The star indicates that the adjoint of an endomorphism in $\mathrm{Hom}(\mathcal{A}_{2,Q}, V_\tau)$ is taken. The transform $\mathcal{F}_Q$ is a continuous linear map from $\mathcal{C}(\mathrm{X} : \tau)$ into the space $\mathcal{S}(i\mathfrak{a}_{Q\mathrm{q}}^*) \otimes \mathcal{A}_{2,Q}$ of Euclidean Schwartz functions on $i\mathfrak{a}_{Q\mathrm{q}}^*$ with values in the finite dimensional Hilbert space $\mathcal{A}_{2,Q}$. The wave packet transform $\mathcal{J}_Q$ is defined as the adjoint of the Fourier transform with respect to the natural L^2-type inner products on the spaces involved; see [12], § 20. It is a continuous linear map $\mathcal{S}(i\mathfrak{a}_{Q\mathrm{q}}^*) \otimes \mathcal{A}_{2,Q} \to \mathcal{C}(\mathrm{X} : \tau)$, given by the formula

$$\mathcal{J}_Q \varphi(x) = \int_{i\mathfrak{a}_{Q\mathrm{q}}^*} E^\circ(Q : \nu : x)\, \varphi(\nu)\, d\nu,$$

for $\varphi \in \mathcal{S}(i\mathfrak{a}_{Q\mathrm{q}}^*) \otimes \mathcal{A}_{2,Q}$ and $x \in \mathrm{X}$. Here $d\nu$ is Lebesgue measure on $i\mathfrak{a}_{Q\mathrm{q}}^*$, suitably normalized.

Two parabolic subgroups $P, Q \in \mathcal{P}_\sigma$ are called associated if their σ-split components $\mathfrak{a}_{P\mathrm{q}}$ and $\mathfrak{a}_{Q\mathrm{q}}$ are conjugate under the Weyl group W of the root system of $\mathfrak{a}_\mathrm{q}$ in $\mathfrak{g}$. The notion of associatedness defines an equivalence relation $\sim$ on $\mathcal{P}_\sigma$. Let $\mathbf{P}_\sigma$ be a choice of representatives in $\mathcal{P}_\sigma$ for the classes in $\mathcal{P}_\sigma / \sim$. Then the Plancherel formula for functions in $\mathcal{C}(\mathrm{X} : \tau)$ takes the form

$$f = \sum_{Q \in \mathbf{P}_\sigma} [W : W_Q^*]\, \mathcal{J}_Q \mathcal{F}_Q f, \qquad (f \in \mathcal{C}(\mathrm{X} : \tau)),$$

with W_Q^* the normalizer in W of $\mathfrak{a}_{Q\mathrm{q}}$. The operator $[W : W_Q^*]\mathcal{J}_Q \mathcal{F}_Q$ is a continuous projection operator onto a closed subspace $\mathcal{C}_Q(\mathrm{X} : \tau)$ of $\mathcal{C}(\mathrm{X} : \tau)$. Moreover,

$$\mathcal{C}(\mathrm{X} : \tau) = \oplus_{Q \in \mathbf{P}_\sigma} \quad \mathcal{C}_Q(\mathrm{X} : \tau),$$

with orthogonal summands. It follows from the above that $[W : W_Q^*]^{1/2} \mathcal{F}_Q$ extends to a partial isometry from $L^2(\mathrm{X} : \tau)$ to $L^2(i\mathfrak{a}_{Q\mathrm{q}}^*) \otimes \mathcal{A}_{2,Q}$. Its adjoint extends $[W : W_Q^*]^{1/2} \mathcal{J}_Q$ to a partial isometry in the opposite direction.

In the present paper, we build the Plancherel decomposition for $(L, L^2(\mathrm{X}))$ from the above results for all τ. For this it is crucial to relate the Eisenstein integral $E^\circ(Q:\nu)$ to the generalized principal series representations $\pi_{Q,\xi,-\nu}$.

In [19], Delorme has *defined* a normalized Eisenstein integral ${}^\backprime E^\circ(Q:\nu)$ essentially as a matrix coefficient of the generalized principal series. One way to establish the wanted relationship of $E^\circ(Q:\nu)$ with the generalized principal series would thus be to prove the following identity of meromorphic functions in the variable $\nu \in \mathfrak{a}_{Q\mathrm{q}\mathbb{C}}^*$:

$$E^\circ(Q:\nu) = {}^\backprime E^\circ(Q:-\nu). \tag{1.2}$$

In view of the vanishing theorem of [11], the Eisenstein integral $E^\circ(Q:\nu)$ can be uniquely characterized in terms of its annihilating ideal in $\mathbb{D}(\mathrm{X})$ and its asymptotic behavior towards infinity on X; see [12], Def. 13.7 and Prop. 13.6. The identity (1.2) would follow if not only the Eisenstein integral on the left-hand side but also the Eisenstein integral on the right-hand side satisfied these characterizing conditions. For the latter to be true one needs that, for $\psi \in \mathcal{A}_{2,Q}$, the family $\nu \mapsto {}^\backprime E^\circ(Q:-\nu)\psi$ belongs to the space $\mathcal{E}_Q^{\mathrm{hyp}}(\mathrm{X}:\tau)$ of [12], Prop. 13.6. For this in turn, the full set of exponents of the family ${}^\backprime E^\circ(Q:-\nu)\psi$ in its asymptotic expansion along P_0 must be of a certain form; see [12], Defs. 6.3 and 6.1. We have not been able to deduce this type of information from Delorme's work. Nevertheless, by following a different strategy we have been able to establish (1.2), but only at the end of the paper, in Corollary 11.21, after a relation of our Eisenstein integrals with the principal series has been established.

More precisely, the mentioned characterization of the Eisenstein integral $E^\circ(Q:\nu)$ is used to construct certain embeddings of $(\mathfrak{g}, K)$-modules

$$\pi_{Q,\xi,-\nu} \hookrightarrow (L, C^\infty(\mathrm{X})). \tag{1.3}$$

The existence of these embeddings, on the level of $(\mathfrak{g}, K)$-modules, is sufficient to establish the Plancherel decomposition in the sense of representation theory, Theorem 10.9. Further details will be given at a later stage in this introduction.

At the end of the paper we invoke the automatic continuity theorem, Theorem 11.1, due to W. Casselman and N.R. Wallach, see [18] and [29], to show that the embedding (1.3) extends to a G-homomorphism. This implies that our Eisenstein integrals are essentially generalized matrix coefficients of K-finite and H-fixed distribution vectors of principal series representations. From this information combined with results of [16], the identity (1.2) can then be established.

After this motivation, we shall now give an outline of the paper, in particular describing how the Eisenstein integrals give rise to the embeddings (1.3).

In Sect. 3 we show that the discrete part $L^2_d(\mathrm{X}:\tau)$ of $L^2(\mathrm{X}:\tau)$ is finite dimensional. This fact can be derived from the description of the discrete

series by T. Oshima and T. Matsuki in [26]. We show that it can be obtained from [12] and weaker information on the discrete series, also due to [26], namely the rank condition and the fact that the $\mathbb{D}(X)$-characters of $L^2_d(X)$ are real and regular. The mentioned result implies that the parameter space $\mathcal{A}_{2,Q}(\tau)$ of the Eisenstein integral equals $L^2_d(X_Q : \tau_Q)$. Accordingly, it may be decomposed in an orthogonal finite dimensional sum of isotypical subspaces $\mathcal{A}_{2,Q}(\tau)_\xi$, where $\xi \in X^\wedge_{Q,ds}$, the collection of discrete series represetations for X_Q.

In Sect. 4 we explain the connection of the Eisenstein integrals with the principal series. Let $\widehat{K}$ be the unitary dual of K, i.e., the collection of equivalence classes of irreducible unitary representations of K. If V is a locally convex space equipped with a continuous representation of K, then by V_K we denote the subspace of K-finite vectors; for $\vartheta \subset \widehat{K}$ a finite subset we denote by V_ϑ the subspace of V_K consisting of vectors whose K-types belong to ϑ. Let $\vartheta \subset \widehat{K}$ be a finite subset. We define $\mathbf{V}_\vartheta$ to be the space of continuous functions $K \to \mathbb{C}$ that are left K-finite with types contained in the set ϑ. Moreover, we define τ_ϑ to be the restriction of the right regular representation of K to $\mathbf{V}_\vartheta$. Let $\delta_e : \mathbf{V}_\vartheta \to \mathbb{C}$ be evaluation in e. Then $F \mapsto \delta_e \circ F$ is a natural isomorphism from $L^2(X : \tau_\vartheta)$ onto $L^2(X)_\vartheta$. Its inverse, called sphericalization, is denoted by ς_ϑ.

For $\xi \in X^\wedge_{Q,ds}$, we denote by $\bar{V}(\xi)$ the space of continuous linear M_Q-equivariant maps $\mathcal{H}_\xi \to L^2(X_Q)$. This space is a finite dimensional Hilbert space. We denote by $L^2(K : \xi)$ the space of the induced representation $\mathrm{Ind}^K_{K\cap M_Q}(\xi|_{K\cap M_Q})$. It is well known that the induced representation (1.1) may be realized as a ν-dependent representation in $L^2(K : \xi)$, which we shall denote by $\pi_{Q,\xi,\nu}$ as well; this is the so-called compact picture of (1.1).

If $\vartheta \subset \widehat{K}$ is a finite subset, there is a natural isometry from $\bar{V}(\xi) \otimes L^2(K : \xi)_\vartheta$ into $\mathcal{A}_{2,Q}(\tau_\vartheta)$, denoted $T \mapsto \psi_T$. We show in Sect. 4 that we may use the Eisenstein integrals to define a map $J_{Q,\xi,\nu} : \bar{V}(\xi) \otimes L^2(K : \xi)_K \to C^\infty(X)_K$ by the formula

$$J_{Q,\xi,\nu}(T)(x) = \delta_e\big[E^\circ_\vartheta(Q : \nu : x)\psi_T\big]. \tag{1.4}$$

Here $\vartheta \subset \widehat{K}$ is any finite subset such that $T \in \bar{V}(\xi) \otimes C^\infty(K : \xi)_\vartheta$ and E°_ϑ denotes the Eisenstein integral with $\tau = \tau_\vartheta$. The map $J_{Q,\xi,\nu}$ is a priori well-defined for ν in the complement of the union of a certain set $\mathcal{H}(Q, \xi)$ of hyperplanes in $\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$. This union is disjoint from $i\mathfrak{a}^*_{Q\mathrm{q}}$.

The main result of the section is Theorem 4.6. It asserts that $\mathcal{H}(Q, \xi)$ is locally finite and that, for ν in the complement of $\cup\mathcal{H}(Q, \xi)$, the map $J_{Q,\xi,\nu}$ is $(\mathfrak{g}, K)$-equivariant for the infinitesimal representations associated with $1 \otimes \pi_{Q,\xi,-\nu}$ and L. The proof of this result is given in the next two sections. In the first of these we prepare for the proof by showing that $\pi_{Q,\xi,\nu}$ is finitely generated, with local uniformity in the parameter ν, see Proposition 5.1. This result is needed for the proof of the local finiteness of $\mathcal{H}(Q, \xi)$.

In Sect. 6 the $(\mathfrak{g}, K)$-equivariance of the map $J_{Q,\xi,\nu}$ is established. The K-equivariance readily follows from the definitions. For the $\mathfrak{g}$-equivariance it is necessary to compute derivatives of the Eisenstein integral of the form $L_X E^\circ_\tau(Q:\nu)\psi$, for $\psi \in \mathcal{A}_{2,Q}(\tau)$ and $X \in \mathfrak{g}$. The computation is achieved by introducing a meromorphic family of spherical functions $\widetilde{F}\colon \mathfrak{a}^*_{Q\mathfrak{q}_\mathbb{C}} \times \mathrm{X} \to \mathfrak{g}^*_\mathbb{C} \otimes V_\tau$ by the formula

$$\widetilde{F}_\nu(x)(Z) = L_Z\big[E^\circ_\tau(Q:\nu:\,\cdot\,)\psi\big](x),$$

for $\nu \in \mathfrak{a}^*_{Q\mathfrak{q}_\mathbb{C}}$, $x \in \mathrm{X}$ and $Z \in \mathfrak{g}_\mathbb{C}$. The function $\widetilde{F}_\nu$ is $\widetilde{\tau}$-spherical, with $\widetilde{\tau} := \mathrm{Ad}_K^\vee \otimes \tau$ and $\mathrm{Ad}_K := \mathrm{Ad}|_K$. It has the same annihilating ideal in $\mathbb{D}(\mathrm{X})$ as the Eisenstein integral $E^\circ_\tau(Q:\nu)\psi$. Moreover, its asymptotic behavior on X can be expressed in terms of that of $E^\circ_\tau(Q:\nu)$. By the mentioned characterization of Eisenstein integrals this enables us to show that $\widetilde{F}_\nu$ equals an Eisenstein integral of the form $E^\circ_{\widetilde{\tau}}(Q:\nu)\partial_Q(\nu)\psi$, with $\partial_Q(\nu)$ an explicitly given differential operator $\mathcal{A}_{2,Q}(\tau) \to \mathcal{A}_{2,Q}(\widetilde{\tau})$, see Theorem 6.12. The $\mathfrak{g}$-equivariance of $J_{Q,\xi,\nu}$ is then obtained by computing the action of $\partial_Q(\nu)$ on ψ_T, for $T \in \bar{V}(\xi) \otimes C^\infty(K:\xi)_\vartheta$; see Lemma 6.13 and Proposition 6.15. At the end of the section we complete the proof of Theorem 4.6 by establishing the local finiteness of $\mathcal{H}(Q,\xi)$, combining the results of Sects. 5 and 6; see Proposition 6.16.

In Sect. 7 we define a Fourier transform $f \mapsto \hat{f}(Q:\xi:\nu)$ from $C^\infty_c(\mathrm{X})_K$ to $\bar{V}(\xi) \otimes L^2(K:\xi)_K$ by transposition of the map $J_{Q,\xi,-\bar{\nu}}$. It is given by the formula

$$\langle \hat{f}(Q:\xi:\nu) \mid T\rangle = \int_X f(x)\overline{J_{Q,\xi,-\bar{\nu}}(T)(x)}\,dx$$

and intertwines the $(\mathfrak{g}, K)$-module of L with that of $1 \otimes \pi_{Q,\xi,-\nu}$. In view of (1.4), the transform $f \mapsto \hat{f}$ is related to the spherical Fourier transform by the formula

$$\langle \hat{f}(Q:\xi:\nu) \mid T\rangle = \langle \mathcal{F}_Q(\varsigma_\vartheta f)(\nu) \mid \psi_T\rangle, \tag{1.5}$$

for $f \in C^\infty_c(X)_\vartheta$.

The established relation (1.5) combined with the spherical Plancherel formula implies that the Fourier transform $f \mapsto \hat{f}(Q:\xi:\nu)$ defines an isometry from $L^2(\mathrm{X})$ into the direct integral

$$\pi = \sum_{Q\in\mathbf{P}_\sigma} \sum_{\xi\in \mathrm{X}^\wedge_{Q,ds}} [W\colon W^*_Q] \int_{i\mathfrak{a}^*_{Q\mathfrak{q}}} 1 \otimes \pi_{Q,\xi,-\nu}\,d\nu, \tag{1.6}$$

realized in a Hilbert space $\mathfrak{L}^2$. The continuous parts of this direct integral are studied in Sect. 8. In Sect. 9 it is first shown, in Theorem 9.5, that the Fourier transform $f \mapsto \hat{f}$ extends to an isometry $\mathfrak{F}$ from $L^2(\mathrm{X})$ into $\mathfrak{L}^2$. Moreover, its restriction to $C^\infty_c(\mathrm{X})_K$ is a $(\mathfrak{g}, K)$-module map into $\mathfrak{L}^{2\infty}$. By an argument involving continuity and density, it is then shown that $\mathfrak{F}$ is

G-equivariant, see Theorem 9.6. At this stage we have established that $\mathfrak{F}$ maps the regular representation L isometrically into a direct integral decomposition. For this to give the Plancherel decomposition, we need to show that the image of $\mathfrak{F}$ is a direct integral with representations that are irreducible and mutually inequivalent outside a set of Plancherel measure zero. This is done in Lemma 10.5 and Proposition 10.8. In the process we use results of F. Bruhat and Harish-Chandra on irreducibility and equivalence of unitarily parabolically induced representations, see Theorem 10.7. The Plancherel theorem is formulated in Theorem 10.9. Finally, in Theorem 10.11 a precise description of the image of $\mathfrak{F}$ is given.

At this point it is still not clear that our description of the Plancherel decomposition uses the same parametrizations as the one in Delorme's paper [21]. It is the object of the last section to show that this is indeed the case. As said, a key idea is to use the automatic continuity theorem, Theorem 11.1, due to Casselman and Wallach, see [18] and [29]. It implies that the map $J_{Q,\xi,\nu}$ has a continuous linear extension, hence can be realized by taking the matrix coefficient with an H-fixed distribution vector of $\mathrm{Ind}_Q^G(\xi \otimes \nu \otimes 1)$. By means of the description of such vectors in [16], combined with an asymptotic analysis, it is shown that our Eisenstein integral is related to Delorme's by the identity (1.2), see Corollary 11.21.

Finally, the constants $[W\colon W_Q^*]$ occurring in our formula (1.6) differ from those in the similar formula of Delorme. This is due to different choices of normalizations of measures, as is explained in the final part of the paper.

2 Notation and preliminaries

Throughout this paper, we use all notation and preliminaries from [12], Sect. 2. In particular, G is a group of Harish-Chandra's class, σ an involution of G and H an open subgroup of G^σ, the group of fixed points for σ. The associated reductive symmetric space is denoted by

$$\mathrm{X} = G/H.$$

All occurring measures will be normalized according to the conventions described in [12], end of Sect. 5.

Apart from the references just given, we shall give precise references to [12] for additional notation, definitions and results.

3 A property of the discrete series

In this section we discuss an important result on the discrete part of $L^2(\mathrm{X})$, which is a consequence of the classification of the discrete series by T. Oshima and T. Matsuki in [26]. In our approach to the Plancherel formula via the residue calculus, we obtain it as a consequence of the rank condition

and the regularity of the infinitesimal character, also due to [26], see [12], Rem. 16.2.

In the rest of this section we assume that (τ, V_τ) is a finite dimensional unitary representation of K. A function $f: \mathrm{X} \to V_\tau$ is called τ-spherical if $f(kx) = \tau(k) f(x)$, for all $x \in \mathrm{X}$ and $k \in K$. The Hilbert space of square integrable τ-spherical functions is denoted by $L^2(\mathrm{X}: \tau)$. Its discrete part, denoted $L^2_d(\mathrm{X}: \tau)$, is defined as in [12], § 12. The Fréchet space of τ-spherical Schwartz functions, denoted $\mathcal{C}(\mathrm{X}: \tau)$, is defined as in [12], Eqn. (12.1). The subspace of $\mathbb{D}(\mathrm{X})$-finite functions in $\mathcal{C}(\mathrm{X}: \tau)$ is denoted by $\mathcal{A}_2(\mathrm{X}: \tau)$.

Proposition 3.1 *Let (τ, V_τ) be a finite dimensional unitary representation of K. Then*

$$L^2_d(\mathrm{X}: \tau) = \mathcal{A}_2(\mathrm{X}: \tau). \tag{3.1}$$

Moreover, each of the spaces above is finite dimensional.

Proof: By the reasoning at the end of the proof of Lemma 12.6 in [12] it follows that the space on the right-hand side of (3.1) is contained in the space on the left-hand side. If the center of G is not compact modulo H, then it follows from [26], see [12], Thm. 16.1, that X has no discrete series. Hence, $L^2_d(\mathrm{X}) = 0$ and we obtain (3.1).

On the other hand, if G has a compact center modulo H the result is part of [12], Lemma 12.6. □

If $(\xi, \mathcal{H}_\xi)$ is an irreducible unitary representation of G, let $\mathrm{Hom}_G(\mathcal{H}_\xi, L^2(\mathrm{X}))$ denote the space of G-equivariant continuous linear maps from $\mathcal{H}_\xi$ into $L^2(\mathrm{X})$. This space is non-trivial if and only if (the class of) ξ belongs to $\mathrm{X}^\wedge_{ds}$, the collection of equivalence classes of discrete series representations of X. If $\xi \in \mathrm{X}^\wedge_{ds}$, then the mentioned space is finite dimensional, by the finite multiplicity of the discrete series, see [1], Thm. 3.1.

For any irreducible unitary representation ξ, the canonical map from the tensor product $\mathrm{Hom}_G(\mathcal{H}_\xi, L^2(\mathrm{X})) \otimes \mathcal{H}_\xi$ to $L^2(\mathrm{X})$ is an embedding, which is G-equivariant for the representations $1 \otimes \xi$ and L, respectively. We denote its image by $L^2(\mathrm{X})_\xi$ and equip the space $\mathrm{Hom}_G(\mathcal{H}_\xi, L^2(\mathrm{X}))$ with the unique inner product that turns the mentioned embedding into an isometric G-equivariant isomorphism

$$m_\xi: \ \mathrm{Hom}_G\big(\mathcal{H}_\xi, L^2(\mathrm{X})\big) \otimes \mathcal{H}_\xi \ \xrightarrow{\sim}\ L^2(\mathrm{X})_\xi. \tag{3.2}$$

Obviously the space on the right-hand side of (3.2) depends on ξ through its class $[\xi]$, and will therefore also be indicated with index $[\xi]$ in place of ξ.

With the notation just introduced, it follows that

$$L^2_d(\mathrm{X}) = \widehat{\oplus}_{\omega \in \mathrm{X}^\wedge_{ds}} L^2(\mathrm{X})_\omega, \tag{3.3}$$

with orthogonal summands. Here and elsewhere, the hat over the summation symbol indicates that the closure of the algebraic direct sum is taken.

If ω is an equivalence class of an irreducible unitary representation of G, we write $L^2(\mathrm{X}:\tau)_\omega := L^2(\mathrm{X}:\tau) \cap [L^2(\mathrm{X})_\omega \otimes V_\tau]$. It is readily seen that this space is non-trivial if and only if ω belongs to $\mathrm{X}_{ds}^\wedge$ and has a K-type in common with the contragredient of τ. The collection of ω with this property is denoted by $\mathrm{X}_{ds}^\wedge(\tau)$.

Lemma 3.2 *The collection $\mathrm{X}_{ds}^\wedge(\tau)$ is finite. Moreover,*

$$L^2_d(\mathrm{X}:\tau) = \oplus_{\omega\in \mathrm{X}_{ds}^\wedge(\tau)} \; L^2(\mathrm{X}:\tau)_\omega, \tag{3.4}$$

where the direct sum is orthogonal and all the summands are finite dimensional.

Proof: That the direct sum decomposition is orthogonal and has closure $L^2_d(\mathrm{X}:\tau)$ follows from the similar properties of (3.3). The space on the left-hand side of (3.4) is finite dimensional, by Proposition 3.1. Since all summands on the right-hand side are non-trivial, the collection parametrizing these summands is finite. □

Remark 3.3 It follows from Proposition 3.1 that the spaces $L^2(\mathrm{X}:\tau)_\omega$, for $\omega \in \mathrm{X}_{ds}^\wedge$, are contained in $\mathcal{A}_2(\mathrm{X}:\tau)$; we therefore also denote them by $\mathcal{A}_2(\mathrm{X}:\tau)_\omega$. Note that $L^2(\mathrm{X}:\tau)_\omega = 0$ for ω an irreducible unitary representation of G that does not belong to $\mathrm{X}_{ds}^\wedge$. Accordingly, we put $\mathcal{A}_2(\mathrm{X}:\tau)_\omega = 0$ for such ω. In view of what has been said, the decomposition (3.4) may be rewritten as

$$\mathcal{A}_2(\mathrm{X}:\tau) = \oplus_{\omega\in \mathrm{X}_{ds}^\wedge(\tau)} \; \mathcal{A}_2(\mathrm{X}:\tau)_\omega. \tag{3.5}$$

Let $C(K)_K$ denote the space of right K-finite continuous functions on K. If ϑ is a finite subset of $\widehat{K}$, the unitary dual of K, then by $C(K)_\vartheta$ we denote the subspace of $C(K)_K$ consisting of functions with right K-types contained in the set ϑ. If $\delta \in \widehat{K}$, then $\delta^\vee$ denotes the contragredient representation. Accordingly, we put $\vartheta^\vee := \{\delta^\vee \mid \delta \in \vartheta\}$. We define

$$\mathbf{V}_\vartheta := C(K)_{\vartheta^\vee} \tag{3.6}$$

and equip this space with the restriction of the right regular representation of K; this restriction is denoted by τ_ϑ. We endow $\mathbf{V}_\vartheta$ with the $L^2(K)$-inner product defined by means of normalized Haar measure. By δ_e we denote the map $\mathbf{V}_\vartheta \to \mathbb{C}$, $\varphi \mapsto \varphi(e)$.

Lemma 3.4 *Let E be a complete locally convex space equipped with a continuous representation of K. Then the map $I \otimes \delta_e$ restricts to a topological linear isomorphism from $(E\otimes \mathbf{V}_\vartheta)^K$ onto E_ϑ. If E is equipped with a continuous pre-Hilbert structure for which K acts unitarily, then the isomorphism is an isometry. In particular, this yields natural isometries*

$$L^2(\mathrm{X}:\tau_\vartheta) \simeq L^2(\mathrm{X})_\vartheta, \quad C^\infty_c(\mathrm{X}:\tau_\vartheta) \simeq C^\infty_c(\mathrm{X})_\vartheta,$$

where the last two spaces are equipped with the inner products inherited from the first two spaces.

Proof: This is well known and easy to prove. □

The inverse of the isomorphism $I \otimes \delta_e$ will be denoted by $\varsigma = \varsigma_\vartheta$; see [7], text before Lemma 5, for similar notation. Given a finite subset $\vartheta \subset \widehat{K}$ we shall write $\mathrm{X}^\wedge_{ds}(\vartheta)$ for $\mathrm{X}^\wedge_{ds}(\tau_\vartheta)$, the set of discrete series representations that have a K-type contained in ϑ. The following result is now an immediate consequence of Lemma 3.2.

Corollary 3.5 *Let $\vartheta \subset \widehat{K}$ be a finite set of K-types. Then $\mathrm{X}^\wedge_{ds}(\vartheta)$ is a finite set.*

We end this section with two simple relations between ς_ϑ and $\varsigma_{\vartheta'}$, for finite subsets $\vartheta, \vartheta' \subset \widehat{K}$ with $\vartheta \subset \vartheta'$. Let E be a complete locally convex space equipped with a continuous representation of K. We denote by $\mathrm{i}_{\vartheta',\vartheta}\colon E_\vartheta \to E_{\vartheta'}$ the natural inclusion map and by $P_{\vartheta,\vartheta'}\colon E_{\vartheta'} \to E_\vartheta$ the K-equivariant projection map. Likewise, the inclusion map $\mathbf{V}_\vartheta \to \mathbf{V}_{\vartheta'}$ and the K-equivariant projection $\mathbf{V}_{\vartheta'} \to \mathbf{V}_\vartheta$ (relative to $\tau_{\vartheta'}, \tau_\vartheta$) are denoted by $\mathrm{i}_{\vartheta',\vartheta}$ and $P_{\vartheta,\vartheta'}$, respectively. By K-equivariance, the maps $I \otimes \mathrm{i}_{\vartheta',\vartheta}$ and $I \otimes P_{\vartheta,\vartheta'}$ induce maps

$$I\otimes\mathrm{i}_{\vartheta',\vartheta}\colon (E\otimes\mathbf{V}_\vartheta)^K \to (E\otimes\mathbf{V}_{\vartheta'})^K, \quad I\otimes P_{\vartheta,\vartheta'}\colon (E\otimes\mathbf{V}_{\vartheta'})^K \to (E\otimes\mathbf{V}_\vartheta)^K.$$

Lemma 3.6 *Let notation be as above. Then*

$$\varsigma_{\vartheta'} \circ (I \otimes \mathrm{i}_{\vartheta',\vartheta}) = \mathrm{i}_{\vartheta',\vartheta} \circ \varsigma_\vartheta, \qquad \varsigma_\vartheta \circ (I \otimes P_{\vartheta,\vartheta'}) = P_{\vartheta,\vartheta'} \circ \varsigma_{\vartheta'}.$$

Proof: The first identity is immediate from the definitions. The second identity follows from the first by using that the maps $P_{\vartheta,\vartheta'}\colon E_{\vartheta'} \to E_\vartheta$ and $P_{\vartheta,\vartheta'}\colon \mathbf{V}_{\vartheta'} \to \mathbf{V}_\vartheta$ may both be characterized by the identities $P_{\vartheta,\vartheta'} \circ \mathrm{i}_{\vartheta',\vartheta} = I$ and $P_{\vartheta,\vartheta'} \circ \mathrm{i}_{\vartheta',\vartheta'\setminus\vartheta} = 0$. □

4 Eisenstein integrals and induced representations

Let $Q \in \mathcal{P}_\sigma$. We denote by $\mathrm{X}^\wedge_{Q,*,ds}$ the collection of equivalence classes of unitary irreducible representations $\xi \in M_Q$ such that ξ is a discrete series representation of $\mathrm{X}_{Q,v}$, for some $v \in N_K(\mathfrak{a}_\mathrm{q})$.

In this section we describe the relation of the normalized Eisenstein integral $E^\circ(Q : \nu)$ with the induced representations $\mathrm{Ind}_Q^G(\xi \otimes \nu \otimes 1)$, where $\nu \in \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$ and $\xi \in \mathrm{X}^\wedge_{Q,*,ds}$. In the rest of this section we assume $\xi \in \mathrm{X}^\wedge_{Q,*,ds}$ to be fixed.

Let ${}^Q\mathcal{W} \subset N_K(\mathfrak{a}_\mathrm{q})$ be a choice of representatives for $W_Q\backslash W/W_{K\cap H}$, see [12], text after Eqn. (2.2). For $v \in {}^Q\mathcal{W}$, we equip $\mathrm{X}_{Q,v}$ with the left M_Q-invariant measure $dx_{Q,v}$, specified at the end of [12], Sect. 5. Moreover, we define $\bar{V}(Q,\xi,v) = \bar{V}(\xi,v)$ by

$$\bar{V}(\xi,v) := \mathrm{Hom}_{M_Q}\big(\mathcal{H}_\xi, L^2(\mathrm{X}_{Q,v})\big). \tag{4.1}$$

As mentioned in Sect. 3, this space is finite dimensional. In accordance with the mentioned section, we equip it with the unique inner product that turns the natural map

$$m_{\xi,v}: \quad \bar{V}(\xi, v) \otimes \mathcal{H}_\xi \quad \xrightarrow{\simeq} \quad L^2(\mathrm{X}_{Q,v})_\xi, \tag{4.2}$$

into an isometric M_Q-equivariant isomorphism. We define the formal direct sums

$$\bar{V}(\xi) := \oplus_{v \in {}^Q\mathcal{W}} \ \bar{V}(\xi, v), \qquad L^2_{Q,\xi} := \oplus_{v \in {}^Q\mathcal{W}} \ L^2(\mathrm{X}_{Q,v})_\xi \tag{4.3}$$

and equip them with the direct sum inner products. The first of these direct sums will also be denoted by $\bar{V}(Q, \xi)$. The second of these direct sums is a unitary M_Q-module. The direct sum of the maps $m_{\xi,v}$ as v ranges over ${}^Q\mathcal{W}$, is an isometric isomorphism

$$m_\xi: \quad \bar{V}(\xi) \otimes \mathcal{H}_\xi \quad \xrightarrow{\simeq} \quad L^2_{Q,\xi} \tag{4.4}$$

that intertwines the natural M_Q-representations.

Remark 4.1 If Q is minimal, then $\mathrm{X}^\wedge_{Q,*,ds}$ coincides with the set $\widehat{M}_{\mathrm{ps}}$, defined in [3], p. 368. Moreover, ${}^Q\mathcal{W} = \mathcal{W}$ is a choice of representatives for $W/W_{K\cap H}$ in $N_K(\mathfrak{a}_\mathfrak{q})$. If $v \in \mathcal{W}$, and $\eta \in \mathcal{H}_\xi^{M\cap vHv^{-1}}$, then the map $j_\eta: \mathcal{H}_\xi \to L^2(M/M\cap vHv^{-1})$, defined by $j_\eta(v)(m) = \langle v \,|\, \xi(m)\eta)\rangle$, is an M-equivariant map. Moreover, $\eta \mapsto j_\eta$ defines an anti-linear map from $V(\xi, v)$ onto $\mathrm{Hom}_M(\mathcal{H}_\xi, L^2(M/M\cap vHv^{-1}))$. This gives an identification of $\overline{V(\xi, v)}$ with $\bar{V}(\xi, v)$. We recall from [3], p. 378, that we equipped $V(\xi, v) = \mathcal{H}_\xi^{M\cap vHv^{-1}}$ with the restriction of the inner product from $\mathcal{H}_\xi$. By the Schur orthogonality relations this implies that the inner product on $\overline{V(\xi, v)}$ coincides with $\dim(\xi)$ times the inner product on $\bar{V}(\xi, v)$. Let $V(\xi)$ be defined as in [3], Eqn. (5.1). Then $\overline{V(\xi)} \simeq \bar{V}(\xi)$ and the inner product on $\overline{V(\xi)}$ coincides with $\dim(\xi)$ times the inner product on $\bar{V}(\xi)$.

For $\nu \in \mathfrak{a}^*_{Q\mathfrak{q}\mathbb{C}}$, let $L^2(Q : \xi : \nu)$ denote the space of measurable functions $G \to \mathcal{H}_\xi$, transforming according to the rule

$$\varphi(manx) = a^{\nu+\rho_Q}\, \xi(m)\, \varphi(x), \qquad (x \in G,\ (m, a, n) \in M_Q \times A_Q \times N_Q),$$

and satisfying $\int_K \|\varphi(k)\|^2_\xi \, dk < \infty$. As usual we identify measurable functions that are equal almost everywhere. The space $L^2(Q : \xi : \nu)$ is a Hilbert space for the inner product given by

$$\langle \varphi \,|\, \psi \rangle = \int_K \langle \varphi(k) \,|\, \psi(k)\rangle_\xi \, dk. \tag{4.5}$$

The restriction of the right regular representation of G to this space is denoted by $\mathrm{Ind}_Q^G(\xi \otimes \nu \otimes 1)$, or more briefly by $\pi_{Q,\xi,\nu} = \pi_{\xi,\nu}$.

Let $C^\infty(Q:\xi:\nu)$ denote the subspace of $L^2(Q:\xi:\nu)$ consisting of functions that are smooth $G \to \mathcal{H}_\xi^\infty$. This subspace is G-invariant; the associated G-representation in it is continuous for the usual Fréchet topology.

Remark 4.2 It follows from [13], § III.7, that the Fréchet G-module $C^\infty(Q:\xi:\nu)$ equals the G-module of smooth vectors for the representation $\pi_{Q,\xi,\nu}$, equipped with its natural Fréchet topology.

It will be convenient to work with the compact picture of the induced representation $\pi_{\xi,\nu}$. Let $L^2(K:\xi)$ denote the space of square integrable functions $\varphi: K \to \mathcal{H}_\xi$ that transform according to the rule

$$\varphi(mk) = \xi(m)\varphi(k), \qquad (k \in K, m \in K_Q). \tag{4.6}$$

Multiplication induces a diffeomorphism $Q \times_{K_Q} K \simeq G$. Hence, restriction to K induces an isometry from $L^2(Q:\xi:\nu)$ onto $L^2(K:\xi)$. This isometry restricts to a topological linear isomorphism from $C^\infty(Q:\xi:\nu)$ onto the subspace $C^\infty(K:\xi)$ of functions in $L^2(K:\xi)$ that are smooth $K \to \mathcal{H}_\xi^\infty$, where the latter space is equipped with the usual Fréchet topology. Via the isometric restriction map we transfer $\pi_{\xi,\nu}$ to a G-representation in $L^2(K:\xi)$, also denoted by $\pi_{Q,\xi,\nu} = \pi_{\xi,\nu}$.

Let (τ, V_τ) be a finite dimensional unitary representation of K. We define

$$L^2(K:\xi:\tau) := \left[L^2(K:\xi)\otimes V_\tau\right]^K. \tag{4.7}$$

By finite dimensionality of τ, the space in (4.7) is finite dimensional and contained in $C(K, \mathcal{H}_\xi)\otimes V_\tau$.

Let ev_e denote the evaluation map $C(K, \mathcal{H}_\xi) \to \mathcal{H}_\xi$, $\varphi \mapsto \varphi(e)$, and let $\mathrm{ev}_e \otimes I$ denote the induced map $L^2(K:\xi:\tau) \to \mathcal{H}_\xi \otimes V_\tau$.

Lemma 4.3

(a) *The map* $\mathrm{ev}_e \otimes I$ *defines an isometric isomorphism from* $L^2(K:\xi:\tau)$ *onto the space* $(\mathcal{H}_\xi \otimes V_\tau)^{K_Q}$.

(b) *The space* $L^2(K:\xi:\tau)$ *equals its subspace* $C^\infty(K:\xi:\tau) := [C^\infty(K:\xi)\otimes V_\tau]^K$.

Proof: Observe that $L^2(K:\xi)$ is the representation space for $\mathrm{Ind}_{K_Q}^K(\xi|_{K_Q})$. Hence (a) follows by Frobenius reciprocity. It is readily checked that $\mathrm{ev}_e \otimes I$ maps $C^\infty(K:\xi:\tau)$ onto $(\mathcal{H}_\xi^\infty \otimes V_\tau)^{K_Q}$. The latter space equals $(\mathcal{H}_{\xi K_Q} \otimes V_\tau)^{K_Q} = (\mathcal{H}_\xi \otimes V_\tau)^{K_Q}$; hence (b) follows. □

Given $T \in \bar{V}(\xi) \otimes L^2(K:\xi:\tau)$ we may now define the element $\psi_T \in L^2_{Q,\xi} \otimes V_\tau$ by

$$\psi_T = [m_\xi \otimes I] \circ [I \otimes \mathrm{ev}_e \otimes I](T).$$

We agree to denote the map $\mathrm{ev}_e \otimes I\colon\ L^2(K:\xi:\tau) \to (\mathcal{H}_\xi \otimes V_\tau)^{K_Q}$ also by $\varphi \mapsto \varphi(e)$. With this notation, if $T = \eta \otimes \varphi$, with $\eta \in \bar{V}(\xi)$ and $\varphi \in L^2(K:\xi:\tau)$, then

$$\psi_{T,v} = [\eta_v \otimes I](\varphi(e)), \qquad (v \in {}^Q\mathcal{W}). \tag{4.8}$$

We recall from Remark 3.3, applied to the space $\mathrm{X}_{Q,v}$ in place of X, for $v \in {}^Q\mathcal{W}$, that $[L^2(\mathrm{X}_{Q,v})_\xi \otimes V_\tau]^{K_Q} \simeq \mathcal{A}_2(\mathrm{X}_{Q,v}:\tau_Q)_\xi$, naturally and isometrically. The space

$$\mathcal{A}_{2,Q}(\tau)_\xi := \oplus_{v\in {}^Q\mathcal{W}}\ \ \mathcal{A}_2(\mathrm{X}_{Q,v}:\tau_Q)_\xi \tag{4.9}$$

is a subspace of the space $\mathcal{A}_{2,Q}(\tau)$, defined in [12], Eqn. (13.1), as the similar direct sum without the indices ξ on the summands. It follows from the above discussion combined with (4.3) that summation over ${}^Q\mathcal{W}$ naturally induces an isometric isomorphism

$$\left(L^2_{Q,\xi} \otimes V_\tau\right)^{K_Q} \simeq \mathcal{A}_{2,Q}(\tau)_\xi, \tag{4.10}$$

via which we shall identify.

Lemma 4.4 *The map $T \mapsto \psi_T$ is an isometry from $\bar{V}(\xi) \otimes L^2(K:\xi:\tau)$ onto $\mathcal{A}_{2,Q}(\tau)_\xi$.*

Proof: It follows from Lemma 4.3 that

$$I \otimes \mathrm{ev}_e \otimes I\colon\quad \bar{V}(\xi) \otimes L^2(K:\xi:\tau)\ \to\ \bar{V}(\xi) \otimes [\mathcal{H}_\xi \otimes V_\tau]^{K_Q} \tag{4.11}$$

is an isometric isomorphism. The map $m_\xi \otimes I$ is an isometry from $\bar{V}(\xi) \otimes \mathcal{H}_\xi \otimes V_\tau$ onto $L^2_{Q,\xi} \otimes V_\tau$, which intertwines the K_Q-actions $1 \otimes \xi|_{K_Q} \otimes \tau_Q$ and $L|_{K_Q} \otimes \tau_Q$. Therefore, it induces an isometry between the subspaces of K_Q-invariants, which by (4.10) is identified with an isometry

$$m_\xi \otimes I\colon\quad \bar{V}(\xi) \otimes [\mathcal{H}_\xi \otimes V_\tau]^{K_Q} \xrightarrow{\simeq} \mathcal{A}_{2,Q}(\tau)_\xi. \tag{4.12}$$

Since $T \mapsto \psi_T$ is the composition of (4.11) with (4.12), the result follows. □

It follows from Lemma 3.4 that

$$L^2(K:\xi:\tau_\vartheta) \simeq L^2(K:\xi)_\vartheta,$$

with an isometric isomorphism. The latter space is equal to $C^\infty(K:\xi)_\vartheta$, in view of Lemmas 4.3 (b) and 3.4. Accordingly, the map $T \mapsto \psi_T$, defined for $\tau = \tau_\vartheta$, may naturally be viewed as an isometric isomorphism

$$T \mapsto \psi_T, \qquad \bar{V}(Q,\xi) \otimes C^\infty(K:\xi)_\vartheta \ \xrightarrow{\simeq}\ \mathcal{A}_{2,Q}(\tau_\vartheta)_\xi. \tag{4.13}$$

Moreover, it is given by the following formula, for $T = \eta \otimes \varphi \in \bar{V}(Q,\xi) \otimes C^\infty(K:\xi)_\vartheta$;

$$\mathrm{pr}_v \psi_T = \eta_v(\varphi(e)), \qquad (v \in {}^Q\mathcal{W}).$$

We now come to the connection with the normalized Eisenstein integral $E^\circ_\tau(Q:\nu) = E^\circ(Q:\nu)$, defined as in [12], Def. 13.7. The Eisenstein integral is meromorphic in the variable $\nu \in \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$, as a function with values in $C^\infty(\mathrm{X}) \otimes \mathrm{Hom}(\mathcal{A}_{2,Q}, V_\tau)$. If $\psi \in \mathcal{A}_{2,Q}$, we agree to write $E^\circ(Q:\psi:\nu:\cdot) = E^\circ(Q:\nu:\cdot)\psi$. Then $E^\circ(Q:\psi:\nu) \in C^\infty(\mathrm{X}:\tau)$, for generic $\nu \in \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$.

We need a 'functorial' property of the normalized Eisenstein integral that we shall now describe. Let $(\tau', V_{\tau'})$ be a second finite dimensional unitary representation of K, and let $S: V_\tau \to V_{\tau'}$ be a K-equivariant linear map. Then via action on the last tensor component, S naturally induces linear maps $C^\infty(K:\xi:\tau) \to C^\infty(K:\xi:\tau')$, $\mathcal{A}_{2,Q}(\tau)_\xi \to \mathcal{A}_{2,Q}(\tau')_\xi$ and $C^\infty(\mathrm{X}:\tau) \to C^\infty(\mathrm{X}:\tau')$ that we all denote by $I \otimes S$.

Lemma 4.5 *Let $S: V_\tau \to V_{\tau'}$ be a K-equivariant map as above.*

(a) *Let $T \in \bar{V}(\xi) \otimes C^\infty(K:\xi:\tau)$. Then $\psi_{[I\otimes I\otimes S]T} = [I \otimes S]\psi_T$.*
(b) *Let $\psi \in \mathcal{A}_{2,Q}(\tau)$. Then*

$$[I \otimes S]E^\circ_\tau(Q:\psi:\nu) = E^\circ_{\tau'}(Q:[I\otimes S]\psi:\nu),$$

as a meromorphic $C^\infty(\mathrm{X}:\tau)$-valued identity in the variable $\nu \in \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$.

Proof: (a) is a straightforward consequence of the definitions. Assertion (b) follows from the characterization of the Eisenstein integral in [12], Def. 13.7. More precisely, it follows from the mentioned definition and [12], Prop. 13.6 (a), that the family $f = E^\circ(Q:\psi)$ belongs to $\mathcal{E}^{\mathrm{hyp}}_Q(\mathrm{X}:\tau)$. See [12], Def. 6.6, for the definition of the latter space. Moreover, still by [12], Prop. 13.6, for ν in a non-empty open subset Ω of $\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$, each $v \in {}^Q\mathcal{W}$ and all $X \in \mathfrak{a}_{Q\mathrm{q}}$ and $m \in \mathrm{X}_{Q,v,+}$,

$$q_{v-\rho_Q}(Q, v \mid f_\nu, X, m) = \psi_v(m). \tag{4.14}$$

It readily follows from the definitions that $g: (\nu, x) \mapsto S(f(\nu,x))$ belongs to $\mathcal{E}^{\mathrm{hyp}}_Q(\mathrm{X}:\tau')$; moreover, (4.14) implies that

$$q_{v-\rho_Q}(Q, v \mid g_\nu, X, m) = S(\psi_v(m)) = [\mathrm{pr}_v[I\otimes S]\psi](m),$$

for all $\nu \in \Omega$, each $v \in {}^Q\mathcal{W}$, and all $X \in \mathfrak{a}_{Q\mathrm{q}}$ and $m \in \mathrm{X}_{Q,v,+}$. In view of [12], Def. 13.7 and Prop. 13.6 (a), this implies that $g = E^\circ(Q:[I\otimes S]\psi)$. □

If $\vartheta \subset \widehat{K}$ is a finite subset and $\psi \in \mathcal{A}_{2,Q}(\tau_\vartheta)$, we denote the associated normalized Eisenstein integral $E^\circ_{\tau_\vartheta}(Q:\psi:\nu)$ also by $E^\circ_\vartheta(Q:\psi:\nu)$. This Eisenstein integral is a smooth τ_ϑ-spherical function, depending meromorphically on the parameter $\nu \in \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$.

Lemma 4.5 implies an obvious relation between the Eisenstein integrals $E^\circ_\vartheta(Q:\psi:\nu)$ for different subsets ϑ. If $\vartheta \subset \vartheta'$ are finite subsets of $\widehat{K}$, then $\mathbf{V}_\vartheta \subset \mathbf{V}_{\vartheta'}$. The associated inclusion map is denoted by $\mathrm{i}_{\vartheta',\vartheta}$; it intertwines τ_ϑ with $\tau_{\vartheta'}$. From Lemmas 3.6 and 4.5 (a) it follows that

$$\begin{aligned}\psi_{[I\otimes \mathrm{i}_{\vartheta',\vartheta}]T} &= \psi_{[I\otimes I\otimes \mathrm{i}_{\vartheta',\vartheta}][I\otimes\varsigma_\vartheta]T}\\ &= [I\otimes \mathrm{i}_{\vartheta',\vartheta}]\psi_T, \qquad \big(T\in \bar V(\xi)\otimes C^\infty(K:\xi)_\vartheta\big).\end{aligned} \tag{4.15}$$

Moreover, from Lemma 4.5 (b) it follows that

$$E^\circ_{\vartheta'}(Q:[I\otimes \mathrm{i}_{\vartheta',\vartheta}]\psi:\nu) = [I\otimes \mathrm{i}_{\vartheta',\vartheta}]E^\circ_\vartheta(Q:\psi:\nu), \quad (\psi\in \mathcal{A}_{2,Q}(\tau_\vartheta)). \tag{4.16}$$

We have similar formulas for the K-equivariant projection operator $P_{\vartheta,\vartheta'}$: $\mathbf{V}_{\vartheta'}\to \mathbf{V}_\vartheta$. From Lemmas 3.6 and 4.5 it follows that

$$\psi_{[I\otimes I\otimes P_{\vartheta,\vartheta'}]T} = [I\otimes P_{\vartheta,\vartheta'}]\psi_T, \qquad \big(T\in \bar V(\xi)\otimes C^\infty(K:\xi)_{\vartheta'}\big), \tag{4.17}$$

and

$$E^\circ_\vartheta(Q:[I\otimes P_{\vartheta,\vartheta'}]\psi:\nu) = [I\otimes P_{\vartheta,\vartheta'}]E^\circ_{\vartheta'}(Q:\psi:\nu), \quad (\psi\in \mathcal{A}_{2,Q}(\tau_{\vartheta'})). \tag{4.18}$$

We recall from [12], § 4, that a $\Sigma_r(Q)$-hyperplane in $\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$ is a hyperplane of the form $(\alpha^\perp)_{\mathbb{C}}+\xi$, with $\alpha\in\Sigma_r(Q)$ and $\xi\in\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$. The hyperplane is said to be real if ξ may be chosen from $\mathfrak{a}^*_{Q\mathrm{q}}$. If $\vartheta\subset\widehat{K}$ is a finite subset, then by [12], Prop. 13.14, there exists a locally finite collection $\mathcal{H}$ of real $\Sigma_r(Q)$-hyperplanes in $\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$ such that for each $T\in\bar V(\xi)\otimes C^\infty(K:\xi)_\vartheta$ the function $\nu\mapsto E^\circ_\vartheta(Q:\psi_T:\nu)$ has a singular locus contained in $\cup\mathcal{H}$. We denote by $\mathcal{H}(Q,\xi,\vartheta)$ the minimal collection with this property. It follows from the definition just given that $\vartheta\subset\vartheta'\Rightarrow\mathcal{H}(Q,\xi,\vartheta)\subset\mathcal{H}(Q,\xi,\vartheta')$. Let $\mathcal{H}(Q,\xi)$ denote the union of the collections $\mathcal{H}(Q,\xi,\vartheta)$, as ϑ ranges over the collection of finite subsets of $\widehat{K}$. Then

$$i\mathfrak{a}^*_{Q\mathrm{q}}\cap\cup\mathcal{H}(Q,\xi)=\emptyset, \tag{4.19}$$

by the regularity theorem for the normalized Eisenstein integral, see [12], Thm. 18.8.

For $\nu\in\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}\setminus\cup\mathcal{H}(Q,\xi)$, we define the linear map

$$J_{Q,\xi,\nu}=J_{\xi,\nu}\colon\ \bar V(\xi)\otimes C^\infty(K:\xi)_K\to C^\infty(\mathrm{X})_K$$

by

$$J_{\xi,\nu}(T)(x)=E^\circ_\vartheta(Q:\psi_T:\nu:x)(e), \qquad (x\in\mathrm{X}), \tag{4.20}$$

for $\vartheta\subset\widehat{K}$ a finite subset and $T\in\bar V(\xi)\otimes C^\infty(K:\xi)_\vartheta$. This definition is unambiguous in view of (4.15) and (4.16).

Theorem 4.6 *Let $Q \in \mathcal{P}_\sigma$ and $\xi \in \mathrm{X}^\wedge_{Q,*,ds}$. The collection $\mathcal{H}(Q,\xi)$ consists of real $\Sigma_r(Q)$-hyperplanes and is locally finite. Its union is disjoint from $i\mathfrak{a}^*_{Q\mathrm{q}}$. Let $\nu \in \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$ be in the complement of this union. Then $J_{Q,\xi,\nu}$ is a $(\mathfrak{g},K)$-intertwining map from $\bar{V}(\xi)\otimes C^\infty(K:\xi)_K$, equipped with the induced representation $1\otimes\pi_{Q,\xi,-\nu}$, to $C^\infty(\mathrm{X})_K$, equipped with the $(\mathfrak{g},K)$-module structure induced by the left regular representation of G in $C^\infty(\mathrm{X})$.*

The proof of this theorem will be given in the next two sections. In Sect. 5 we investigate uniformity of generators for $\pi_{Q,\xi,\nu}$ relative to the parameter ν. In Sect. 6 we shall investigate the effect of left differentiations on left spherical functions.

5 Generators of induced representations

In this section we show that the parabolically induced representations, introduced in Sect. 4, are generated by finitely many K-finite vectors, with local uniformity in the continuous induction parameter.

Proposition 5.1 *Let $Q \in \mathcal{P}_\sigma$ and let ξ be a unitary representation of M_Q of finite length. Assume that $\Omega \subset \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$ is a bounded subset. Then there exists a finite subset $\vartheta \subset \widehat{K}$ such that, for all $\nu \in \Omega$,*

$$\pi_{Q,\xi,\nu}(U(\mathfrak{g}))C^\infty(K:\xi)_\vartheta = C^\infty(K:\xi)_K. \tag{5.1}$$

Remark 5.2 In particular, the result holds for $\sigma = \theta$; then Q is an arbitrary parabolic subgroup of G and $\mathfrak{a}_{Q\mathrm{q}}$ equals its usual Langlands split component $\mathfrak{a}_Q$.

Proof: It suffices to prove the result for ξ irreducible. We shall do this by a method given for ξ tempered in [28], § 5.5.5. Let

$$\omega := \{\nu \in \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}} \mid \quad \langle \mathrm{Re}\,\nu - \rho_Q\,,\,\alpha\rangle > 0, \quad \forall\,\alpha \in \Delta_r(Q)\}.$$

Then for $\nu \in \omega$ we may define the standard intertwining operator $A(\nu) = A(\bar{Q}:Q:\xi:\nu)$ from $C^\infty(Q:\xi:\nu)$ to $C^\infty(\bar{Q}:\xi:\nu)$, by

$$A(\nu)f(x) = \int_{\bar{N}_Q} f(\bar{n}\,x)\,d\bar{n}, \qquad (x \in G),$$

where $d\bar{n}$ denotes a choice of Haar measure on $\bar{N}_Q$. The integral is absolutely convergent; this follows by an argument that involves estimates completely analogous to the ones given for Q minimal in [4], proof of Lemma 15.6. It also follows from these estimates that, for $f \in C^\infty(K:\xi)$, the function $A(\nu)f \in C^\infty(K:\xi)$ depends holomorphically on $\nu \in \omega$.

Lemma 5.3 *Let $f, g \in C^\infty(K:\xi)$, $\nu \in \omega$ and $X \in \mathfrak{a}^+_{Q\mathrm{q}}$. Then*

$$\lim_{t\to\infty} e^{t(-\nu+\rho_Q)(X)}\langle \pi_{Q,\xi,\nu}(m\exp tX)f \mid g\rangle = \langle \xi(m)[A(\nu)f](e) \mid g(e)\rangle_\xi. \tag{5.2}$$

Proof: See [29], Lemma 10.5.1. □

Completion of the proof of Prop. 5.1: From (5.2) it can be deduced, by an argument due to Langlands [24], Lemma 3.13, see also Milicic [25], Proof of Thm. 1, that if $f \in C^\infty(Q:\xi:\nu)_K$ and $A(\nu)f \neq 0$, then f is a cyclic vector for $\pi_{\xi,\nu}$ in the sense that the $(\mathfrak{g}, K)$-module generated by f equals $C^\infty(Q:\xi:\nu)_K$. See also [29], Cor. 10.5.2. We can now prove the result in the case that the closure of Ω is contained in ω. Indeed, assume this to be the case and let $\nu_0 \in \overline{\Omega}$. Since $f \mapsto A(\nu_0)f(e)$ can be expressed as a convolution operator with non-trivial kernel, there exists a finite set $\vartheta \subset \widehat{K}$ and a function $f \in C^\infty(K:\xi)_\vartheta$ such that $A(\nu_0)f(e) \neq 0$; by continuity in the parameter ν there exists an open neighborhood ω_0 of ν_0 in ω such that $A(\nu)f(e) \neq 0$ for all $\nu \in \omega_0$. From what we said above, it follows that (5.1) holds for all $\nu \in \omega_0$. By compactness of the set $\overline{\Omega}$, the result now readily follows in case $\overline{\Omega}$ is contained in ω.

We shall now use tensoring with a finite dimensional representation to extend the result to an arbitrary bounded subset $\Omega \subset \mathfrak{a}_{Q\mathfrak{q}\mathbb{C}}^*$.

Let $P \in \mathcal{P}_\sigma^{\min}$ be such that $P \subset Q$. Let $\Delta_Q(P) := \{\alpha \in \Delta(P) \mid \alpha|_{\mathfrak{a}_{Q\mathfrak{q}}} = 0\}$ and put $\Delta(Q) = \Delta(P) \setminus \Delta_Q(P)$. We fix $n \in \mathbb{N}$ such that $\langle \mathrm{Re}\,\nu - \rho_Q\,,\,\alpha\rangle/\langle\alpha\,,\,\alpha\rangle > -8n$ for all $\nu \in \bar{\Omega}$ and $\alpha \in \Delta(Q)$. We fix $\mu \in \mathfrak{a}_\mathfrak{q}^*$ with the property that $\langle\mu\,,\,\alpha\rangle/\langle\alpha\,,\,\alpha\rangle$ equals $8n$ for all $\alpha \in \Delta(Q)$ and zero for all $\alpha \in \Delta_Q(P)$. Then $\mu + \overline{\Omega} \subset \omega$. Hence there exists a finite subset $\vartheta' \subset \widehat{K}$ such that $\pi_{\xi,\nu+\mu}(U(\mathfrak{g}))C^\infty(K:\xi)_{\vartheta'} = C^\infty(K:\xi)_K$, for all $\nu \in \overline{\Omega}$.

It follows from the condition on μ that $\langle\mu\,,\,\alpha\rangle/2\langle\alpha\,,\,\alpha\rangle \in 4\mathbb{Z}$ for all $\alpha \in \Delta(P)$. Since Σ is a possibly non-reduced root system, this implies that $\langle\mu\,,\,\alpha\rangle/2\langle\alpha\,,\,\alpha\rangle \in 2\mathbb{Z}$ for all $\alpha \in \Sigma$. According to [4], Cor. 5.7 and Prop. 5.5, there exists a class one finite dimensional irreducible G-module (F,π) of $\Delta(P)$-highest $\mathfrak{a}_\mathfrak{q}$-weight μ; the highest weight space F_μ is one dimensional, and $M_\sigma = M_{P\sigma}$ acts trivially on it. Since $M_{Q\sigma}$ centralizes $\mathfrak{a}_{Q\mathfrak{q}}$, it normalizes F_μ. By compactness it follows that $(K_Q)_e$ acts trivially on F_μ. Since μ vanishes on ${}^*\mathfrak{a}_{Q\mathfrak{q}} = \mathfrak{a}_\mathfrak{q} \cap \mathfrak{m}_Q$, it follows that ${}^*A_{Q\mathfrak{q}}$ also acts trivially on F_μ. Finally, since $M_{Q\sigma}$ is generated by M_σ, $(K_Q)_e$ and ${}^*A_{Q\mathfrak{q}}$, it follows that $M_{Q\sigma}$ acts by the identity on F_μ.

Let $e_\mu \in F_\mu$ be a non-trivial highest weight vector. Then the map $m\colon F^* \to C^\infty(G)$ defined by $m(v)(x) = v(\pi(x^{-1})e_\mu)$ is readily seen to be an equivariant map from F^* into $C^\infty(Q:1:-\mu)$. The map $M_\nu\colon C^\infty(Q:\xi:\nu+\mu)\otimes F^* \to C^\infty(Q:\xi:\nu)$ given by $M_\nu(\varphi\otimes v) = m(v)\varphi$ is G-equivariant, for every $\nu \in \mathfrak{a}_{Q\mathfrak{q}\mathbb{C}}^*$.

Let $v_K \in F^*$ be a non-trivial K-fixed vector. Then, since $G = QK$, the function $m(v_K)$ is nowhere vanishing. From this we see that M_ν is surjective, for every $\nu \in \mathfrak{a}_{Q\mathfrak{q}\mathbb{C}}^*$. It follows that the $U(\mathfrak{g})$-module generated by $V_\nu := M_\nu(C^\infty(Q:\xi:\nu+\mu)_{\vartheta'}\otimes F^*)$ equals $C^\infty(Q:\xi:\nu)_K$, for all $\nu \in \overline{\Omega}$. Let $\vartheta \subset \widehat{K}$ be the collection of all K-types occurring in $\delta\otimes F^*$ for some $\delta \in \vartheta'$. Then ϑ is a finite set and $V_\nu \subset C^\infty(Q:\xi:\nu)_\vartheta$, for all $\nu \in \mathfrak{a}_{Q\mathfrak{q}\mathbb{C}}^*$. Hence, (5.1) follows for all $\nu \in \Omega$. □

6 Differentiation of spherical functions

In this section we assume (τ, V_τ) to be a finite dimensional unitary representation of K. We shall investigate the action of L_Z, for $Z \in \mathfrak{g}$, on the Eisenstein integral $E^\circ(Q:\nu)$. Here L denotes the infinitesimal left regular representation. As a preparation, we shall first investigate the action of L_Z on functions from the space $C^\infty(\mathrm{X}_+:\tau)$, defined in [12], § 6. Secondly, we shall investigate the action of L_Z on families from $\mathcal{E}_Q^{\mathrm{hyp}}(\mathrm{X}:\tau)$, defined in [12], Def. 6.6.

Given a function $F \in C^\infty(\mathrm{X}_+:\tau)$, we define the function $\widetilde{F}:\mathrm{X}_+ \to \mathfrak{g}_{\mathbb{C}}^* \otimes V_\tau \simeq \mathrm{Hom}(\mathfrak{g}_{\mathbb{C}}, V_\tau)$ by

$$\widetilde{F}(x)(Z) = L_Z F(x), \qquad (x \in \mathrm{X}_+,\ Z \in \mathfrak{g}_{\mathbb{C}}).$$

One readily checks that

$$\widetilde{F}(kx)(Z) = \tau(k)\widetilde{F}(x)(\mathrm{Ad}(k^{-1})Z), \qquad (x \in \mathrm{X}_+,\ k \in K,\ Z \in \mathfrak{g}_{\mathbb{C}}).$$

Hence, $\widetilde{F}$ is a spherical function of its own right. In fact, let $\mathrm{Ad}_K^\vee$ denote the restriction to K of the coadjoint representation of G in $\mathfrak{g}_{\mathbb{C}}^*$ and put $\widetilde{\tau} := \mathrm{Ad}_K^\vee \otimes \tau$. Then

$$\widetilde{F} \in C^\infty(\mathrm{X}_+:\widetilde{\tau}).$$

Our first objective is to show that if F has a certain converging expansion towards infinity along (Q, v), for $Q \in \mathcal{P}_\sigma$ and $v \in N_K(\mathfrak{a}_{\mathrm{q}})$, then $\widetilde{F}$ has a similar expansion, which can be computed in terms of that of F. As a preparation, we study sets consisting of points of the form mav, where $v \in N_K(\mathfrak{a}_{\mathrm{q}})$, $m \in M_{Q\sigma}$ and $a \to \infty$ in $A_{Q\mathrm{q}}^+$. They describe regions of convergence for the expansions involved, in the spirit of [11], § 3. We will also describe decompositions of elements of $\mathfrak{g}$ along such sets, in a fashion similar to [11], § 4. These will be needed to compute the expansion of $\widetilde{F}$.

Let $Q \in \mathcal{P}_\sigma$. We define the function $R_{Q,v}: M_{1Q} \to]0, \infty[$ as in [11], Sect. 3. Recall that $R_{Q,v}$ is left K_Q- and right $M_{1Q} \cap vHv^{-1}$-invariant; thus, it may be viewed as a function on $\mathrm{X}_{1Q,v}$. If $Q = G$, then $R_{Q,v}$ equals the constant function 1 and if $Q \neq G$, then according to [11], Lemma 3.2, it is given by

$$R_{Q,v}(au) = \max_{\alpha \in \Sigma(Q)} a^{-\alpha},$$

for $a \in A_{\mathrm{q}}$ and $u \in N_{K_Q}(\mathfrak{a}_{\mathrm{q}})$. The inclusion map $M_Q \to M_{1Q}$ induces an embedding via which we may identify $\mathrm{X}_{Q,v}$ with a sub M_Q-manifold of $\mathrm{X}_{1Q,v}$. From [11], Lemma 3.2, we recall that $R_{Q,v} \geq 1$ on $\mathrm{X}_{Q,v}$.

Lemma 6.1 *Let $v \in N_K(\mathfrak{a}_{\mathrm{q}})$ and put $Q' = v^{-1}Qv$. Then*

$$R_{Q,v}(m) = R_{Q',1}(v^{-1}mv), \qquad (m \in M_{1Q}).$$

Proof: This follows immediately from the characterization of $R_{Q,v}$ given above. □

In accordance with [11], Eqn. (3.7), we define, for $v \in N_K(\mathfrak{a}_\mathrm{q})$ and $R > 0$,

$$M_{1Q,v}[R] := \{m \in M_{1Q} \mid R_{Q,v}(m) < R\},$$

and $M_{Q\sigma,v}[R] := M_{Q\sigma} \cap M_{1Q,v}[R]$. Note that $M_{1Q,1}[R]$ and $M_{Q\sigma,1}[R]$ equal the sets $M_{1Q}[R]$ and $M_{Q\sigma}[R]$, defined in [11], text preceding Lemma 4.7, respectively. Finally, for $R > 0$ we define

$$A^+_{Q\mathrm{q}}(R) := \big\{a \in A_{Q\mathrm{q}} \mid a^{-\alpha} < R \quad \text{for all} \quad \alpha \in \Delta_r(Q)\big\}. \tag{6.1}$$

Lemma 6.2 *Let $v \in N_K(\mathfrak{a}_\mathrm{q})$ and put $Q' = v^{-1}Qv$. Let $R > 0$.*

(a) $M_{1Q,v}[R] = vM_{1Q'}[R]v^{-1}, \quad M_{Q\sigma,v}[R] = vM_{Q'\sigma}[R]v^{-1}$.
(b) $A^+_{Q\mathrm{q}}(R) = vA^+_{Q'\mathrm{q}}(R)v^{-1}$.

Proof: Assertion (a) follows readily from combining Lemma 6.1 with the definitions of the sets involved. Assertion (b) is clear from (6.1). □

We define the open dense subset M'_{1Q} of M_{1Q} as in [11], Eqn. (4.3). Write $\mathfrak{g}^\pm := \ker(-I \pm \theta\sigma)$ and put $\mathfrak{g}^\pm_\alpha := \mathfrak{g}_\alpha \cap \mathfrak{g}^\pm$, for $\alpha \in \Sigma$. Write $H_{1Q} := M_{1Q} \cap H$. Then by [11], Cor. 4.2,

$$\begin{aligned} M'_{1Q} &= K_Q\,[M'_{1Q} \cap A_\mathrm{q}]\,H_{1Q}, \\ M'_{1Q} \cap A_\mathrm{q} &= \big\{a \in A_\mathrm{q} \mid a^\alpha \neq 1 \text{ for all } \alpha \in \Sigma(Q) \text{ with } \mathfrak{g}^+_\alpha \neq 0\big\}. \end{aligned} \tag{6.2}$$

In particular, M'_{1Q} is a left K_Q- and right H_{1Q}-invariant open dense subset of M_{1Q}. If $v \in N_K(\mathfrak{a}_\mathrm{q})$, then by $M'_{1Q,v}$ we denote the analogue of the set M'_{1Q} for the pair (G, vHv^{-1}).

Lemma 6.3 *Let $v \in N_K(\mathfrak{a}_\mathrm{q})$ and put $Q' = v^{-1}Qv$. Then $M'_{1Q,v} := vM'_{1Q'}v^{-1}$.*

Proof: This readily follows from the definition. □

Lemma 6.4 *Let $v \in N_K(\mathfrak{a}_\mathrm{q})$.*

(a) $M_{1Q,v}[1] \subset M'_{1Q,v}$.
(b) *Let $R_1, R_2 > 0$. Then $M_{Q\sigma,v}[R_1]A^+_{Q\mathrm{q}}(R_2) \subset M_{1Q,v}[R_1R_2]$.*

Proof: For $v = 1$, the results are given in [11], Lemma 4.7. Let now v be arbitrary and put $Q' = v^{-1}Qv$. Using Lemma 6.2 (a) with $R = 1$ and Lemma 6.3 we obtain (a) from the similar statement with Q', 1 in place of Q, v. Likewise, assertion (b) follows by application of Lemma 6.2. □

We now come to the investigation of decompositions in $\mathfrak{g}$, needed for the study of the asymptotic behavior of $\widetilde{F}$. Write $\mathfrak{k}_{(Q)} := \mathfrak{k} \cap (\mathfrak{n}_Q + \bar{\mathfrak{n}}_Q)$. Then $I + \theta\colon X \mapsto X + \theta X$ is a linear isomorphism from $\bar{\mathfrak{n}}_Q$ onto $\mathfrak{k}_{(Q)}$. For $\alpha \in \Sigma$ we put $\mathfrak{k}^\pm_\alpha := (I + \theta)(\mathfrak{g}^\pm_{-\alpha})$. Then $\mathfrak{k}_{(Q)}$ is the direct sum of the spaces $\mathfrak{k}^\pm_\alpha$, for $\alpha \in \Sigma(Q)$.

Lemma 6.5 *Let $v \in N_K(\mathfrak{a}_\mathfrak{q})$. If $m \in M'_{1Q,v}$, then $\mathfrak{n}_Q \subset \mathfrak{k}_{(Q)} \oplus \mathrm{Ad}(mv)\mathfrak{h}$.*

Proof: For $v = 1$ this follows from [11], Lemma 4.3 (b), with $\bar{Q}$ in place of Q. If v is arbitrary, put $Q' = v^{-1}Qv$. Then for $m \in M'_{1Q,v}$ we have $v^{-1}mv \in M'_{1Q'}$, hence $\mathrm{Ad}(v^{-1})\mathfrak{n}_Q = \mathfrak{n}_{Q'} \subset \mathfrak{k}_{(Q')} \oplus \mathrm{Ad}(v^{-1}mv)\mathfrak{h}$, and the result follows by application of $\mathrm{Ad}(v)$. □

By the above lemma, for $m \in M'_{1Q,v}$ we may define a linear map $\Phi(m) = \Phi_{Q,v}(m) \in \mathrm{Hom}(\mathfrak{n}_Q, \mathfrak{k}_{(Q)})$ by

$$X \in \Phi(m)X + \mathrm{Ad}(mv)\mathfrak{h}, \qquad (X \in \mathfrak{n}_Q). \tag{6.3}$$

It is readily seen that $\Phi_{Q,v}$ is an analytic $\mathrm{Hom}(\mathfrak{n}_Q, \mathfrak{k}_{(Q)})$-valued function on $M'_{1Q,v}$.

Lemma 6.6 *If $m \in M'_{1Q,v}$, $k \in K_Q$ and $h \in M_{1Q} \cap vHv^{-1}$, then*

$$\Phi(kmh) = \mathrm{Ad}(k) \circ \Phi(m) \circ \mathrm{Ad}(k)^{-1}.$$

Proof: Since M_{1Q} normalizes $\mathfrak{n}_Q$ and K_Q normalizes $\mathfrak{k}_{(Q)}$ the result is an immediate consequence of the definition in equation (6.3). □

Lemma 6.7 *Let $v \in N_K(\mathfrak{a}_\mathfrak{q})$ and put $Q' = v^{-1}Qv$. Then, for all $m \in M'_{1Q,v}$,*

$$\Phi_{Q,v}(m) = \mathrm{Ad}(v) \circ \Phi_{Q',1}(v^{-1}mv) \circ \mathrm{Ad}(v)^{-1}.$$

Proof: This follows from (6.3), by the same reasoning as in the proof of Lemma 6.5. □

Let $\Psi = \Psi_Q \colon M'_{1Q} \to \mathrm{Hom}(\bar{\mathfrak{n}}_Q, \mathfrak{k}_{(Q)})$ be defined as in [11], Eqn. (4.4). Then, for $X \in \bar{\mathfrak{n}}_Q$ and $m \in M'_{1Q}$,

$$X \in \mathrm{Ad}(m)^{-1}\Psi(m)X + \mathfrak{h}. \tag{6.4}$$

Lemma 6.8 *Let $m \in M'_{1Q}$. Then*

$$\Phi_{Q,1}(m) = -\Psi(m) \circ \sigma \circ \mathrm{Ad}(m^{-1}). \tag{6.5}$$

Proof: If $X \in \mathfrak{n}_Q$ and $m \in M'_{1Q}$, then $\sigma\mathrm{Ad}(m^{-1})X \in \bar{\mathfrak{n}}_Q$, so that $\sigma\mathrm{Ad}(m^{-1})X$ belongs to $\mathrm{Ad}(m^{-1})\Psi(m)\sigma\mathrm{Ad}(m^{-1})X + \mathfrak{h}$. Since $\mathrm{Ad}(m^{-1})X \in -\sigma\mathrm{Ad}(m^{-1})X + \mathfrak{h}$, this implies that

$$\mathrm{Ad}(m^{-1})X \in -\mathrm{Ad}(m^{-1})\Psi(m)\sigma\mathrm{Ad}(m^{-1})X + \mathfrak{h}.$$

Comparing with the definition of $\Phi_{Q,1}(m)$ given in (6.3) with $v = 1$, we obtain the desired identity. □

In the formulation of the next result we use the terminology of neat convergence of exponential polynomial series, introduced in [11], § 1.

Proposition 6.9 *Let $v \in N_K(\mathfrak{a}_\mathrm{q})$. There exist unique real analytic $\mathrm{Hom}(\mathfrak{n}_Q, \mathfrak{k}_{(Q)})$-valued functions $\Phi_\mu = \Phi_{Q,v,\mu}$ on $M_{Q\sigma}$, for $\mu \in \mathbb{N}\Delta_r(Q)$, such that, for every $m \in M_{Q\sigma}$ and all $a \in A^+_{Q\mathrm{q}}(R_{Q,v}(m)^{-1})$,*

$$\Phi_{Q,v}(ma) = \sum_{\mu \in \mathbb{N}\Delta_r(Q)} a^{-\mu} \Phi_\mu(m), \tag{6.6}$$

with absolutely convergent series. Moreover, $\Phi_0 = 0$. Finally, for every $R > 1$ the series in (6.6) converges neatly on $A^+_{Q\mathrm{q}}(R^{-1})$ as a $\Delta_r(Q)$-power series with coefficients in $C^\infty(M_{Q\sigma,v}[R]) \otimes \mathrm{Hom}(\mathfrak{n}_Q, \mathfrak{k}_{(Q)})$.

Proof: We first assume that $v = 1$. Let $\Psi_\mu \colon M_{Q\sigma} \to \mathrm{End}(\bar{\mathfrak{n}}_Q)$ be as in [11], Prop. 4.8. Then it follows from combining the mentioned proposition with (6.5) that, for $m \in M_{Q\sigma}$ and $a \in A^+_{Q\mathrm{q}}(R_{Q,1}(m)^{-1})$,

$$\Phi(ma) = -(I + \theta) \circ \sum_{\mu \in \mathbb{N}\Delta_r(Q)} a^{-\mu} \Psi_\mu(m) \circ \sigma \circ \mathrm{Ad}(ma)^{-1},$$

with absolutely convergent series. We now see that the restriction of $\Phi(ma)$ to $\mathfrak{g}_\alpha$, for $\alpha \in \Sigma(Q)$, equals

$$-(I + \theta) \circ \sum_{\mu \in \mathbb{N}\Delta_r(Q)} a^{-\mu-\alpha} \Psi_\mu(m) \circ \sigma \circ \mathrm{Ad}(m)|_{\mathfrak{g}_\alpha}.$$

Put $\Phi_0 = 0$ and, for $\nu \in \mathbb{N}\Delta_r(Q) \setminus \{0\}$, define $\Phi_\nu(m) \in \mathrm{Hom}(\mathfrak{n}_Q, \mathfrak{k}_{(Q)})$ by

$$\Phi_\nu(m)|_{\mathfrak{g}_\alpha} := -(I + \theta) \circ \Psi_{\nu-\alpha}(m) \circ \sigma \circ \mathrm{Ad}(m)|_{\mathfrak{g}_\alpha}$$

if $\nu - \alpha \in \mathbb{N}\Delta_r(Q)$, and by $\Phi_\nu(m)|_{\mathfrak{g}_\alpha} = 0$ otherwise. Then (6.6) follows with absolute convergence. All remaining assertions about convergence follow from the analogous assertions in [11], Prop. 4.8.

We now turn to the case that v is general. Let $Q' = v^{-1}Qv$, and define

$$\Phi_{Q,v,\mu}(m) = \mathrm{Ad}(v) \circ \Phi_{Q',1,\mathrm{Ad}(v)^{-1}\mu}(v^{-1}mv) \circ \mathrm{Ad}(v)^{-1},$$

for $\mu \in \mathbb{N}\Delta_r(Q)$ and $m \in M_{Q\sigma}$. Then all assertions follow from the similar assertions with Q', 1 in place of Q, v, by application of Lemmas 6.7 and 6.2. □

We now come to the behavior of L_Z, for $Z \in \mathfrak{g}_\mathbb{C}$, at points of the form mav, with $v \in N_K(\mathfrak{a}_\mathrm{q})$, $m \in M_{Q\sigma}$ and $a \to \infty$ in $A^+_{Q\mathrm{q}}$. We start by observing that

$$\mathfrak{g} = \mathfrak{n}_Q \oplus \mathfrak{a}_{Q\mathrm{q}} \oplus (\mathfrak{m}_{Q\sigma} \cap \mathfrak{p}) \oplus \mathfrak{k}, \tag{6.7}$$

as a direct sum of linear spaces. Accordingly, we write, for $Z \in \mathfrak{g}_\mathbb{C}$,

$$Z = Z_\mathrm{n} + Z_\mathrm{a} + Z_\mathrm{m} + Z_\mathrm{k}, \tag{6.8}$$

with terms in the complexifications of the summands in (6.7), respectively. If $\mathfrak{l}$ is a real Lie algebra, then by $U(\mathfrak{l})$ we denote the universal enveloping algebra of its complexification, and by $U_k(\mathfrak{l})$, for $k \in \mathbb{N}$, the subspace of elements of order at most k. For $Z \in \mathfrak{g}_{\mathbb{C}}$ we define the element $D_0(Z) = D_{Q,v,0}(Z)$ of $U_1(\mathfrak{m}_{Q\sigma}) \otimes U_1(\mathfrak{a}_{Q\mathrm{q}}) \otimes \mathrm{End}(V_\tau)$ by

$$D_0(Z) := Z_{\mathrm{m}} \otimes I \otimes I + I \otimes Z_{\mathrm{a}} \otimes I + I \otimes I \otimes \tau(\check{Z}_{\mathrm{k}}), \tag{6.9}$$

where $X \mapsto \check{X}$ denotes the canonical anti-automorphism of $U(\mathfrak{g})$. If, moreover, $m \in M_{Q\sigma}$, we define, for $\mu \in \mathbb{N}\Delta_r(Q) \setminus \{0\}$, the element $D_\mu(Z,m) = D_{Q,v,\mu}(Z,m)$ of $U_1(\mathfrak{m}_{Q\sigma}) \otimes U_1(\mathfrak{a}_{Q\mathrm{q}}) \otimes \mathrm{End}(V_\tau)$ by

$$D_\mu(Z,m) := I \otimes I \otimes \tau(\Phi_{Q,v,\mu}(m)\check{Z}_{\mathrm{n}}).$$

Finally, if $m \in M_{Q\sigma}$ and $a \in A^+_{Q\mathrm{q}}(R_{Q,v}(m)^{-1})$, we define the element $D_{Q,v}(Z,a,m) \in U_1(\mathfrak{m}_{Q\sigma}) \otimes U_1(\mathfrak{a}_{Q\mathrm{q}}) \otimes \mathrm{End}(V_\tau)$ by

$$D_{Q,v}(Z,a,m) = \sum_{\mu \in \mathbb{N}\Delta_r(Q)} a^{-\mu} D_\mu(Z,m), \tag{6.10}$$

where we have put $D_0(Z,m) = D_0(Z)$. We also agree to write

$$D^+_{Q,v}(Z,a,m) := D_{Q,v}(Z,a,m) - D_0(Z).$$

It follows from Prop. 6.9 that, for each $R > 1$, the series (6.10) is neatly convergent on $A^+_{Q\mathrm{q}}(R^{-1})$ as a $\Delta_r(Q)$-exponential series with values in $C^\infty(M_{Q\sigma,v}[R]) \otimes U_1(\mathfrak{m}_{Q\sigma}) \otimes U_1(\mathfrak{a}_{Q\mathrm{q}}) \otimes \mathrm{End}(V_\tau)$. Moreover,

$$D^+_{Q,v}(Z,a,m) = I \otimes I \otimes \tau(\Phi_{Q,v}(ma)\check{Z}_{\mathrm{n}}). \tag{6.11}$$

In the formulation of the following result we use the notation of the paper [11], Sects. 1–3. Via the left regular representation, we view $U(\mathfrak{m}_{Q\sigma}) \otimes U(\mathfrak{a}_{Q\mathrm{q}}) \otimes \mathrm{End}(V_\tau)$ as the algebra of right-invariant differential operators on $M_{1Q} \simeq M_{Q\sigma} \times A_{Q\mathrm{q}}$, with coefficients in $\mathrm{End}(V_\tau)$.

Proposition 6.10 *Let $F \in C^{\mathrm{ep}}(\mathrm{X}_+ : \tau)$. Then $\widetilde{F} \in C^{\mathrm{ep}}(\mathrm{X}_+ : \widetilde{\tau})$. Moreover, if $Q \in \mathcal{P}_\sigma$ and $v \in N_K(\mathfrak{a}_{\mathrm{q}})$, then* $\mathrm{Exp}\,(Q, v \mid \widetilde{F}) \subset \mathrm{Exp}\,(Q, v \mid F) - \mathbb{N}\Delta_r(Q)$. *Finally, for every $Z \in \mathfrak{g}_{\mathbb{C}}$, the $\Delta_r(Q)$-exponential expansion*

$$\widetilde{F}(mav)(Z) = \sum_\xi a^\xi \, q_\xi(Q, v \mid \widetilde{F}, \log a, m)(Z) \tag{6.12}$$

along (Q, v) arises from the similar expansion

$$F(mav) = \sum_\xi a^\xi \, q_\xi(Q, v \mid F, \log a, m) \tag{6.13}$$

by the formal application of the expansion (6.10). In particular, if ξ is a leading exponent of F along (Q, v), then, for every $Z \in \mathfrak{g}_{\mathbb{C}}$,

$$q_\xi(Q, v \mid \widetilde{F}, \log(\cdot), \cdot)(Z) = [D_{Q,v,0}(Z) - \xi(Z_{\mathrm{a}})]\, q_\xi(Q, v \mid F, \log(\cdot), \cdot). \tag{6.14}$$

Proof: It is obvious that $\widetilde{F} \in C^\infty(\mathrm{X}_+ : \tau)$. We shall investigate its expansion along (Q, v), for $Q \in \mathcal{P}_\sigma$ and $v \in N_K(\mathfrak{a}_\mathrm{q})$. We start by observing that, for $R > 1$, the expansion (6.13) converges neatly on $A_\mathrm{q}^+(R^{-1})$ as a $\Delta_r(Q)$-exponential polynomial expansion in the variable a, with coefficients in the space $C^\infty(\mathrm{X}_{Q,v,+}[R] : \tau_Q)$, see [11], Thm. 3.4.

If φ is a smooth function on a Lie group L, with values in a complete locally convex space, then for $X \in \mathfrak{l}$ and $x \in L$ we put $\varphi(X; x) := d/dt\, \varphi(\exp tXx)|_{t=0}$. Accordingly, it follows from (6.8) that for $Z \in \mathfrak{g}_\mathbb{C}$, and $m \in M_{Q\sigma}$ and $a \in A_{Q\mathrm{q}}$ with $mav \in \mathrm{X}_+$, we have

$$\begin{aligned}\tilde{F}(mav)(Z) &= F(\check{Z}; mav)\\ &= \tau(\check{Z}_\mathrm{k})F(mav) + F(\check{Z}_\mathrm{m}; mav)\\ &\quad + F(\check{Z}_\mathrm{a}; mav) + F(\check{Z}_\mathrm{n}; mav). \end{aligned} \tag{6.15}$$

The sum of the first three terms allows an expansion that is obtained by the termwise formal application of $D_{Q,v,0}(Z)$ to the expansion (6.13), by [11], Lemmas 1.9 and 1.10. Moreover, the resulting expansion converges on $A_\mathrm{q}^+(R^{-1})$ as a $\Delta_r(Q)$-exponential polynomial expansion in the variable a, with coefficients in the space $C^\infty(\mathrm{X}_{Q,v,+}[R], V_\tau)$. Thus, it remains to discuss the last term in (6.15). Since F is right H-invariant and left τ-spherical, we see by application of (6.3) and (6.11) that the mentioned term may be rewritten as

$$\begin{aligned}F(\check{Z}_\mathrm{n}; mav) &= F(\Phi_{Q,v}(ma)\check{Z}_\mathrm{n}; mav)\\ &= \tau(\Phi_{Q,v}(ma)\check{Z}_\mathrm{n})F(mav)\\ &= D_{Q,v}^+(Z, a, m)F(\cdot\, v)(ma).\end{aligned}$$

It follows from Proposition 6.9 that the series for $D_{Q,v}^+(Z)$ converges neatly on $A_\mathrm{q}^+(R^{-1})$ as a $\Delta_r(Q)$-exponential polynomial expansion in the variable a, with coefficients in the space $C^\infty(M_{Q,\sigma,v}[R]) \otimes \mathrm{End}(V_\tau)$. From [11], Lemma 1.10, it now follows that $F(\check{Z}_\mathrm{n}; mav)$ admits a $\Delta_r(Q)$-exponential polynomial expansion that is obtained by the obvious formal application of the series for $D_{Q,v}^+(Z, a, m)$ to the series for $F(mav)$. The resulting series converges neatly on $A_{Q\mathrm{q}}^+(R^{-1})$ as a $\Delta_r(Q)$-exponential polynomial expansion in the variable a with coefficients in $C^\infty(M_{Q\sigma,v}[R], V_\tau)$. It follows that $\tilde{F}(mav)(Z)$ has an expansion of the type asserted along (Q, v), with exponents as indicated.

In particular, if Q is minimal, it follows that $\tilde{F}(mav)$ allows a neatly converging $\Delta(Q)$-exponential polynomial expansion in the variable $a \in A_\mathrm{q}^+(Q)$, with coefficients in $C^\infty(\mathrm{X}_{Q,v}) \otimes \mathfrak{g}_\mathbb{C}^* \otimes V_\tau$. This implies that $\widetilde{F}$ belongs to the space $C^{\mathrm{ep}}(\mathrm{X}_+ : \tilde{\tau})$, defined in [11], Def. 2.1.

It remains to prove the assertion about the leading exponent ξ for F along (Q, v). From the above discussion we readily see that the term in the expansion (6.12) with exponent ξ is obtained from the application of

the constant term $D_{Q,v,0}(Z)$ of $D_{Q,v}(Z,a,m)$ to the term in the expansion (6.13) with exponent ξ. This yields

$$a^{\xi} q_{\xi}(Q,v \mid \widetilde{F}, \log a, m)(Z) \\ = D_{Q,v,0}(Z)\big[(m,a) \mapsto a^{\xi} q_{\xi}(Q,v \mid F, \log a, m)\big].$$

Now use that $a^{-\xi} \circ D_{Q,v,0}(Z) \circ a^{\xi} = D_{Q,v,0}(Z) + \xi(\check{Z}_{\mathrm{a}})$ to obtain (6.14). □

We can now describe the action of L_Z, for $Z \in \mathfrak{g}_{\mathbb{C}}$, on families from the space $\mathcal{E}^{\mathrm{hyp}}_Q(\mathrm{X}:\tau)$, defined in [12], Def. 6.6.

Theorem 6.11 *Let $F \in \mathcal{E}^{\mathrm{hyp}}_Q(\mathrm{X}:\tau)$. Then the family $\widetilde{F}\colon \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}} \times \mathrm{X} \to \mathfrak{g}^*_{\mathbb{C}} \otimes V_\tau$, defined by $(\widetilde{F})_\nu = (F_\nu)^{\sim}$ belongs to $\mathcal{E}^{\mathrm{hyp}}_Q(\mathrm{X}:\widetilde{\tau})$. Moreover, for every $Z \in \mathfrak{g}_{\mathbb{C}}$ and all ν in an open dense subset of $\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$,*

$$q_{\nu-\rho_Q}(Q,v \mid \widetilde{F}_\nu : \log(\cdot) : \cdot)(Z) \\ = [D_{Q,v,0}(Z) - (\nu-\rho_Q)(Z_{\mathrm{a}})]\, q_{\nu-\rho_Q}(Q,v \mid F_\nu : \log(\cdot) : \cdot). \tag{6.16}$$

Proof: There exist $\delta \in \mathrm{D}_Q$ and a finite subset $Y \subset {}^*\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$ such that $F \in \mathcal{E}^{\mathrm{hyp}}_{Q,Y}(\mathrm{X}:\tau:\delta)$. Let $\mathcal{H} = \mathcal{H}_F$, $d = d_F$ and $k = \deg_a F$ be defined as in the text following [12], Def. 6.1. Then F satisfies all conditions of the mentioned definition. It follows from the characterization of the expansions for $\widetilde{F}$ in Proposition 6.10 that $\widetilde{F}$ satisfies the hypotheses of [12], Def. 6.1 with $\widetilde{\tau}$ in place of τ, with the same $Y, \mathcal{H}, d, k$. In particular, $\widetilde{F}$ belongs to $C^{\mathrm{ep,hyp}}_{Q,Y}(\mathrm{X}_+ : \widetilde{\tau})$.

Since F_ν is annihilated by the ideal $I_{\delta,\nu}$ for generic $\nu \in \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$, the same holds for $\widetilde{F}_\nu$, and we see that $\widetilde{F} \in \mathcal{E}^{\mathrm{hyp}}_{Q,Y}(\mathrm{X}_+ : \widetilde{\tau} : \delta)$, see [12], Def. 6.3.

Let now $s \in W$, $P \in \mathcal{P}^1_\sigma$ such that $s(\mathfrak{a}_{Q\mathrm{q}}) \not\subset \mathfrak{a}_{P\mathrm{q}}$ and $v \in N_K(\mathfrak{a}_{\mathrm{q}})$. Then there exists an open dense subset $\Omega \subset \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$ such that F satisfies the condition stated in [12], Def. 6.4. It follows from Proposition 6.10 and the fact that the functions $m \mapsto D_{P,v,\mu}(Z,m)$ are smooth on all of $M_{P\sigma}$, for $Z \in \mathfrak{g}_{\mathbb{C}}$, $\mu \in \mathbb{N}\Delta_r(P)$, that $\tilde{F}$ also satisfies the condition of [12], Def. 6.4, with the same set Ω. We conclude that $\widetilde{F} \in \mathcal{E}^{\mathrm{hyp}}_{Q,Y}(\mathrm{X}_+ : \widetilde{\tau} : \delta)_{\mathrm{glob}}$. In view of [12], Lemma 6.9, $\nu \mapsto F_\nu$ is a meromorphic $C^\infty(\mathrm{X}:\tau)$-valued function on $\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$. Hence, $\nu \mapsto \widetilde{F}_\nu$ is a meromorphic $C^\infty(\mathrm{X}:\widetilde{\tau})$-valued function on $\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$. In view of [12], Def. 6.6, we now infer that $\widetilde{F} \in \mathcal{E}^{\mathrm{hyp}}_Q(\mathrm{X}:\widetilde{\tau})$.

Finally, for ν in an open dense subset of $\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$, the element $\nu - \rho_Q$ is a leading exponent for F along (Q,v). Thus, (6.16) follows from (6.14). □

Next, we apply the above result to the normalized Eisenstein integral $E^\circ(Q:\psi:\nu)$, defined for $\psi \in \mathcal{A}_{2,Q}$. Let $v \in {}^Q\mathcal{W}$. Given a function $\psi_v \in \mathcal{A}_2(\mathrm{X}_{Q,v}:\tau_Q)$ and an element $\nu \in \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$ we define the function

$$\partial_{Q,v}(\nu)\psi_v\colon\ \mathrm{X}_{Q,v} \to \mathfrak{g}^*_{\mathbb{C}} \otimes V_\tau \tag{6.17}$$

by

$$\partial_{Q,v}(\nu)\psi_v(x)(Z) = [L_{Z_{\mathrm{m}}} - (\nu - \rho_Q)(Z_{\mathrm{a}})]\psi_v(x) - \tau(Z_{\mathrm{k}})[\psi_v(x)],$$

for $x \in \mathrm{X}_{Q,v}$ and $Z \in \mathfrak{g}_{\mathbb{C}}$. Clearly, the function $\partial_{Q,v}(\nu)\psi_v$ is a $\mathbb{D}(\mathrm{X}_{Q,v})$-finite Schwartz function with values in $\mathfrak{g}_{\mathbb{C}}^* \otimes V_\tau$. Since K_Q normalizes the decomposition (6.7) and centralizes $\mathfrak{a}_{Q\mathrm{q}}$, one readily checks that the function is $\widetilde{\tau}_Q$-spherical, with $\widetilde{\tau}_Q := \widetilde{\tau}|_{K_Q}$. Hence,

$$\partial_{Q,v}(\nu)\psi_v \in \mathcal{A}_2(\mathrm{X}_{Q,v} : \widetilde{\tau}_Q).$$

We define the map $\partial_Q(\nu) : \mathcal{A}_{2,Q}(\tau) \to \mathcal{A}_{2,Q}(\widetilde{\tau})$ as the direct sum, for $v \in {}^Q\mathcal{W}$, of the maps $\partial_{Q,v}(\nu) : \mathcal{A}_2(\mathrm{X}_{Q,v} : \tau_Q) \to \mathcal{A}_2(\mathrm{X}_{Q,v} : \widetilde{\tau}_Q)$.

Theorem 6.12 *Let $\psi \in \mathcal{A}_{2,Q}(\tau)$ and let the family $F : \mathfrak{a}_{Q\mathrm{q}\mathbb{C}}^* \times \mathrm{X} \to V_\tau$ be defined by*

$$F(\nu, x) = E_\tau^\circ(Q : \psi : \nu : x).$$

Then the family $\widetilde{F} : \mathfrak{a}_{Q\mathrm{q}\mathbb{C}}^* \times \mathrm{X} \to \mathfrak{g}_{\mathbb{C}}^* \otimes V_\tau$, defined by $(\widetilde{F})_\nu = (F_\nu)^\sim$, is given by

$$\widetilde{F}(\nu, x) = E_{\widetilde{\tau}}^\circ(Q : \partial_Q(\nu)\psi : \nu : x).$$

Proof: It follows from [12], Def. 13.7 and Prop. 13.6, that the family F belongs to $\mathcal{E}_Q^{\mathrm{hyp}}(\mathrm{X} : \tau)$ and that the family $G := E^\circ(Q : \partial_Q(\nu)\psi)$ belongs to $\mathcal{E}_Q^{\mathrm{hyp}}(\mathrm{X} : \widetilde{\tau})$. Let $v \in {}^Q\mathcal{W}$. Then it follows from the mentioned proposition, combined with [11], Thm. 7.7, Eqn. (7.14), that, for ν in an open dense subset of $\mathfrak{a}_{Q\mathrm{q}\mathbb{C}}^*$ and all $X \in \mathfrak{a}_{Q\mathrm{q}}$ and $m \in \mathrm{X}_{Q,v,+}$,

$$q_{v-\rho_Q}(Q, v \mid F_\nu, X, m) = \psi_v(m), \tag{6.18}$$
$$q_{v-\rho_Q}(Q, v \mid G_\nu, X, m) = \partial_{Q,v}(\nu)\psi_v(m). \tag{6.19}$$

From Theorem 6.11 we see that $\widetilde{F} \in \mathcal{E}_Q^{\mathrm{hyp}}(\mathrm{X} : \widetilde{\tau})$. Moreover, combining (6.18) and (6.16) we infer that, for $Z \in \mathfrak{g}_{\mathbb{C}}$, ν in an open dense subset of $\mathfrak{a}_{Q\mathrm{q}\mathbb{C}}^*$ and all $X \in \mathfrak{a}_{Q\mathrm{q}}$ and $m \in \mathrm{X}_{Q,v,+}$,

$$q_{v-\rho_Q}(Q, v \mid \widetilde{F}_\nu, X, m)(Z) = [D_{Q,v,0}(Z) - (\nu - \rho_Q)(Z_{\mathrm{a}})][(m, a) \mapsto \psi_v(m)].$$

From (6.9) we see that the expression on the right-hand side of this equation equals $[\partial_{Q,v}(\nu)\psi_v(m)](Z)$; hence

$$q_{v-\rho_Q}(Q, v \mid \widetilde{F}_\nu, X, m) = \partial_{Q,v}(\nu)\psi_v(m). \tag{6.20}$$

Comparing (6.20) with (6.19) we deduce that the family $\widetilde{F} - G \in \mathcal{E}_Q^{\mathrm{hyp}}(\mathrm{X} : \widetilde{\tau})$ satisfies the hypothesis of the vanishing theorem, [12], Thm. 6.11. Hence, $\widetilde{F} = G$. □

Given $\nu \in \mathfrak{a}_{Q\mathfrak{q}\mathbb{C}}^*$ and $\varphi \in C^\infty(K:\xi:\tau)$ we define the function

$$d(Q,\xi,\nu)\,\varphi \in C^\infty(K:\xi)\otimes \mathfrak{g}_{\mathbb{C}}^* \otimes V_\tau$$

by

$$[d(Q,\xi,\nu)\,\varphi](k,Z) = ([\pi_{\xi,-\nu}(Z)\otimes I]\,\varphi)(k), \tag{6.21}$$

for $k \in K$ and $Z \in \mathfrak{g}_{\mathbb{C}}$. One readily verifies that $d(Q,\xi,\nu)\varphi \in C^\infty(K:\xi:\widetilde{\tau})$.

Lemma 6.13 *Let $T \in \bar{V}(\xi)\otimes C^\infty(K:\xi:\tau)$. Then, for all $\nu \in \mathfrak{a}_{Q\mathfrak{q}\mathbb{C}}^*$,*

$$\psi_{[I\otimes d(Q,\xi,\nu)]T} = \partial_Q(\nu)\psi_T.$$

Proof: By linearity it suffices to prove this for $T = \eta\otimes\varphi$, with $\eta \in \bar{V}(\xi)$ and $\varphi \in C^\infty(K:\xi:\tau)$. Let $v \in {}^Q\mathcal{W}$ and $Z \in \mathfrak{g}_{\mathbb{C}}$. Then combining (6.21) with the decomposition (6.8) we infer that

$$[d(Q,\xi,\nu)\varphi](e)(Z) = [\xi(Z_{\mathrm{m}})\otimes I - I\otimes\tau(Z_{\mathrm{k}})]\varphi(e) - (\nu-\rho_Q)(Z_{\mathrm{a}})\varphi(e).$$

By equivariance, η_v maps $\mathcal{H}_\xi^\infty$ into $L^2(\mathrm{X}_{Q,v})_\xi^\infty \subset C^\infty(\mathrm{X}_{Q,v})$, intertwining the $(\mathfrak{m}_Q, K_Q)$-actions. Using formula (4.8) we now obtain that

$$\begin{aligned}
&\psi_{[I\otimes d(Q,\xi,\nu)]T,v}(\,\cdot\,)(Z)\\
&\quad = [\eta_v\otimes I]([\xi(Z_{\mathrm{m}})\otimes I]\varphi(e))\\
&\qquad -[(\nu-\rho_Q)(Z_{\mathrm{a}})I\otimes I + I\otimes\tau(Z_{\mathrm{k}})](\eta_v\otimes I)(\varphi(e))\\
&\quad = [L_{Z_{\mathrm{m}}} - (\nu-\rho_Q)(Z_{\mathrm{a}})](\psi_{T,v})(\,\cdot\,) - \tau(Z_{\mathrm{k}})[\psi_{T,v}(\,\cdot\,)]\\
&\quad = (\partial(Q,\nu)\psi_T)_v(\,\cdot\,)(Z).
\end{aligned}$$

□

Corollary 6.14 *Let $T \in \bar{V}(\xi)\otimes C^\infty(K:\xi:\tau)$ and let the family $F\colon \mathfrak{a}_{Q\mathfrak{q}\mathbb{C}}^*\times \mathrm{X} \to V_\tau$ be defined by*

$$F_\nu = E^\circ(Q:\psi_T:\nu).$$

Then the family $\widetilde{F}\colon \nu \mapsto (F_\nu)^\sim$ is given by

$$\widetilde{F}_\nu = E^\circ(Q:\psi_{[I\otimes d(Q,\xi,\nu)]T}:\nu). \tag{6.22}$$

Proof: This follows from Theorem 6.12 and Lemma 6.13. □

As a consequence of the above, we can now express derivatives of the normalized Eisenstein integral in a form needed for the proof of Theorem 4.6.

Proposition 6.15 *Let $\vartheta \subset \widehat{K}$ be a finite subset, and let $\vartheta' \subset \widehat{K}$ be the union of the collections of K-types occurring in $\mathrm{Ad}_K\otimes\delta$, as $\delta\in\vartheta$. Let $T \in \bar{V}(\xi)\otimes C^\infty(K:\xi)_\vartheta$. Then $(I\otimes\pi_{\xi,-\nu}(Z))T \in \bar{V}(\xi)\otimes C^\infty(K:\xi)_{\vartheta'}$, for all $Z\in\mathfrak{g}_{\mathbb{C}}$ and $\nu\in\mathfrak{a}_{Q\mathfrak{q}\mathbb{C}}^*$. Moreover, for all $Z\in\mathfrak{g}_{\mathbb{C}}$, $x\in\mathrm{X}$ and $k\in K$,*

$$L_{\mathrm{Ad}(k)^{-1}Z}E_\vartheta^\circ(Q:\psi_T:\nu)(x)(k) = E_{\vartheta'}^\circ(Q:\psi_{[I\otimes\pi_{\xi,-\nu}(Z)]T}:\nu)(x)(k), \tag{6.23}$$

as a meromorphic identity in $\nu\in\mathfrak{a}_{Q\mathfrak{q}\mathbb{C}}^*$.

Proof: Let $\tau = \tau_\vartheta$. We shall use the natural identification $C^\infty(K:\xi)_\vartheta \simeq C^\infty(K:\xi:\tau)$ of Lemma 3.4, so that ψ_T may be viewed as an element of $\bar{V}(\xi) \otimes C^\infty(K:\xi:\tau)$.

Define the family F as in Corollary 6.14. We shall derive the identity (6.23) from (6.22) by using the functorial properties of Lemma 4.5.

Fix $Z \in \mathfrak{g}_\mathbb{C}$. We define the matrix coefficient map $m_Z: \mathfrak{g}_\mathbb{C}^* \to C^\infty(K)$ by

$$m_Z(\zeta)(k) = \zeta(\mathrm{Ad}(k^{-1})Z), \qquad (\zeta \in \mathfrak{g}_\mathbb{C}^*,\ k \in K).$$

The map m_Z intertwines the representation $\mathrm{Ad}_K^\vee$ of K in $\mathfrak{g}_\mathbb{C}^*$ with the right regular representation of K in $C(K)$. In particular, it maps into the finite dimensional space $C(K)_{\vartheta_0^\vee}$, with $\vartheta_0 \subset \widehat{K}$ the set of K-types in Ad_K. We define the equivariant map

$$S_1 := m_Z \otimes I:\ \ \mathfrak{g}_\mathbb{C}^* \otimes \mathbf{V}_\vartheta \to C(K)_{\vartheta_0^\vee} \otimes \mathbf{V}_\vartheta.$$

On the other hand, we define the map $S_2: C(K)_{\vartheta_0^\vee} \otimes \mathbf{V}_\vartheta \to C(K)$ by $\phi \otimes \psi \mapsto \phi\psi$. This map intertwines $\tau_{\vartheta_0} \otimes \tau_\vartheta$ with the right regular representation of K in $C(K)$, hence maps into $C(K)_{\vartheta'^\vee}$. The space $C(K)_{\vartheta_0^\vee} \otimes \mathbf{V}_\vartheta$ may be naturally identified with a finite dimensional K-submodule of $C(K \times K)$, the latter being equipped with the diagonal K-action from the right. Under this identification the map S_2 corresponds with the restriction of the map $\Delta^*: C(K \times K) \to C(K)$ given by $\Delta^*\varphi(k) = \varphi(k, k)$.

The map $S = S_2 \circ S_1:\ \mathfrak{g}_\mathbb{C}^* \otimes V_\tau \to \mathbf{V}_{\vartheta'}$ is K-equivariant. We shall apply $I \otimes S$ to both sides of the identity (6.22). Application of $I \otimes S_1$ to the left-hand side yields $(I \otimes S_1)[\widetilde{F}_\nu(\,\cdot\,)](k) = \widetilde{F}_\nu(\,\cdot\,, \mathrm{Ad}(k^{-1})Z)$, which in turn equals $L_{\mathrm{Ad}(k^{-1})Z}F_\nu$. By application of $I \otimes S_2$ to the latter function we find

$$\begin{aligned}(I \otimes S)[\widetilde{F}_\nu(\,\cdot\,)](k) &= L_{\mathrm{Ad}(k^{-1})Z}F_\nu(\,\cdot\,)(k)\\ &= L_{\mathrm{Ad}(k^{-1})Z}E^\circ(Q:\psi_T:\nu)(\,\cdot\,)(k). \qquad (6.24)\end{aligned}$$

On the other hand, from Lemma 4.5 we see that application of $I \otimes S$ to the expressions on both sides of (6.22) yields

$$(I \otimes S)\widetilde{F}_\nu = E^\circ(Q:\psi_{[I\otimes I\otimes S][I\otimes d(Q,\xi,\nu)]T}:\nu). \qquad (6.25)$$

We observe that $(I \otimes S) \circ d(Q, \xi, \nu)$ is a linear map from $C^\infty(K:\xi:\tau_\vartheta)$ to $C^\infty(K:\xi:\tau_{\vartheta'})$ and claim that the following diagram commutes, for every $\nu \in \mathfrak{a}_{Q\mathbb{C}}^*$,

$$\begin{array}{ccc} C^\infty(K:\xi:\tau_\vartheta) & \xrightarrow{(I\otimes S)\circ d(Q,\xi,\nu)} & C^\infty(K:\xi:\tau_{\vartheta'}) \\ \downarrow & & \downarrow \\ C^\infty(K:\xi)_\vartheta & \xrightarrow{\pi_{\xi,-\nu}(Z)} & C^\infty(K:\xi)_{\vartheta'}\ . \end{array} \qquad (6.26)$$

Here the vertical arrows represent the natural isomorphisms of Lemma 3.4. We denote both of these isomorphisms by $\varphi \mapsto \varphi'$. It suffices to prove the claim, since its validity implies that $\pi_{\xi,-\nu}(Z)$ maps $C^\infty(K:\xi)_\vartheta$ into

$C^\infty(K:\xi)_{\vartheta'}$ and that the expression on the right-hand side of (6.25) equals the one on the right-hand side of (6.23). Combining this with (6.24) we obtain (6.23).

To see that the claim holds, let $\varphi \in C^\infty(K:\xi:\tau_\vartheta) = (C^\infty(K:\xi)\otimes \mathbf{V}_\vartheta)^K$. The associated element $\varphi' \in C^\infty(K:\xi)_\vartheta$ is given by

$$\varphi'(k) = \varphi(k)(e), \qquad (k \in K).$$

The element $(I\otimes S_1)d(Q,\xi,\nu)\varphi$ of $[C^\infty(K:\xi)\otimes C(K)_{\vartheta_0^\vee}\otimes \mathbf{V}_\vartheta]^K$ is given by

$$\begin{aligned}[(I\otimes S_1)d(Q,\xi,\nu)\varphi](k)(k_1) &= [d(Q,\xi,\nu)\varphi(k)]\big(\mathrm{Ad}\big(k_1^{-1}\big)Z\big)\\ &= \big[I\otimes \pi_{\xi,-\nu}\big(\mathrm{Ad}\big(k_1^{-1}\big)Z\big)\big]\varphi(k);\end{aligned}$$

see (6.21). Hence, the element $(I\otimes S)d(Q,\xi,\nu)\varphi \in [C^\infty(K:\xi)\otimes \mathbf{V}_{\vartheta'}]^K$ is given by

$$(I\otimes S)d(Q,\xi,\nu)\varphi(k)(k_1) = \big(I\otimes \pi_{\xi,-\nu}\big(\mathrm{Ad}\big(k_1^{-1}\big)Z\big)\big)\varphi(k)(k_1).$$

The natural isomorphism from $[C^\infty(K:\xi)\otimes \mathbf{V}_{\vartheta'}]^K$ onto $C^\infty(K:\xi)_{\vartheta'}$ is induced by the map $I\otimes\delta_e$, where $\delta_e\colon \mathbf{V}_{\vartheta'}\to\mathbb{C}$ denotes evaluation at e (see (3.6)). Hence,

$$((I\otimes S)d(Q,\xi,\nu)\varphi)'(k) = [(I\otimes \pi_{\xi,-\nu}(Z))\varphi](k)(e) = [\pi_{\xi,-\nu}(Z)\varphi'](k).$$

This establishes the claim. □

We shall apply the above result in combination with Proposition 5.1 to obtain the assertion of Theorem 4.6 about finiteness. If $H\subset \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$ is a $\Sigma_r(Q)$-hyperplane, we denote by α_H the shortest root of $\Sigma_r(Q)$ such that H is a translate of $(\alpha_H^\perp)_{\mathbb{C}}$. Thus, $\langle\alpha_H,\ \cdot\,\rangle$ equals a constant c on H; we denote by l_H the linear polynomial function $\langle\alpha_H,\ \cdot\,\rangle - c$. In accordance with [12], Eqn. (4.3), given a locally finite collection $\mathcal{H}$ of $\Sigma_r(Q)$-hyperplanes in $\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$ and a map $d\colon \mathcal{H}\to\mathbb{N}$, we define, for every bounded subset ω of $\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$, the polynomial $\pi_{\omega,d}$ by

$$\pi_{\omega,d}(\nu) = \prod_{\substack{H\in\mathcal{H}\\ H\cap\omega\neq\emptyset}} l_H^{d(H)}. \tag{6.27}$$

Proposition 6.16 *Let $Q\in\mathcal{P}_\sigma$, $\xi\in \mathrm{X}^\wedge_{Q,*,ds}$. Then $\mathcal{H}(Q,\xi)$ is a locally finite collection of real $\Sigma_r(Q)$-hyperplanes. Moreover, there exists a map $d\colon \mathcal{H}(Q,\xi)\to\mathbb{N}$ such that, for every finite dimensional unitary representation τ of K, every $T\in \bar{V}(\xi)\otimes C^\infty(K:\xi:\tau)$ and every bounded open subset $\omega\subset\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$, the $C^\infty(\mathrm{X}:\tau)$-valued function*

$$\nu\mapsto \pi_{\omega,d}(\nu)E^\circ(Q:\psi_T:\nu) \tag{6.28}$$

is holomorphic on ω. Here $\pi_{\omega,d}$ is defined by (6.27) with $\mathcal{H}=\mathcal{H}(Q,\xi)$.

Proof: Select any bounded open subset $\omega \subset \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$. Let $\vartheta \subset \widehat{K}$ be a finite set associated with $Q, \xi, -\omega$ as in Proposition 5.1. According to [12], Prop. 13.14, there exists a map $d\colon \mathcal{H}(Q,\xi,\vartheta) \to \mathbb{N}$ with the property that, for every $T \in \bar{V}(\xi) \otimes C^\infty(K:\xi)_\vartheta$, the map $\nu \mapsto E^\circ_\vartheta(Q:\psi_T:\nu)$ belongs to $\mathcal{M}(\mathfrak{a}^*_{Q\mathrm{q}}, \mathcal{H}(Q,\xi,\vartheta), d, C^\infty(\mathrm{X}:\tau_\vartheta))$. See [12], § 4, for the definition of the latter space.

Let $\vartheta' \subset \widehat{K}$ be an arbitrary finite subset. Fix $\nu_0 \in \omega$. Then by Proposition 5.1 there exists $k \in \mathbb{N}$ such that the map

$$M_\nu\colon\ U_k(\mathfrak{g}) \otimes C(K:\xi)_\vartheta \to C(K:\xi)_K,\ u \otimes \varphi \mapsto \pi_{\xi,-\nu}(u)\varphi$$

has image containing $C(K:\xi)_{\vartheta'}$ for $\nu = \nu_0$. On the other hand, let $\vartheta'' \subset \widehat{K}$ be the finite collection of K-types occurring in $\delta_1 \otimes \delta_2$, with $\delta_1 \in \widehat{K}$ a K-type occurring in the adjoint representation of K in $U_k(\mathfrak{g})$ and with $\delta_2 \in \vartheta$. Then the image of M_ν is contained in $C^\infty(K:\xi)_{\vartheta''}$ for all $\nu \in \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$. Let $P_{\vartheta',\vartheta''}$ denote the K-equivariant projection from $C(K:\xi)_{\vartheta''}$ onto $C(K:\xi)_{\vartheta'}$. Then $P_{\vartheta',\vartheta''} \circ M_{\nu_0}$ is surjective. Hence there exists a finite dimensional subspace $E \subset U_k(\mathfrak{g}) \otimes C(K:\xi)_\vartheta$ such that $R_\nu := P_{\vartheta',\vartheta''} \circ M_\nu|_E\colon E \to C(K:\xi)_{\vartheta'}$ is a bijection for $\nu = \nu_0$. By continuity and finite dimensionality, there exists an open neighborhood ω_0 of ν_0 in ω such that R_ν is a bijection for all $\nu \in \omega_0$. By Cramer's rule, the inverse $S_\nu := R_\nu^{-1} \in \mathrm{Hom}(C(K:\xi)_{\vartheta'}, E)$ depends holomorphically on $\nu \in \omega_0$. Let $(u_i \mid 1 \le i \le I)$ be a basis of $U_k(\mathfrak{g})$ and $(\varphi_j \mid 1 \le j \le J)$ a basis of $C(K:\xi)_\vartheta$. Then there exist holomorphic $[C^\infty(K:\xi)_{\vartheta'}]^*$-valued functions s_{ij} on ω_0, for $1 \le i \le I,\ 1 \le j \le J$, such that

$$S_\nu(\varphi) = \sum_{\substack{1\le i\le I\\ 1\le j\le J}} s_{ij}(\nu,\varphi)u_i \otimes \varphi_j, \qquad (\nu \in \omega_0),$$

for $\varphi \in C^\infty(K:\xi)_{\vartheta'}$. Let $\varphi \in C(K:\xi)_{\vartheta'}$. Then $\varphi = P_{\vartheta',\vartheta''} \circ M_\nu \circ S_\nu(\varphi)$, hence

$$\varphi = \sum_{i,j} s_{ij}(\nu,\varphi)\ P_{\vartheta',\vartheta''}\pi_{\xi,-\nu}(u_i)\varphi_j.$$

Let $\eta \in \bar{V}(\xi)$. Then it follows from the above by application of (4.17) and (4.18) that

$$\begin{aligned}
&\pi_{\omega,d}(\nu)E^\circ_{\vartheta'}(Q:\psi_{\eta\otimes\varphi}:\nu)\\
&\quad= \sum_{i,j} s_{ij}(\nu,\varphi)\pi_{\omega,d}(\nu)E^\circ_{\vartheta'}(Q:\psi_{\eta\otimes P_{\vartheta',\vartheta''}\pi_{\xi,-\nu}(u_i)\varphi_j}:\nu)\\
&\quad= \sum_{i,j} s_{ij}(\nu,\varphi)(I\otimes P_{\vartheta',\vartheta''})\big[\pi_{\omega,d}(\nu)E^\circ_{\vartheta''}(Q:\psi_{\eta\otimes\pi_{\xi,-\nu}(u_i)\varphi_j}:\nu)\big].
\end{aligned}$$

Applying $I \otimes \delta_e = \varsigma_{\vartheta'}^{-1}$ and using Lemma 3.6 and Proposition 6.15 we infer that

$$\begin{aligned}\pi_{\omega,d}(\nu)E^{\circ}_{\vartheta'}(Q:\psi_{\eta\otimes\varphi}:\nu:\cdot)(e)\\ &= \sum_{i,j} s_{ij}(\nu,\varphi)P_{\vartheta',\vartheta''}\big[\pi_{\omega,d}(\nu)E^{\circ}_{\vartheta''}(Q:\psi_{\eta\otimes\pi_{\xi,-\nu}(u_i)\varphi_j}:\nu:\cdot)(e)\big]\\ &= \sum_{i,j} s_{ij}(\nu,\varphi)P_{\vartheta',\vartheta''}L_{u_i}\big[\pi_{\omega,d}(\nu)E^{\circ}_{\vartheta}(Q:\psi_{\eta\otimes\varphi_j}:\nu:\cdot)(e)\big].\end{aligned} \tag{6.29}$$

From this we conclude that the expression on the left-hand side of the above equation depends holomorphically on $\nu \in \omega_0$, as a function with values in $C^\infty(\mathrm{X})$. Since ν_0 was arbitrary, it follows that the expression on the left-hand side of (6.29) in fact depends holomorphically on $\nu \in \omega$. Hence, every $H \in \mathcal{H}(Q,\xi,\vartheta')$ with $H \cap \omega \neq \emptyset$ must be contained in $\mathcal{H}(Q,\xi,\vartheta)$. This shows that the collection $\mathcal{H}(Q,\xi)$ is locally finite. The argument also shows that there exists a map $d\colon \mathcal{H}(Q,\xi) \to \mathbb{N}$ such that the assertion of the proposition holds for every τ of the form $\tau = \tau_{\vartheta'}$, with $\vartheta' \subset \widehat{K}$ a finite subset. The general result now follows by application of the functorial property of Lemma 4.5. □

Corollary 6.17 *Let $d\colon \mathcal{H}(Q,\xi) \to \mathbb{N}$ be as in Proposition 6.16. Then, for every $T \in \bar{V}(\xi) \otimes C^\infty(K:\xi)_K$ and every bounded open subset $\omega \subset \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$, the function*

$$\nu \mapsto \pi_{\omega,d}(\nu)J_{Q,\xi,\nu}(T)$$

extends to a holomorphic $C^\infty(\mathrm{X})$-valued function on ω.

Proof: This follows from Proposition 6.16 and the definition of $J_{Q,\xi,\nu}$, see (4.20). □

We can now finally give the promised proof.

Proof of Theorem 4.6: The properties of $\mathcal{H}(Q,\xi)$ have been established in Proposition 6.16 and (4.19). Let $\nu \in \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$. That $J_{Q,\xi,\nu}$ is a $\mathfrak{g}$-equivariant map follows from formula (4.20) combined with formula (6.23) with $k = e$. It remains to establish the K-equivariance of $J_{Q,\xi,\nu}$. Let $\vartheta \subset \widehat{K}$ be a finite subset and let $T = \eta \otimes \varphi \in \bar{V}(\xi) \otimes C^\infty(K:\xi)_\vartheta$. We denote the natural isomorphism $C^\infty(K:\xi)_\vartheta \to C^\infty(K:\xi:\tau_\vartheta)$ of Lemma 3.4 by $\varsigma = \varsigma_\vartheta$. Let $k_1 \in K$. Then one readily checks that $\varsigma \circ \pi_{\xi,-\nu}(k_1) = (I \otimes S) \circ \varsigma$, with S the endomorphism of $\mathbf{V}_\vartheta = C(K)_{\vartheta^\vee}$ given by restriction of the left translation L_{k_1}. In particular, S intertwines $\tau_\vartheta = R|_{\mathbf{V}_\vartheta}$ with itself. By the identification discussed in the text before (4.13) we have

$$\begin{aligned}\psi_{[I\otimes\pi_{\xi,-\nu}(k_1)]T} &= \psi_{[I\otimes\varsigma\pi_{\xi,-\nu}(k_1)]T}\\ &= \psi_{(I\otimes I\otimes S)(I\otimes\varsigma)T}.\end{aligned}$$

By Lemma 4.5 (a), combined with the identification mentioned above, the latter expression equals

$$(I \otimes S)\psi_{(I\otimes\zeta)T} = (I \otimes S)\psi_T.$$

Applying Lemma 4.5 (b) we now find that

$$\begin{aligned} J_{Q,\xi,\nu}([I \otimes \pi_{\xi,-\nu}(k_1)]T) &= E^\circ_\vartheta(Q : \psi_{[I\otimes\pi_{\xi,-\nu}(k_1)]T} : \nu)(\cdot)(e) \\ &= \big[[I \otimes S]E^\circ_\vartheta(Q : \psi_T : \nu)(\cdot)\big](e) \\ &= E^\circ_\vartheta(Q : \psi_T : \nu)(\cdot)\big(k_1^{-1}\big) \\ &= L_{k_1} J_{Q,\xi,\nu}(T). \end{aligned}$$

□

7 The Fourier transform

Let $Q \in \mathcal{P}_\sigma$ and $\xi \in \mathrm{X}^\wedge_{Q,*,ds}$. We will use the map $J_{Q,\xi}$, introduced in (4.20), to define a $(\mathfrak{g}, K)$-equivariant Fourier transform for functions from $C^\infty_c(\mathrm{X})_K$.

We define the collection $\mathcal{H}^\vee(Q,\xi)$ of hyperplanes in $\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$ by

$$\mathcal{H}^\vee(Q,\xi) := \{-H \mid H \in \mathcal{H}(Q,\xi)\}.$$

Since $\mathcal{H}(Q,\xi)$ is a locally finite collection of real $\Sigma_r(Q)$-hyperplanes, see Theorem 4.6, the same holds for $\mathcal{H}^\vee(Q,\xi)$. It also follows from the mentioned theorem that $\cup\mathcal{H}^\vee(Q,\xi)$ is disjoint from $i\mathfrak{a}^*_{Q\mathrm{q}}$.

Since $\mathcal{H}(Q,\xi)$ consists of real $\Sigma_r(Q)$-hyperplanes, every hyperplane of $\mathcal{H}(Q,\xi)$ is invariant under the complex conjugation $\lambda \mapsto \bar\lambda$ in $\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$, defined with respect to the real form $\mathfrak{a}^*_{Q\mathrm{q}}$. Hence, $\mathcal{H}^\vee(Q,\xi) = \{-\bar H \mid H \in \mathcal{H}(Q,\xi)\}$. This justifies the following definition.

Definition 7.1 Let $f \in C^\infty_c(\mathrm{X})_K$. For $\nu \in \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}} \setminus \cup\mathcal{H}^\vee(Q,\xi)$, the Fourier transform $\hat f(Q : \xi : \nu)$ is defined to be the element of $\bar V(\xi) \otimes C^\infty(K : \xi)_K$ determined by

$$\langle \hat f(Q : \xi : \nu) \mid T\rangle = \int_{\mathrm{X}} f(x)\overline{J_{Q,\xi,-\bar\nu}(T)(x)}\, dx, \tag{7.1}$$

for all $T \in \bar V(\xi) \otimes C^\infty(K : \xi)_K$.

Lemma 7.2 *Let $\nu \in \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}} \setminus \cup\mathcal{H}^\vee(Q,\xi)$. Then the map $f \mapsto \hat f(Q : \xi : \nu)$ from $C^\infty_c(\mathrm{X})_K$ to $\bar V(\xi) \otimes C^\infty(K : \xi)_K$ intertwines the $(\mathfrak{g}, K)$-module structure on $C^\infty(\mathrm{X})_K$ coming from the left regular representation with the $(\mathfrak{g}, K)$-module structure on $\bar V(\xi) \otimes C^\infty(K : \xi)_K$ coming from $1 \otimes \pi_{Q,\xi,-\nu}$.*

Proof: The spaces $C_c^\infty(\mathrm{X})_K$ and $\bar{V}(\xi) \otimes C^\infty(K:\xi)$ are equipped with the natural L^2-type inner products. The first of these inner products is equivariant for the $(\mathfrak{g}, K)$-module structure coming from the left regular representation. The second induces a sesquilinear pairing of $\bar{V}(\xi) \otimes C^\infty(K:\xi)$ with itself that is equivariant for the $(\mathfrak{g}, K)$-module structures coming from $1 \otimes \pi_{Q,\xi,-\nu}$ and $1 \otimes \pi_{Q,\xi,\bar{\nu}}$, respectively. On the other hand, it follows from (7.1) that the map $f \mapsto \hat{f}(Q,\xi,\nu)$ is adjoint to the map $J_{Q,\xi,-\bar{\nu}}$, with respect to the given inner products. Therefore, the $(\mathfrak{g}, K)$-intertwining property of $f \mapsto \hat{f}(Q,\xi,\nu)$ follows by transposition from the similar property for $J_{Q,\xi,-\nu}$, asserted in Theorem 4.6. □

If $d\colon \mathcal{H}(Q,\xi) \to \mathbb{N}$ is a map, we define the map $d^\vee\colon \mathcal{H}^\vee(Q,\xi) \to \mathbb{N}$ by $d^\vee(H) = d(-H)$, for $H \in \mathcal{H}^\vee(Q,\xi)$.

Lemma 7.3 *Let d, $d^\vee$ be as above and let $\omega \subset \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$ be a bounded subset. Then*

$$\overline{\pi_{-\bar{\omega},d}(-\bar{\nu})} = (-1)^N \pi_{\omega,d^\vee}(\nu), \qquad (\nu \in \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}), \tag{7.2}$$

with $N = \sum_H d(H)$, for $H \in \mathcal{H}(Q,v)$, $H \cap \omega \neq \emptyset$.

Proof: In the notation of the text preceding (6.27), we have, for $H \in \mathcal{H}^\vee(Q,\xi)$,

$$\overline{l_H(-\bar{\nu})} = -\langle \alpha_H\,,\, -\nu\rangle - c_H = \langle \alpha_{-H}\,,\, \nu\rangle + c_{-H} = -l_{-H}(\nu). \tag{7.3}$$

Moreover, since H is real, $-\bar{\omega} \cap H \neq \emptyset$ is equivalent to $\omega \cap (-H) \neq \emptyset$. In view of the definition in (6.27), the identity (7.2) follows from (7.3) raised to the power $d^\vee(H) = d(-H)$, by taking the product over all $H \in \mathcal{H}^\vee(Q,\xi)$ with $\omega \cap (-H) \neq \emptyset$. □

Lemma 7.4 *Let $d\colon \mathcal{H}(Q,\xi) \to \mathbb{N}$ be as in Proposition 6.16 and define $d^\vee\colon \mathcal{H}^\vee(Q,\xi) \to \mathbb{N}$ by $d^\vee(H) = d(-H)$. Then, for every bounded open subset $\omega \subset \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$, every finite subset $\vartheta \subset \widehat{K}$ and every $f \in C^\infty(\mathrm{X})_\vartheta$, the function*

$$\nu \mapsto \pi_{\omega,d^\vee}(\nu)\hat{f}(Q,\xi,\nu), \tag{7.4}$$

originally defined on $\omega \setminus \cup\mathcal{H}^\vee(Q,\xi)$, extends to a holomorphic $\bar{V}(\xi) \otimes C^\infty(K:\xi)_\vartheta$-valued function on ω.

Proof: Let f be fixed as above. It follows from Lemma 7.2 that the function (7.4) has values in the finite dimensional space $\bar{V}(\xi) \otimes C^\infty(K:\xi)_\vartheta$. Hence, it suffices to establish the holomorphic continuation of the function that results from (7.4) by taking the inner product with a fixed element T from $\bar{V}(\xi) \otimes C^\infty(K:\xi)_\vartheta$. In view of (7.1) the resulting function equals

$$\nu \mapsto \pi_{\omega,d^\vee}(\nu)\langle f \mid J_{Q,\xi,-\bar{\nu}}(T)\rangle = \langle f \mid \pi_{-\bar{\omega},d}(-\bar{\nu})J_{Q,\xi,-\bar{\nu}}(T)\rangle,$$

see Lemma 7.3. We may now apply Corollary 6.17, with $-\bar{\omega}$ in place of ω, to finish the proof. □

The following result gives the connection between the present Fourier transform and the spherical Fourier transform, defined in [12], § 19.

Lemma 7.5 *Let $\vartheta \subset \widehat{K}$ be a finite subset, let $f \in C^\infty(\mathrm{X})_\vartheta$ and let $F = \varsigma_\vartheta(f) \in C^\infty(\mathrm{X}:\tau_\vartheta)$ be the corresponding spherical function, see Lemma 3.4. Let $\mathcal{F}_Q F$ be the τ_ϑ-spherical Fourier transform of F. Then, for every $T \in \bar{V}(\xi) \otimes C^\infty(K:\xi)_\vartheta$,*

$$\langle \hat{f}(Q:\xi:\nu) \mid T\rangle = \langle \mathcal{F}_Q F(\nu) \mid \psi_T\rangle, \qquad (\nu \in \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}} \setminus \cup\mathcal{H}^\vee(Q,\xi)).$$

Proof: It follows from (7.1) and (4.20) that

$$\langle \hat{f}_\vartheta(Q:\xi:\nu) \mid T\rangle = \int_{\mathrm{X}} f(x)\overline{E^\circ_\vartheta(Q:\psi_T:-\bar{\nu})(x)(e)}\,dx.$$

One may now proceed as in [7], p. 539, proof of Prop. 3, displayed equations, but in reversed order. □

8 A direct integral

In this section we assume that $Q \in \mathcal{P}_\sigma$ and $\xi \in \mathrm{X}^\wedge_{Q,*,ds}$ are fixed. We will define and study a direct integral representation $\pi_{Q,\xi}$ that will later appear as a summand in the Plancherel decomposition.

We equip

$$\mathfrak{H}(Q,\xi) := \bar{V}(\xi) \otimes L^2(K:\xi) \tag{8.1}$$

with the tensor product Hilbert structure and the natural structure of K-module. Moreover, we define

$$\mathfrak{L}^2(Q,\xi) := L^2(i\mathfrak{a}^*_{Q\mathrm{q}},\ \mathfrak{H}(Q,\xi),\ [W:W^*_Q]\,d\mu_Q),$$

the space of square integrable functions $i\mathfrak{a}^*_{Q\mathrm{q}} \to \mathfrak{H}(Q,\xi)$, equipped with the natural L^2-Hilbert structure associated with the indicated measure. Recall that $d\mu_Q$ is Lebesgue measure on $i\mathfrak{a}^*_{Q\mathrm{q}}$, normalized as in [12], end of § 5.

By unitarity of the representations $\pi_{Q,\xi,\nu}$, for $\nu \in i\mathfrak{a}^*_{Q\mathrm{q}}$, the prescription

$$(\pi_{Q,\xi}(x)\varphi)(\nu) := [I \otimes \pi_{Q,\xi,-\nu}(x)]\varphi(\nu), \qquad (\varphi \in \mathfrak{L}^2(Q,\xi),\ x \in G),$$

defines a homomorphism $\pi_{Q,\xi}$ from G into the group $\mathrm{U}(\mathfrak{L}^2(Q,\xi))$ of unitary transformations of $\mathfrak{L}^2(Q,\xi)$.

Lemma 8.1 *The homomorphism $\pi_{Q,\xi}: G \to \mathrm{U}(\mathfrak{L}^2(Q,\xi))$ is a unitary representation of G in $\mathfrak{L}^2(Q,\xi)$.*

Remark 8.2 It follows from the result above that $\pi_{Q,\xi}$ provides a realization of the following direct integral

$$\pi_{Q,\xi} \simeq \int_{i\mathfrak{a}_{Q\mathrm{q}}^*}^{\oplus} I_{\bar{V}(\xi)} \otimes \pi_{Q,\xi,-\nu} \left[W\colon W_Q^*\right] d\mu_Q(\nu).$$

For the proof of Lemma 8.1 it is convenient to define a dense subspace of $\mathcal{L}^2(Q,\xi)$ by

$$\mathcal{L}_0^2(Q,\xi) := C_c\big(i\mathfrak{a}_{Q\mathrm{q}}^*, \bar{V}(\xi) \otimes C^\infty(K:\xi)\big). \tag{8.2}$$

This space is equipped with a locally convex topology in the usual way. Thus, if $\mathcal{A} \subset i\mathfrak{a}_{Q\mathrm{q}}^*$ is compact, let $\mathcal{L}_{0\mathcal{A}}^2(Q,\xi)$ denote the space $C_{\mathcal{A}}(i\mathfrak{a}_{Q\mathrm{q}}^*, \bar{V}(\xi) \otimes C^\infty(K:\xi))$ of continuous functions from (8.2) with support contained in $\mathcal{A}$. This space is equipped with the Fréchet topology determined by the seminorms

$$\varphi \mapsto \sup_{\nu\in\mathcal{A}} s(\varphi(\nu)),$$

where s ranges over the continuous seminorms on $\bar{V}(\xi)\otimes C^\infty(K:\xi)$. Moreover,

$$\mathcal{L}_0^2(Q,\xi) = \cup_{\mathcal{A}}\ \mathcal{L}_{0\mathcal{A}}^2(Q,\xi)$$

is equipped with the direct limit locally convex topology. Thus topologized, $\mathcal{L}_0^2(Q,\xi)$ is a complete locally convex space.

Lemma 8.3

(a) *The space $\mathcal{L}_0^2(Q,\xi)$ is G-invariant.*

(b) *The restriction of $\pi_{Q,\xi}$ to $\mathcal{L}_0^2(Q,\xi)$ is a smooth representation of G. Moreover, if $\varphi \in \mathcal{L}_0^2(Q,\xi)$ and $u \in U(\mathfrak{g})$, then $\pi_{Q,\xi}(u)\varphi$ is given by*

$$[\pi_{Q,\xi}(u)\varphi](\nu) = [I \otimes \pi_{Q,\xi,-\nu}(u)]\ \varphi(\nu). \tag{8.3}$$

Proof: Let $\mathcal{A} \subset i\mathfrak{a}_{Q\mathrm{q}}^*$ be compact. Then it is a straightforward consequence of the definition that the space $\mathcal{L}_{0\mathcal{A}}^2(Q,\xi)$ is G-invariant. In particular, this implies (a).

For (b) it suffices to prove that the restriction of $\pi_{Q,\xi}$ to $\mathcal{L}_{0\mathcal{A}}^2(Q,\xi)$ is smooth and that (8.3) holds for $\varphi \in \mathcal{L}_{0\mathcal{A}}^2(Q,\xi)$.

Let $\varphi \in \mathcal{L}_{0\mathcal{A}}^2(Q,\xi)$. We consider the function $\Phi\colon \mathcal{A}\times G\times K \to \bar{V}(\xi)\otimes \mathcal{H}_\xi^\infty$ defined by

$$\Phi(\nu,x,k) := [I\otimes \mathrm{ev}_k]([\pi_{Q,\xi}(x)\varphi](\nu)) = [I\otimes \mathrm{ev}_k\pi_{Q,\xi,-\nu}(x)]\varphi(\nu),$$

where ev_k denotes the map $C^\infty(K:\xi)\to\mathcal{H}_\xi^\infty$ induced by evaluation in k. We recall that the multiplication map $N_Q\times A_Q\times \exp(\mathfrak{m}_Q\cap\mathfrak{p})\times K\to G$ is a diffeomorphism. Accordingly, we write

$$x = n_Q(x)a_Q(x)m_Q(x)k_Q(x), \qquad (x\in G), \tag{8.4}$$

where ν_Q, a_Q, m_Q and k_Q are smooth maps from G into N_Q, A_Q, $\exp(\mathfrak{m}_Q \cap \mathfrak{p})$ and K_Q, respectively. Applying (8.4) with kx in place of x, we find that

$$\Phi(\nu, x, k) = a_Q(kx)^{-\nu+\rho_Q} \xi(m_Q(kx))\varphi(\nu, k_Q(kx)).$$

From this expression we see that the function Φ is continuous, and smooth in the variables (x, k). Moreover, for every C^∞ differential operator D on $G \times K$, the $\bar{V}(\xi) \otimes \mathcal{H}_\xi^\infty$-valued function

$$(\nu, x, k) \mapsto D[\Phi(\nu)](x, k)$$

is continuous. This implies that the $C_{\mathcal{A}}(i\mathfrak{a}_{Q\mathrm{q}}^*, \bar{V}(\xi) \otimes C^\infty(K : \xi))$-valued function $x \mapsto \Phi(\cdot, x, \cdot)$ is smooth on G; hence φ is a smooth vector for $\pi_{Q,\xi}$. Let D be any C^∞ differential operator on G and let $\nu \in \mathcal{A}$. Then evaluation in ν induces a continuous linear operator $\mathcal{L}^2_{0,\mathcal{A}} \to \bar{V}(\xi) \otimes C^\infty(K : \xi)$. Hence, for all $x \in G$,

$$D[\pi_{Q,\xi}(\,\cdot\,)\varphi](x)(\nu) = D[\mathrm{ev}_\nu(\pi_{Q,\xi}(\,\cdot\,)\varphi)](x) = D(\pi_{Q,\xi,-\nu}(\,\cdot\,)\varphi(\nu))(x).$$

Applying this with $D = R_u$ and $x = e$ we obtain (8.3). □

Proof of Lemma 8.1: Put $\pi := \pi_{Q,\xi}$. It suffices to show that the map $G \times \mathcal{L}^2(Q, \xi) \to \mathcal{L}^2(Q, \xi)$, $(x, \varphi) \mapsto \pi(x)\varphi$ is continuous. Since π is a homomorphism from the group G into $\mathrm{U}(\mathcal{L}^2(Q, \xi))$, it suffices to show that for any fixed $\varphi \in \mathcal{L}^2(Q, \xi)$ we have

$$\lim_{x \to e} \pi(x)\varphi = \varphi \quad \text{in} \quad \mathcal{L}^2(Q, \xi). \tag{8.5}$$

Moreover, again because π maps into $\mathrm{U}(\mathcal{L}^2(Q, \xi))$, it suffices to prove (8.5) for φ in a dense subspace of $\mathcal{L}^2(Q, \xi)$. Let $\varphi \in \mathcal{L}^2_0(Q, \xi)$. Then $\pi(x)\varphi \to \varphi$ in $\mathcal{L}^2_0(Q, \xi)$, as $x \to e$, by Lemma 8.3. By continuity of the inclusion map $\mathcal{L}^2_0(Q, \xi) \hookrightarrow \mathcal{L}^2(Q, \xi)$, this implies (8.5). □

We end this section by establishing some other useful properties of the invariant subspace $\mathcal{L}^2_0(Q, \xi)$. If $\vartheta \subset \widehat{K}$, then one readily verifies that

$$\mathcal{L}^2_0(Q, \xi)_\vartheta = C_c\big(i\mathfrak{a}_{Q\mathrm{q}}^*, \bar{V}(\xi) \otimes C^\infty(K : \xi)_\vartheta\big). \tag{8.6}$$

The space of K-finite vectors in $\mathcal{L}^2_0(Q, \xi)$ equals the union of the spaces (8.6) as ϑ ranges over the collection of finite subsets of $\widehat{K}$. The natural $U(\mathfrak{g})$-module structure of $\mathcal{L}^2_0(Q, \xi)_K$ is given by formula (8.3).

Lemma 8.4 *Let (ρ, W) be a continuous representation of G in a complete locally convex space, and let U be a dense G-invariant subspace of W. If U is contained in W^∞ then it is dense in W^∞ for the C^∞-topology.*

Remark 8.5 For W a Banach space, this result is Thm. 1.3 of [27]. The following proof is an adaptation of the proof given in the mentioned paper.

Proof: Replacing U by its closure in W^∞ if necessary, we may as well assume that U is closed in W^∞. Fix a choice dg of Haar measure on G. If $\varphi \in C_c^\infty(G)$, then the map

$$\rho(\varphi) := \int_G \varphi(g)\rho(g)\, dg$$

is continuous linear from W to W^∞, as can be seen from a straightforward estimation. Moreover, since U is closed in W^∞, the map $\rho(\varphi)$ maps U into itself. Let W_1 be the collection of vectors in W^∞ of the form $\rho(\varphi)w_0$, with $\varphi \in C_c^\infty(G)$ and $w_0 \in W^\infty$. Then W_1 is dense in W^∞. Hence, it suffices to show that W_1 is contained in U. Select $w_1 \in W_1$ and let N_1 be an open neighborhood of 0 in W^∞. Then it suffices to show that $U \cap (w_1 + N_1) \neq \emptyset$.

Write $w_1 = \rho(\varphi)w_0$, with $w_0 \in W^\infty$ and $\varphi \in C_c^\infty(G)$. By the mentioned continuity of $\rho(\varphi)$, there exists an open neighborhood N_0 of 0 in W such that $\rho(\varphi)N_0 \subset N_1$. By density of U in W, the intersection $U \cap (w_0 + N_0)$ is non-empty. Hence,

$$\begin{aligned} \emptyset &\subsetneq \rho(\varphi)[U \cap (w_0 + N_0)] \\ &\subset U \cap \rho(\varphi)(w_0 + N_0) \\ &\subset U \cap (w_1 + N_1). \end{aligned}$$

□

Lemma 8.6 *The space $\mathcal{L}^2_0(Q,\xi)_K$ is dense in $\mathcal{L}^2(Q,\xi)^\infty$ with respect to the natural Fréchet topology of the latter space.*

Proof: The inclusion map $j: \mathcal{L}^2_0(Q,\xi) \to \mathcal{L}^2(Q,\xi)$ is continuous, intertwines the G-actions and has dense image. From Lemma 8.3 it follows that $\mathcal{L}^2_0(Q,\xi)^\infty = \mathcal{L}^2_0(Q,\xi)$. By equivariance of j it follows that $\mathcal{L}^2_0(Q,\xi)$ is contained in $\mathcal{L}^2(Q,\xi)^\infty$. By application of Lemma 8.4 we see that $\mathcal{L}^2_0(Q,\xi)$ is dense in $\mathcal{L}^2(Q,\xi)^\infty$. The conclusion now follows since $\mathcal{L}^2_0(Q,\xi)_K$ is dense in $\mathcal{L}^2_0(Q,\xi)$. □

9 Decomposition of the regular representation

Up till now, for $Q \in \mathcal{P}_\sigma$, the expression $\xi \in \mathrm{X}^\wedge_{Q,*,ds}$ meant, by abuse of language, that ξ is an irreducible unitary representation of M_Q with equivalence class $[\xi]$ contained in $\mathrm{X}^\wedge_{Q,*,ds}$. From now on it will be convenient to distinguish between representations and their classes. For every $Q \in \mathcal{P}_\sigma$ and all $\omega \in \mathrm{X}^\wedge_{Q,*,ds}$ we select an irreducible unitary representation $\xi = \xi_\omega$ in a Hilbert space $\mathcal{H}_\omega := \mathcal{H}_\xi$, with $[\xi] = \omega$. Moreover, we put $\mathfrak{H}(Q,\omega) := \mathfrak{H}(Q,\xi_\omega)$, see (8.1).

For $\vartheta \subset \widehat{K}$ a finite subset, let $\mathrm{X}^\wedge_{Q,*,ds}(\vartheta)$ denote the collection of $\omega \in \mathrm{X}^\wedge_{Q,*,ds}$ that have a K_Q-type in common with τ_ϑ.

Lemma 9.1 *Let $Q \in \mathcal{P}_\sigma$ and let $\vartheta \subset \widehat{K}$ be a finite subset. Then $\mathrm{X}^\wedge_{Q,*,ds}(\vartheta)$ is finite. Moreover,*

$$\mathcal{A}_{2,Q}(\tau_\vartheta) = \oplus_{\omega \in \mathrm{X}^\wedge_{Q,*,ds}(\vartheta)} \ \mathcal{A}_{2,Q}(\tau_\vartheta)_\omega,$$

where the direct sum is finite and orthogonal.

Proof: The collection $\mathrm{X}^\wedge_{Q,*,ds}(\vartheta)$ is finite by Lemma 3.2, applied to the spaces $\mathrm{X}_{Q,v}$, for $v \in {}^Q\mathcal{W}$. Put $\tau = \tau_\vartheta$ and fix $v \in {}^Q\mathcal{W}$. We note that $\mathcal{A}_2(\mathrm{X}_{Q,v} : \tau_Q)_\omega = 0$ for $\omega \in \mathrm{X}^\wedge_{Q,*,ds}(\vartheta) \setminus \mathrm{X}^\wedge_{Q,v,ds}(\tau_Q)$, by Remark 3.3, with $\mathrm{X}_{Q,v}$, τ_Q in place of X, τ. Hence, in view of the same remark,

$$\mathcal{A}_2(\mathrm{X}_{Q,v} : \tau_Q) = \oplus_{\omega \in \mathrm{X}^\wedge_{Q,*,ds}(\vartheta)} \ \mathcal{A}_2(\mathrm{X}_{Q,v} : \tau_Q)_\omega,$$

with orthogonal summands. The result now follows by summation over $v \in {}^Q\mathcal{W}$, in view of (4.9), and [12], Eqn. (13.1). □

Lemma 9.2 *Let $Q \in \mathcal{P}_\sigma$, $\vartheta \subset \widehat{K}$ a finite subset and $\omega \in \mathrm{X}^\wedge_{Q,*,ds}$. Then $\mathfrak{H}(Q,\omega)_\vartheta \neq 0$ if and only if $\omega \in \mathrm{X}^\wedge_{Q,*,ds}(\vartheta)$.*

Proof: We have that $\mathfrak{H}(Q,\omega)_\vartheta = \bar{V}(Q,\xi_\omega) \otimes L^2(K : \xi_\omega)_\vartheta$, with non-trivial first component in the tensor product. Hence, $\mathfrak{H}(Q,\omega)_\vartheta$ is non-trivial if and only if $L^2(K : \xi_\omega)_\vartheta$ is. Since $L^2(K : \xi_\omega)$ is the representation space for $\mathrm{Ind}_{K_Q}^K(\xi_\omega)$, the assertion follows by Frobenius reciprocity. □

If $Q \in \mathcal{P}_\sigma$, we define the Hilbert space

$$\mathfrak{H}(Q) := \widehat{\oplus}_{\omega \in \mathrm{X}^\wedge_{Q,*,ds}} \ \mathfrak{H}(Q,\omega), \tag{9.1}$$

where the hat over the direct sum symbol indicates that the natural Hilbert space completion of the algebraic direct sum is taken. Let $\vartheta \subset \widehat{K}$ be a finite subset. In view of Lemmas 9.1 and 9.2 it follows that we may define a map

$$\Psi_\vartheta = \Psi_{Q,\vartheta} : \ \mathfrak{H}(Q)_\vartheta \to \mathcal{A}_{2,Q}(\tau_\vartheta)$$

as the direct sum of the isometries $T \mapsto \psi_T : \mathfrak{H}(Q,\omega)_\vartheta \to \mathcal{A}_{2,Q}(\tau_\vartheta)_\omega$, for $\omega \in \mathrm{X}^\wedge_{Q,*,ds}(\vartheta)$, see (4.13). The following result is immediate.

Lemma 9.3 *Ψ_ϑ is an isometric isomorphism from $\mathfrak{H}(Q)_\vartheta$ onto $\mathcal{A}_{2,Q}(\tau_\vartheta)$.*

If $f \in C_c^\infty(\mathrm{X})_K$ then for $Q \in \mathcal{P}_\sigma$, $\omega \in \mathrm{X}^\wedge_{Q,*,ds}$ and $\nu \in i\mathfrak{a}^*_{Q\mathrm{q}}$ we define the Fourier transform $\hat{f}(Q : \omega : \nu) \in \bar{V}(\xi_\omega) \otimes C^\infty(K : \xi_\omega)_K$ by

$$\hat{f}(Q : \omega : \nu) := \hat{f}(Q : \xi_\omega : \nu).$$

This definition is justified since it follows from (4.19) that $\nu \notin \cup \mathcal{H}^\vee(Q,\xi)$.

Proposition 9.4 *Let $\vartheta \subset \widehat{K}$ be a finite set of K-types, let $f \in C_c^\infty(\mathrm{X})_\vartheta$ and let $F = \varsigma_\vartheta(f) \in C_c^\infty(\mathrm{X}:\tau_\vartheta)$ be the associated spherical function, see Lemma 3.4. Then for each $Q \in \mathcal{P}_\sigma$ and every $\nu \in i\mathfrak{a}_{Q\mathrm{q}}^*$,*

$$\mathcal{F}_Q F(\nu) = \sum_{\omega \in \mathrm{X}_{Q,*,ds}^\wedge} \psi_{\hat{f}(Q:\omega:\nu)} \qquad \text{in} \qquad \mathcal{A}_{2,Q}(\tau_\vartheta). \tag{9.2}$$

If $\hat{f}(Q:\omega:\nu)$ is non-zero, then $\omega \in \mathrm{X}_{Q,*,ds}^\wedge(\vartheta)$; in particular, the above sum is finite. Finally,

$$\|\mathcal{F}_Q F(\nu)\|^2 = \sum_{\omega \in \mathrm{X}_{Q,*,ds}^\wedge} \|\hat{f}(Q:\omega:\nu)\|^2_{\bar{V}(\xi_\omega)\otimes L^2(K:\xi_\omega)}.$$

Proof: It follows from Lemma 7.2 that $\hat{f}(Q:\omega:\nu)$ is an element of $\mathfrak{H}(Q,\omega)_\vartheta$, for every $\omega \in \mathrm{X}_{Q,*,ds}^\wedge$. Hence, if this element is non-zero, then ω belongs to the finite set $\mathrm{X}_{Q,*,ds}^\wedge(\vartheta)$.

The identity (9.2) follows from Lemma 7.5, since Ψ_ϑ is a surjective isometry. The final assertion follows by once more using that Ψ_ϑ is an isometry. □

The following result will turn out to be the Plancherel identity for K-finite functions. We recall from [12], Def. 13.4, that two parabolic subgroups $P, Q \in \mathcal{P}_\sigma$ are said to be associated if their σ-split components $\mathfrak{a}_{P\mathrm{q}}$ and $\mathfrak{a}_{Q\mathrm{q}}$ are conjugate under W. The equivalence relation of associatedness on $\mathcal{P}_\sigma$ is denoted by $\sim$. Let $\mathbf{P}_\sigma \subset \mathcal{P}_\sigma$ be a choice of representatives for $\mathcal{P}_\sigma/\sim$. If $Q \in \mathcal{P}_\sigma$, then W_Q^* denotes the normalizer of $\mathfrak{a}_{Q\mathrm{q}}$ in W.

Theorem 9.5 *Let $f \in C_c^\infty(\mathrm{X})_K$. Then*

$$\|f\|_2^2 = \sum_{Q\in\mathbf{P}_\sigma} \sum_{\omega\in \mathrm{X}_{Q,*,ds}^\wedge} \big[W: W_Q^*\big] \int_{i\mathfrak{a}_{Q\mathrm{q}}^*} \|\hat{f}(Q:\omega:\nu)\|^2 \, d\mu_Q(\nu).$$

Proof: This follows from [12], Thm. 23.4, combined with Proposition 9.4. □

Our next goal is to show that the above indeed corresponds with a direct integral decomposition for the left regular representation L of G in $L^2(\mathrm{X})$.

Let $Q \in \mathbf{P}_\sigma$. For $\omega \in \mathrm{X}_{Q,*,ds}^\wedge$, the direct integral representation $\pi_{Q,\omega} := \pi_{Q,\xi_\omega}$ of G in $\mathfrak{L}^2(Q,\omega) := \mathfrak{L}^2(Q,\xi_\omega)$ is unitary, see Lemma 8.1. We define

$$\mathfrak{L}^2(Q) := \widehat{\oplus}_{\omega\in \mathrm{X}_{Q,*,ds}^\wedge} \mathfrak{L}^2(Q,\omega), \tag{9.3}$$

where the hat over the direct sum sign has the same meaning as in (9.1). We note that $\mathfrak{L}^2(Q)$ is naturally isometrically isomorphic with the Hilbert space of $\mathfrak{H}(Q)$-valued L^2-functions on $i\mathfrak{a}_{Q\mathrm{q}}^*$, relative to the measure $[W: W_Q^*]\, d\mu_Q$.

Let π_Q be the associated direct sum of the representations $\pi_{Q,\omega}$. Then π_Q is a unitary representation of G in $\mathfrak{L}^2(Q)$.

Finally, we define

$$\mathfrak{L}^2 := \oplus_{Q\in\mathbf{P}_\sigma}\ \mathfrak{L}^2(Q) \tag{9.4}$$

and equip it with the direct sum inner product. The direct sum being finite, $\mathfrak{L}^2$ becomes a Hilbert space in this way. The associated direct sum $\pi = \oplus_{Q\in\mathbf{P}_\sigma}\pi_Q$ is a unitary representation of G in $\mathfrak{L}^2$.

For $Q \in \mathbf{P}_\sigma$ and $\omega \in \mathrm{X}^\wedge_{Q,*,ds}$ we denote the natural inclusion map $\mathfrak{L}^2(Q,\omega) \to \mathfrak{L}^2$ by $\mathrm{i}_{Q,\omega}$; its adjoint $\mathrm{pr}_{Q,\omega}\colon \mathfrak{L}^2 \to \mathfrak{L}^2(Q,\omega)$ is the natural projection map. If $\varphi \in \mathfrak{L}^2$, we denote its component $\mathrm{pr}_{Q,\omega}\varphi \in \mathfrak{L}^2(Q:\omega)$ by $\varphi(Q:\omega:\,\cdot\,)$. Thus,

$$\|\varphi\|^2_{\mathfrak{L}^2} = \sum_{Q\in\mathbf{P}_\sigma}\sum_{\omega\in\mathrm{X}^\wedge_{Q,*,ds}} \big[W\colon W^*_Q\big] \int_{i\mathfrak{a}^*_{Q\mathrm{q}}} \|\varphi(Q:\omega:\nu)\|^2\, d\mu_Q.$$

It follows from Theorem 9.5 that the Fourier transform $\hat f$ of an element $f \in C^\infty_c(\mathrm{X})_K$ belongs to $\mathfrak{L}^2$. Moreover,

$$\|f\|_{L^2(\mathrm{X})} = \|\hat f\|_{\mathfrak{L}^2}.$$

Theorem 9.6 *The map $f \mapsto \hat f$ has a unique extension to a continuous linear map $\mathfrak{F}\colon L^2(\mathrm{X}) \to \mathfrak{L}^2$. The map $\mathfrak{F}$ is isometric and intertwines the G-representations $(L, L^2(X))$ and $(\pi, \mathfrak{L}^2)$.*

Proof: The first two assertions are obvious from the discussion preceding the theorem. It remains to prove the intertwining property. Fix $Q \in \mathbf{P}_\sigma$ and $\omega \in \mathrm{X}^\wedge_{Q,*,ds}$; then it suffices to prove that $\mathfrak{F}_{Q,\omega} := \mathrm{pr}_{Q,\omega}\circ\mathfrak{F}$ intertwines L with $\pi_{Q,\omega}$. We will do this in a number of steps. For convenience, we write $\xi = \xi_\omega$.

Lemma 9.7 *Let $f \in C^\infty_c(\mathrm{X})$.*

(a) *If $k \in K$, then $\mathfrak{F}_{Q,\omega}(L_k f) = \pi_{Q,\omega}(k)\mathfrak{F}_{Q,\omega}f$.*

(b) *If $u \in U(\mathfrak{g})$, then*

$$\langle \mathfrak{F}_{Q,\omega}(L_u f)\,|\,\varphi\rangle = \langle \mathfrak{F}_{Q,\omega}(f)\,|\,\pi(\check u)\varphi\rangle \tag{9.5}$$

for all $\varphi \in \mathfrak{L}^2(Q,\omega)^\infty$.

Proof: We first assume that f is K-finite. Assertion (a) is an immediate consequence of the K-equivariance asserted in Lemma 7.2.

In view of Lemma 8.6 it suffices to prove assertion (b) for $\varphi \in \mathfrak{L}^2_0(Q,\omega)$. In view of Lemma 8.3 we may as well assume that $u = X \in \mathfrak{g}$. Then the expression on the left-hand side of (9.5) equals

$$\int_{i\mathfrak{a}^*_{Q\mathrm{q}}} \langle (L_X f)\hat{}(Q:\xi:\nu)\,|\,\varphi(\nu)\rangle_{\mathfrak{H}(Q,\omega)}\, d\mu_Q(\nu).$$

The integrand is continuous and compactly supported as a function of ν. By the $\mathfrak{g}$-equivariance asserted in Lemma 7.2 and unitarity of the representations $\pi_{Q,\xi,-\nu}$ for all $\nu \in i\mathfrak{a}_{Q\mathrm{q}}^*$, we see that the integral equals

$$\int_{i\mathfrak{a}_{Q\mathrm{q}}^*} \langle \hat{f}(Q:\xi:\nu) \,|\, [I \otimes \pi_{Q,\xi,-\nu}(\check{X})]\varphi(\nu)\rangle \, d\mu_Q(\nu)$$

$$= \int_{i\mathfrak{a}_{Q\mathrm{q}}^*} \langle \hat{f}(Q:\xi:\nu) \,|\, [\pi_{Q,\omega}(\check{X})\varphi](\nu)\rangle \, d\mu_Q(\nu),$$

see also Lemma 8.3. The expression on the right-hand side of the latter equality equals the one on the right-hand side of (9.5). This establishes the result for f in the dense subspace $C_c^\infty(\mathrm{X})_K$ of $C_c^\infty(\mathrm{X})$. The idea is to extend the result by an argument involving continuity.

For assertion (a) we proceed as follows. Fix $k \in K$. Then the map $f \mapsto L_k f$ is continuous from $C_c^\infty(\mathrm{X})$ to $L^2(\mathrm{X})$. Since $\mathcal{F}_{Q,\omega}$ is continuous $L^2(\mathrm{X}) \to \mathfrak{L}^2(Q,\omega)$, by the first part of the proof of Theorem 9.6, whereas $\pi_{Q,\omega}$ is a unitary representation, it follows that both $f \mapsto \mathcal{F}_{Q,\omega} L_k f$ and $f \mapsto \pi_{Q,\omega}(k)\mathcal{F}_{Q,\omega} f$ are continuous maps from $C_c^\infty(\mathrm{X})$ to $\mathfrak{L}^2(Q,\omega)$. Hence, (a) follows by continuity and density.

Finally, for the proof of (b) we fix $u \in U(\mathfrak{g})$ and $\varphi \in \mathfrak{L}^2(Q,\omega)^\infty$. Then the map $f \mapsto L_u f$ is continuous from $C_c^\infty(\mathrm{X})$ to $L^2(\mathrm{X})$, whereas $\mathcal{F}_{Q,\omega}$ is continuous from $L^2(\mathrm{X})$ to $\mathfrak{L}^2(Q,\omega)$ as said above. It follows that the inner product on the left-hand side of (9.5) depends continuous linearly on $f \in C_c^\infty(\mathrm{X})$. Since $\pi_{Q,\omega}(\check{u})\varphi$ is a fixed element of $\mathfrak{L}^2(Q,\omega)$, the same holds for the inner product on the right-hand side of (9.5). Thus, (b) follows by continuity and density from the similar statement for K-finite functions. □

Lemma 9.8 *Let $f \in C_c^\infty(\mathrm{X})$. Then*

$$\mathfrak{F}_{Q,\omega}(L_x f) = \pi_{Q,\omega}(x)\mathfrak{F}_{Q,\omega}(f)$$

for all $x \in G$.

Proof: By Lemma 9.7 (a) it suffices to prove the identity for x in the connected component of G containing e. Hence it suffices to establish the identity for $x \in \exp(\mathfrak{g})$. Write $\pi = \pi_{Q,\omega}$ and fix $X \in \mathfrak{g}$. Then it suffices to show that $\pi(\exp tX)^{-1}\mathfrak{F}_{Q,\omega}(L_{\exp tX} f)$ is a constant function of $t \in \mathbb{R}$ with value $\mathfrak{F}_{Q,\omega}(f)$. For this it suffices to show that, for every $\varphi \in \mathfrak{L}^2(Q,\omega)^\infty$, the function

$$\Phi: t \mapsto \langle \pi(\exp tX)^{-1}\mathfrak{F}_{Q,\omega}(L_{\exp tX} f) \,|\, \varphi\rangle$$

is differentiable with derivative zero. We observe that

$$\Phi(t) = \langle \mathfrak{F}_{Q,\omega}(L_{\exp tX} f) \,|\, \pi(\exp tX)\varphi\rangle.$$

The $L^2(\mathrm{X})$-valued function $t \mapsto L_{\exp tX} f$ is C^1 on $\mathbb{R}$, with derivative $t \mapsto L_X L_{\exp tX} f$. Moreover, since φ is a smooth vector, the $\mathfrak{L}^2(Q,\omega)$-

valued function $t \mapsto \pi(\exp tX)\varphi$ is also C^1 on $\mathbb{R}$, with derivative $t \mapsto \pi(X)\pi(\exp tX)$. By continuity of $\mathfrak{F}_{Q,\omega}$ and the inner product on $\mathcal{L}^2(Q,\omega)$, it follows that $\Phi(t)$ is C^1 with derivative given by

$$\Phi'(t) = \langle \mathfrak{F}_{Q,\omega}(L_X L_{\exp tX} f) \,|\, \pi(\exp tX)\varphi\rangle + \langle \mathfrak{F}_{Q,\omega}(L_{\exp tX} f) \,|\, \pi(X)\pi(\exp tX)\varphi\rangle.$$

The latter expression equals zero by Lemma 9.7 (b), applied with $L_{\exp tX} f$ and $\pi(\exp tX)\varphi$ in place of f and φ, respectively. □

End of proof of Theorem 9.6: In the beginning of the proof of the theorem, we established that $\mathcal{F}_{Q,\omega}$ is a continuous linear map from $L^2(\mathrm{X})$ to $\mathcal{L}^2(Q,\omega)$. By density of $C_c^\infty(\mathrm{X})_K$ in $L^2(\mathrm{X})$ and continuity of the representations L and $\pi_{Q,\omega}$, the G-equivariance of $\mathfrak{F}_{Q,\omega}$ follows from Lemma 9.8. □

Let $Q \in \mathbf{P}_\sigma$ and let $\vartheta \subset \widehat{K}$ be a finite subset. We recall from [12], Thm. 23.4, that the spherical Fourier transform $\mathcal{F}_Q$ associated with $\tau = \tau_\vartheta$, originally defined as a continuous linear map $\mathcal{C}(\mathrm{X}\colon \tau_\vartheta) \to \mathcal{S}(i\mathfrak{a}_{Q\mathrm{q}}^*) \otimes \mathcal{A}_{2,Q}(\tau_\vartheta)$, has a unique extension to a continuous linear map $L^2(\mathrm{X}\colon \tau_\vartheta) \to L^2(i\mathfrak{a}_{Q\mathrm{q}}^*) \otimes \mathcal{A}_{2,Q}(\tau_\vartheta)$, denoted by the same symbol. For application in the next section, we state the relation between the extended spherical Fourier transform $\mathcal{F}_Q$ and $\mathfrak{F}$ in a lemma. Let pr_Q denote the projection operator $\mathcal{L}^2 \to \mathcal{L}^2(Q)$ associated with the decomposition (9.4).

Lemma 9.9 *Let $Q \in \mathbf{P}_\sigma$ and let $\vartheta \subset \widehat{K}$ be a finite subset. Let $f \in L^2(\mathrm{X})_\vartheta$ and let $F = \varsigma_\vartheta(f) \in L^2(\mathrm{X}\colon \tau_\vartheta)$ be the associated spherical function, see Lemma 3.4. Then*

$$\mathcal{F}_Q F(\nu) = \Psi_{Q,\vartheta}(\mathrm{pr}_Q \circ \mathfrak{F} f(\nu)), \tag{9.6}$$

for almost all $\nu \in i\mathfrak{a}_{Q\mathrm{q}}^*$. Here $\Psi_{Q,\vartheta}$ is the isometry of Lemma 9.3.

Proof: For $f \in C_c^\infty(\mathrm{X})_\vartheta$ we have $\mathrm{pr}_Q \circ \mathfrak{F} f(\nu) = \hat{f}(Q\colon \cdot : \nu)$, so that (9.6) follows from (9.2) and the definition of $\Psi_{Q,\vartheta}$ before Lemma 9.3. The general result follows from this by density of $C_c^\infty(\mathrm{X})_\vartheta$ in $L^2(\mathrm{X})_\vartheta$ and continuity of the maps $\mathcal{F}_Q \circ \varsigma_\vartheta$ and $(I \otimes \Psi_{Q,\vartheta}) \circ \mathrm{pr}_Q \circ \mathfrak{F}$ from $L^2(\mathrm{X})_\vartheta$ to $L^2(i\mathfrak{a}_{Q\mathrm{q}}^*) \otimes \mathcal{A}_{2,Q}(\tau_\vartheta)$, see [12], Thm. 23.4, and Lemmas 3.4, 9.3 and Theorem 9.6. □

10 The Plancherel decomposition

Our goal in this section is to establish the Plancherel decomposition. For this we need to characterize the image of the transform $\mathfrak{F}$, defined in the previous section. To achieve this we shall first decompose $\mathfrak{F}$ into pieces corresponding with the parabolic subgroups from $\mathbf{P}_\sigma$.

Let $Q \in \mathbf{P}_\sigma$. For $\vartheta \subset \widehat{K}$ a finite subset, we defined in [12], text before Thm. 23.1, a subspace $\mathcal{C}_Q(\mathrm{X}:\tau_\vartheta)$ of $\mathcal{C}(\mathrm{X}:\tau_\vartheta)$, as the image of the wave packet transform $\mathcal{J}_Q$. In [12], text before Cor. 23.3, we defined $L^2_Q(\mathrm{X}:\tau_\vartheta)$ as the closure of $\mathcal{C}_Q(\mathrm{X}:\tau_\vartheta)$ in $L^2(\mathrm{X}:\tau_\vartheta)$. Accordingly, we denote by $L^2_Q(\mathrm{X})_\vartheta$ the canonical image of $L^2_Q(\mathrm{X}:\tau_\vartheta)$ in $L^2(\mathrm{X})_\vartheta$, cf. Lemma 3.4. Finally, we denote by $L^2_Q(\mathrm{X})$ the L^2-closure of the union of such spaces for all ϑ. Then it follows from [12], Cor. 23.3, that

$$L^2(\mathrm{X}) = \oplus_{Q\in\mathbf{P}_\sigma}\ L^2_Q(\mathrm{X}), \tag{10.1}$$

with orthogonal K-invariant direct summands.

Lemma 10.1 *Let $Q \in \mathbf{P}_\sigma$. The space $L^2_Q(\mathrm{X})$ is G-invariant. Moreover, $\mathfrak{F}$ maps $L^2_Q(\mathrm{X})$ into $\mathfrak{L}^2(Q)$.*

Proof: We shall first prove the second assertion. Fix $P \in \mathbf{P}_\sigma$, $P \neq Q$, and assume that $\vartheta \subset \widehat{K}$ is a finite subset. Then it follows from [12], Cor. 23.3 and Thm. 23.4 (c), that $\mathcal{F}_P = 0$ on $L^2_Q(\mathrm{X}:\tau_\vartheta)$. In view of Lemma 9.9 this implies that $\mathrm{pr}_P \circ \mathfrak{F}(f) = 0$ for every $f \in L^2_Q(\mathrm{X})_\vartheta$; here pr_P denotes the orthogonal projection $\mathfrak{L}^2 \to \mathfrak{L}^2(P)$. By density of $L^2_Q(\mathrm{X})_K$ in $L^2_Q(\mathrm{X})$ and continuity of the map $\mathrm{pr}_P \circ \mathfrak{F}\colon L^2(\mathrm{X}) \to \mathfrak{L}^2(P)$, see Theorem 9.6, it follows that $\mathrm{pr}_P \circ \mathfrak{F}$ vanishes on $L^2_Q(\mathrm{X})$, for every $P \in \mathbf{P}_\sigma \setminus \{Q\}$. The second assertion now follows by orthogonality of the decomposition (9.4).

Since $\mathfrak{F}$ is an isometry, its adjoint $\mathfrak{F}^*$ is surjective from $\mathfrak{L}^2$ onto $L^2(\mathrm{X})$. Moreover, since $\mathfrak{F}$ is compatible with the decompositions (10.1) and (9.4), it follows by orthogonality of the mentioned decompositions that

$$L^2_Q(\mathrm{X}) = \mathfrak{F}^*(\mathfrak{L}^2(Q)).$$

By G-equivariance of $\mathfrak{F}$ and unitarity of the representations L and π, the map $\mathfrak{F}^*$ is G-equivariant. If follows that $L^2_Q(\mathrm{X})$ is G-invariant. □

We denote by $\mathfrak{F}_Q$ the restriction of $\mathfrak{F}$ to $L^2_Q(\mathrm{X})$, viewed as a map into $\mathfrak{L}^2(Q)$.

Corollary 10.2 *The map $\mathfrak{F}$ is the direct sum of the maps $\mathfrak{F}_Q$, for $Q \in \mathbf{P}_\sigma$.*

If $\mathcal{H}$ is a Hilbert space, we denote by $\mathrm{End}(\mathcal{H})$ the space of continuous linear endomorphisms of $\mathcal{H}$, equipped with the operator norm. By $\mathrm{U}(\mathcal{H})$ we denote the subspace of unitary endomorphisms. If $P \in \mathcal{P}_\sigma$, we define $W(\mathfrak{a}_{P\mathrm{q}}) = W(\mathfrak{a}_{P\mathrm{q}} \mid \mathfrak{a}_{P\mathrm{q}})$ as in [12], § 3. Then by [12], Cor. 3.5,

$$W(\mathfrak{a}_{P\mathrm{q}}) \simeq W_P^*/W_P. \tag{10.2}$$

Proposition 10.3 *For each $s \in W(\mathfrak{a}_{Q\mathrm{q}})$ there exists a measurable map $\mathfrak{C}_{Q,s}: i\mathfrak{a}_{Q\mathrm{q}}^* \to \mathrm{End}(\mathfrak{H}(Q))$, which is almost everywhere uniquely determined, such that $\nu \mapsto \|\mathfrak{C}_{Q,s}(\nu)\|$ is essentially bounded, and such that for every $f \in C_c^\infty(\mathrm{X})$,*

$$\mathfrak{F}_Q f(s\nu) = \mathfrak{C}_{Q,s}(\nu)\, \mathfrak{F}_Q f(\nu), \tag{10.3}$$

for almost all $\nu \in i\mathfrak{a}_{Q\mathrm{q}}^*$. For almost every $\nu \in i\mathfrak{a}_{Q\mathrm{q}}^*$ the map $\mathfrak{C}_{Q,s}(\nu)$ is unitary. Moreover, for all $s, t \in W(\mathfrak{a}_{Q\mathrm{q}})$,

$$\mathfrak{C}_{Q,st}(\nu) = \mathfrak{C}_{Q,s}(t\nu) \circ \mathfrak{C}_{Q,t}(\nu). \tag{10.4}$$

In particular, $\mathfrak{C}_{Q,1}(\nu) = I$ and $\mathfrak{C}_{Q,s}(\nu)^{-1} = \mathfrak{C}_{Q,s^{-1}}(s\nu)$, for all $s \in W(\mathfrak{a}_{Q\mathrm{q}})$.

For Q minimal this result is part of Prop. 18.6 in [8]. In the present more general setting, we initially reason in a similar way. For $\Omega \subset i\mathfrak{a}_{Q\mathrm{q}}^*$ a measurable subset, we denote by $\mathfrak{L}^2_\Omega(Q)$ the closed G-invariant subspace of $\mathfrak{L}^2(Q)$ consisting of square integrable functions $i\mathfrak{a}_{Q\mathrm{q}}^* \to \mathfrak{H}(Q)$ that vanish almost everywhere outside Ω. The orthogonal projection onto this subspace is denoted by $\varphi \mapsto \varphi_\Omega$.

The uniqueness statement of Proposition 10.3 follows from the following lemma, which generalizes [8], Lemma 18.7. We denote by $\mathfrak{a}_{Q\mathrm{q}}^{*\mathrm{reg}}$ the collection of elements $H \in \mathfrak{a}_{Q\mathrm{q}}^*$ whose parabolic equivalence class relative to $(\mathfrak{a}_{\mathrm{q}}^*, \Sigma)$ is open in $\mathfrak{a}_{Q\mathrm{q}}^*$. The set $\mathfrak{a}_{Q\mathrm{q}}^{*\mathrm{reg}}$ consists of finitely many connected components, called chambers. The group $W(\mathfrak{a}_{Q\mathrm{q}})$ acts freely, but in general not transitively, on the collection of chambers; therefore, there exists an open and closed fundamental domain for $W(\mathfrak{a}_{Q\mathrm{q}})$ in $\mathfrak{a}_{Q\mathrm{q}}^{*\mathrm{reg}}$.

Lemma 10.4 *Let $\Omega \subset i\mathfrak{a}_{Q\mathrm{q}}^{*\mathrm{reg}}$ be an open and closed fundamental domain for $W(\mathfrak{a}_{Q\mathrm{q}})$. Then $f \mapsto (\mathrm{pr}_Q \mathfrak{F} f)_\Omega$ maps $C_c^\infty(\mathrm{X})$ onto a dense subspace of $\mathfrak{L}^2_\Omega(Q)$, and $C_c^\infty(\mathrm{X})_\vartheta$ onto a dense subspace of $\mathfrak{L}^2_\Omega(Q)_\vartheta$, for every finite set $\vartheta \subset \widehat{K}$.*

Proof: The proof is similar to the proof of Lemma 18.7 of [8]. Fix a finite subset $\vartheta \subset K$; then it suffices to prove the statement about $C_c^\infty(\mathrm{X})_\vartheta$, by density of the K-finite vectors. Let $T \in \mathfrak{L}^2_\Omega(Q)_\vartheta$, and suppose that $\langle \mathrm{pr}_Q \mathfrak{F} f \mid T\rangle = 0$ for all $f \in C_c^\infty(\mathrm{X})_\vartheta$. Then it suffices to show that $T = 0$. Put $T(\nu) = \sum_\omega T(\omega\colon \nu)$ with $T(\omega\colon \nu) \in \mathfrak{H}(Q, \omega)_\vartheta$. Note that this sum is finite by Lemmas 9.1 and 9.2. Let $\tau = \tau_\vartheta$, then $\psi_{T(\omega\colon \nu)} \in \mathcal{A}_{2,Q}(\tau)$. We put

$$\Psi(\nu) := \Psi_\vartheta(T(\nu)) = \sum_{\omega \in \mathrm{X}^\wedge_{Q,*,ds}} \psi_{T(\omega\colon \nu)} \in \mathcal{A}_{2,Q}(\tau).$$

Note that for Q minimal, the constants d_ω that occur in [8] are absent here, see Remark 4.1. Let $F \in C_c^\infty(\mathrm{X}\colon \tau)$, and let $f = F(\cdot)(e)$. Then

$f \in C_c^\infty(\mathrm{X})_\vartheta$ and $F = \varsigma_\vartheta(f)$, see Lemma 3.4. Moreover, as in [8], proof of Lemma 18.7,

$$\langle \mathcal{F}_Q F \mid \Psi \rangle = \langle \hat{f}(Q) \mid T \rangle = \langle \mathrm{pr}_Q \mathfrak{F} f \mid T \rangle = 0. \tag{10.5}$$

Let the space $[L^2(i\mathfrak{a}_{Q\mathrm{q}}^*) \otimes \mathcal{A}_{2,Q}(\tau)]^{W(\mathfrak{a}_{Q\mathrm{q}})}$ be defined as in [12], text before Cor. 22.2. It follows from the definition of this space that the restriction map $\varphi \mapsto \varphi|_\Omega$ is a bijection from it onto $L^2(\Omega) \otimes \mathcal{A}_{2,Q}(\tau)$.

The image of $C_c^\infty(\mathrm{X}\colon \tau)$ under $\mathcal{F}_Q$ is dense in the space $[L^2(i\mathfrak{a}_{Q\mathrm{q}}^*) \otimes \mathcal{A}_{2,Q}(\tau)]^{W(\mathfrak{a}_{Q\mathrm{q}})}$, by [12], Thm. 23.4 (c). Combining this with (10.5) we see that Ψ is perpendicular to the mentioned space. Since $\Psi = 0$ outside Ω, we infer that $\Psi|_\Omega$ is perpendicular to $L^2(\Omega) \otimes \mathcal{A}_{2,Q}(\tau)$. We conclude that Ψ, hence T, is zero. □

Proof of Proposition 10.3: We fix a finite subset $\vartheta \subset \widehat{K}$ and put $\tau = \tau_\vartheta$. We will first prove that there exists a measurable map $\mathfrak{C}_{Q,s,\vartheta}\colon i\mathfrak{a}_{Q\mathrm{q}}^* \to \mathrm{End}(\mathfrak{H}(Q)_\vartheta)$, such that (10.3) is valid with $\mathfrak{C}_{Q,s,\vartheta}$ in place of $\mathfrak{C}_{Q,s}$, for every $f \in C_c^\infty(\mathrm{X})_\vartheta$. We define

$$\mathfrak{C}_{Q,s,\vartheta}(\nu) = \Psi_\vartheta^{-1} \circ C_{Q|Q}^\circ(s\colon \nu) \circ \Psi_\vartheta, \tag{10.6}$$

where Ψ_ϑ is the isometry from $\mathfrak{H}(Q)_\vartheta$ onto $\mathcal{A}_{2,Q}(\tau)$ defined in the text preceding Lemma 9.3 and where the C-function is defined as in [12], Def. 17.6, with $\tau = \tau_\vartheta$. We note that the $\mathrm{End}(\mathfrak{H}(Q)_\vartheta)$-valued function $\mathfrak{C}_{Q,s,\vartheta}$ is analytic on $i\mathfrak{a}_{Q\mathrm{q}}^*$, by [12], Cor. 18.6. It follows from Lemma 9.3 combined with the Maass–Selberg relations for the C-function, see [12], Thm. 18.3, that $\mathfrak{C}_{Q,s,\vartheta}(\nu)$ maps $\mathfrak{H}(Q)_\vartheta$ unitarily into itself, for $\nu \in i\mathfrak{a}_{Q\mathrm{q}}^*$. From (10.6) and [12], Lemma 22.1 with $P = R = Q$, we deduce that (10.4) is valid with everywhere the index ϑ added.

In view of Lemma 10.4, the function $\mathfrak{C}_{Q,s,\vartheta}$ is uniquely determined by the requirements in the beginning of this proof. If $\vartheta' \subset \widehat{K}$ is a second finite subset with $\vartheta \subset \vartheta'$, let $\mathrm{i}_{\vartheta',\vartheta}$ denote the inclusion map $\mathfrak{H}(Q)_\vartheta \to \mathfrak{H}(Q)_{\vartheta'}$, and let $\mathrm{pr}_{\vartheta,\vartheta'}$ denote the orthogonal projection $\mathfrak{H}(Q)_{\vartheta'} \to \mathfrak{H}(Q)_\vartheta$. Then it follows from the uniqueness that

$$\mathrm{pr}_{\vartheta,\vartheta'} \circ \mathfrak{C}_{Q,s,\vartheta'}(\nu) \circ \mathrm{i}_{\vartheta',\vartheta} = \mathfrak{C}_{Q,s,\vartheta}(\nu),$$

for every $\nu \in i\mathfrak{a}_{Q\mathrm{q}}^*$. By unitarity of the endomorphisms $\mathfrak{C}_{Q,s,\vartheta}(\nu)$ and $\mathfrak{C}_{Q,s,\vartheta'}(\nu)$ this implies that $\mathfrak{C}_{Q,s,\vartheta'}(\nu)$ leaves the subspace $\mathfrak{H}(Q)_\vartheta$ of $\mathfrak{H}(Q)_{\vartheta'}$ invariant, and equals $\mathfrak{C}_{Q,s,\vartheta}(\nu)$ on it. Thus, we may define the endomorphism $\mathfrak{C}_{Q,s}(\nu)$ of $\mathfrak{h}(Q)$ by requiring it to be equal to $\mathfrak{C}_{Q,s,\vartheta}(\nu)$ on $\mathfrak{h}(Q)_\vartheta$, for every finite subset $\vartheta \subset \widehat{K}$. The endomorphism defined depends measurably on ν, has essentially bounded norm and satisfies (10.3). We asserted already that it is uniquely determined by these properties, in view of Lemma 10.4. The remaining asserted properties of $\mathfrak{C}_{Q,s}(\nu)$ follow from the discussion above. □

Lemma 10.5 *Let $Q \in \mathbf{P}_\sigma$ and let Ω be an open and closed fundamental domain for the $W(\mathfrak{a}_{Q\mathrm{q}})$-action in $i\mathfrak{a}_{Q\mathrm{q}}^{*\mathrm{reg}}$. Then the map $f \mapsto |W(\mathfrak{a}_{Q\mathrm{q}})|^{1/2}(\mathfrak{F}_Q f)_\Omega$ defines an isometric isomorphism from $L^2_Q(\mathrm{X})$ onto $\mathfrak{L}^2(Q)_\Omega$, intertwining the restriction of L to $L^2_Q(\mathrm{X})$ with the direct integral representation*

$$\widehat{\oplus}_{\omega\in \mathrm{X}^\wedge_{Q,*,ds}} \int_\Omega^\oplus 1_{\bar{V}(Q,\omega)} \otimes \pi_{Q,\xi,-\nu} \left[W\colon W_Q^*\right] d\mu_Q(\nu) \tag{10.7}$$

of G in $\mathfrak{L}^2(Q)_\Omega$.

Proof: In view of Theorem 9.6 and Lemma 10.1, the map $\mathfrak{F}_Q$ is an isometry from $L^2_Q(\mathrm{X})$ into $\mathfrak{L}^2(Q)$, intertwining the restriction of L to $L^2_Q(\mathrm{X})$ with $\pi_Q := \pi|_{\mathfrak{L}^2(Q)}$. The map $\varphi \mapsto \varphi_\Omega$ from $\mathfrak{L}^2(Q)$ to $\mathfrak{L}^2(Q)_\Omega$ intertwines π_Q with the direct integral (10.7). Thus, it remains to be shown that the map $T\colon f \mapsto |W(\mathfrak{a}_{Q\mathrm{q}})|^{1/2}(\mathfrak{F}_Q f)_\Omega$ from $L^2_Q(\mathrm{X})$ to $\mathfrak{L}^2(Q)_\Omega$ is isometric and onto.

To establish the first property, we note that, for $f \in L^2_Q(\mathrm{X})$ and $s \in W(\mathfrak{a}_{Q\mathrm{q}})$,

$$\|\mathfrak{F}_Q f(s\nu)\| = \|\mathfrak{C}_{Q,s}(\nu)\mathfrak{F}_Q f(\nu)\| = \|\mathfrak{F}_Q f(\nu)\|,$$

for almost every $\nu \in i\mathfrak{a}^*_{Q\mathrm{q}}$, by Proposition 10.3. Hence,

$$\begin{aligned}
\|f\|_2^2 = \|\mathfrak{F} f\|^2_{\mathfrak{L}^2} &= \|\mathfrak{F}_Q f\|^2_{\mathfrak{L}^2(Q)} \\
&= \int_{i\mathfrak{a}^*_{Q\mathrm{q}}} \|\mathfrak{F}_Q f(\nu)\|^2 \left[W\colon W_Q^*\right] d\mu_Q(\nu) \\
&= \sum_{s\in W(\mathfrak{a}_{Q\mathrm{q}})} \int_\Omega \|\mathfrak{F}_Q f(s\nu)\|^2 \left[W\colon W_Q^*\right] d\mu_Q(\nu) \\
&= |W(\mathfrak{a}_{Q\mathrm{q}})| \int_\Omega \|\mathfrak{F}_Q f(\nu)\|^2 \left[W\colon W_Q^*\right] d\mu_Q(\nu).
\end{aligned}$$

It follows that T is an isometry. On the other hand, T has dense image in view of Corollary 10.2 and Lemma 10.4. We conclude that T is surjective. □

We shall now investigate irreducibility and equivalence of the occurring representations $\pi_{Q,\omega,\nu}$.

Lemma 10.6 *Let $\pi \in \mathrm{X}^\wedge_{ds}$. Then π has a real infinitesimal $Z(\mathfrak{g})$-character in the following sense. Let $\mathfrak{j}$ be a Cartan subalgebra of $\mathfrak{g}$, $W(\mathfrak{j})$ the Weyl group of the root system $\Sigma(\mathfrak{j})$ of $\mathfrak{j}_\mathbb{C}$ in $\mathfrak{g}_\mathbb{C}$. Let $[\Lambda] \in \mathfrak{j}^*_\mathbb{C}/W(\mathfrak{j})$ be the infinitesimal character of π. Then $\langle \Lambda\,,\, \alpha\rangle$ is real for every $\alpha \in \Sigma(\mathfrak{j})$.*

Proof: Let $\mathfrak{b}$ a θ-stable Cartan subspace of $\mathfrak{q}$, let $\Sigma(\mathfrak{b})$ be the root system of $\mathfrak{b}$ in $\mathfrak{g}_\mathbb{C}$, $W(\mathfrak{b})$ the associated Weyl group and $I(\mathfrak{b})$ the algebra of $W(\mathfrak{b})$-invariants in $S(\mathfrak{b})$, the symmetric algebra of $\mathfrak{b}_\mathbb{C}$. Let $\gamma_\mathfrak{b}\colon \mathbb{D}(\mathrm{X}) \to I(\mathfrak{b})$ be the associated Harish-Chandra isomorphism. Let $L^2(\mathrm{X})_\pi$ be defined as in (3.2)

with π in place of ξ. We may fix a non-zero simultaneous eigenfunction f for $\mathbb{D}(X)$ in $[L^2(X)_\pi]^\infty$. Let $\lambda \in \mathfrak{b}_\mathbb{C}^*$ be such that $Df = \gamma_\mathfrak{b}(D:\lambda)f$ for $D \in \mathbb{D}(X)$. Then in particular, for each element Z of $\mathfrak{Z}$, the center of $U(\mathfrak{g})$,

$$R_Z f = \gamma_\mathfrak{b}(Z:\lambda)f. \tag{10.8}$$

On the other hand, $R_Z f = L_{\check{Z}} f$ may be expressed in terms of the infinitesimal character of π as follows. Let $\mathfrak{j}$ be a θ-stable Cartan subalgebra of $\mathfrak{g}$ containing $\mathfrak{b}$, let $\Sigma(\mathfrak{j})$ and $W(\mathfrak{j})$ be as in the lemma, and let $I(\mathfrak{j})$ denote the algebra of $W(\mathfrak{j})$-invariants in $S(\mathfrak{j})$. We denote the canonical isomorphism $\mathfrak{Z} \to I(\mathfrak{j})$ by $\gamma_\mathfrak{j}$. Let $\Lambda \in \mathfrak{j}_\mathbb{C}$ be as in the lemma, then

$$R_Z f = L_{\check{Z}} f = \gamma_\mathfrak{j}(\check{Z}:\Lambda)f = \gamma_\mathfrak{j}(Z:-\Lambda)f. \tag{10.9}$$

From (10.8) and (10.9) we obtain that

$$\gamma_\mathfrak{j}(Z:-\Lambda) = \gamma_\mathfrak{b}(Z:\lambda), \qquad (Z \in \mathfrak{Z}). \tag{10.10}$$

Let $\mathfrak{l}$ be the centralizer of $\mathfrak{b}$ in $\mathfrak{g}$, let $\Sigma(\mathfrak{l}_\mathbb{C}, \mathfrak{j})$ be the root system of $\mathfrak{j}$ in $\mathfrak{l}_\mathbb{C}$, $\Sigma^+(\mathfrak{l}_\mathbb{C}, \mathfrak{j})$ a choice of positive roots and $\delta_\mathfrak{l} \in \mathfrak{j}_\mathbb{C}^*$ the associated half sum of the positive roots. By a standard computation in the universal enveloping algebra, involving the definitions of $\gamma_\mathfrak{j}$ and $\gamma_\mathfrak{b}$, it follows that $\gamma_\mathfrak{b}(Z:\lambda) = \gamma_\mathfrak{j}(Z:\lambda-\delta_\mathfrak{l})$, for all $Z \in \mathfrak{Z}$. Combining this with (10.10) we obtain that $-\Lambda$ and $\lambda - \delta_\mathfrak{l}$ are $W(\mathfrak{j})$-conjugate.

Now $\langle \lambda\,,\, \alpha\rangle$ is real, for each $\alpha \in \Sigma(\mathfrak{b})$, by [26], see also [12], Thm. 16.1. It readily follows that Λ is real. □

The following result is due to F. Bruhat [14] for minimal parabolic subgroups and to Harish-Chandra in general. A proof is essentially given in [23].

Theorem 10.7 *For $j = 1, 2$, let $P_j = M_j A_j N_j$ be a parabolic subgroup of G, with the indicated Langlands decomposition. Moreover, let ξ_j be an irreducible unitary representation of M_j with real infinitesimal character and let $\nu_j \in i\mathfrak{a}_j^*$ be regular with respect to the roots of $\mathfrak{a}_j$ in P_j. Let π_j denote the unitarily induced representation* $\mathrm{Ind}_{P_j}^G(\xi_j \otimes \nu_j \otimes 1)$.

(a) *The representation π_j is irreducible, for $j = 1, 2$.*
(b) *The representations π_1 and π_2 are equivalent if and only if the data $(\mathfrak{a}_j, \nu_j, [\xi_j])$, for $j = 1, 2$, are conjugate under K. The latter condition means that there exists $k \in K$ such that $\mathrm{Ad}(k)\mathfrak{a}_1 = \mathfrak{a}_2$, $\nu_1 \circ \mathrm{Ad}(k)^{-1} = \nu_2$ and $\xi_1^k \sim \xi_2$, where $\xi_1^k := \xi_1(k^{-1}(\,\cdot\,)k)|_{M_2}$.*

Proof: Taking into account the actions of the centers of M_j and G, one readily checks that it suffices to prove this result for G connected semisimple and with finite center. Thus, let us assume this to be the case.

Assertion (a) follows from [23], Thm. 4.11. Thus, it remains to prove assertion (b). We first establish the 'if' part. If, in addition to the hypothesis, $P_2 = kP_1k^{-1}$, then the equivalence of π_1, π_2 is a trivial consequence

of conjugating all induction data. Thus, by applying conjugation we may reduce to the case that $\mathfrak{a}_1 = \mathfrak{a}_2$, $\nu_1 = \nu_2$ and $\xi_1 \sim \xi_2$. Then P_1 and P_2 have the same split component. It now follows from [22], Prop. 8.5 (v), that there exists a unitary intertwining operator from π_1 onto π_2. Hence $\pi_1 \sim \pi_2$.

We shall now prove the 'only' if part. Assume that $\pi_1 \sim \pi_2$. By conjugating all induction data of π_1 with an element of K, we see that we may restrict ourselves to the situation that P_1 and P_2 contain a fixed minimal parabolic subgroup P_0 of G, with split component A_0. In particular, $\mathfrak{a}_j \subset \mathfrak{a}_0$ for $j = 1, 2$. It now follows from [23], p. 94, text under the heading 'equivalence', that there exists a $k \in N_K(\mathfrak{a}_0)$ such that $\mathrm{Ad}(k)\mathfrak{a}_1 = \mathfrak{a}_2$ and $\nu_1 \circ \mathrm{Ad}(k)^{-1} = \nu_2$. Conjugating all data of π_1 with k we see that we may as well assume that $\mathfrak{a}_1 = \mathfrak{a}_2$ and $\nu_1 = \nu_2$. Moreover, applying [22], Prop. 8.5 (v), as in the first part of this proof we see that in addition we may assume that $P_1 = P_2$. We now claim that $\xi_1 \sim \xi_2$. This assertion is essentially proved in [23], proof of Thm. 4.11, but not explicitly stated as a result. We shall indicate how to modify the mentioned proof. We use the notation of [23]. In particular, $\xi_j = {}^\circ\sigma_j$. We follow the proof of [23], Thm. 4.11, after the heading 'equivalence', but with $P_1 = P_2 = P$ and $\nu_1 = \nu_2 = \nu$. From $i(\pi_1, \pi_2^*) > 0$ it follows, by application of [23], Thm. 4.10, that

$$\left(M_1^{(0)} \otimes \left(E_1^0 \bar{\otimes} E_2^{0\prime}\right)' \otimes \mathbb{C}_1'\right)^{(P \otimes P)}$$

has positive dimension. Now $M_1^{(0)}$ equals $C^\infty(P)$, equipped with the left times right action of P (see [23], Eqn. (2.6)). Hence the above space is naturally isomorphic with the space of $\mathrm{diag}(P \times P)$-invariants in $(E_1^0 \bar{\otimes} E_2^{0\prime})'$ which in turn is naturally isomorphic with $\mathrm{Hom}_P(E_2^0, E_1^0) = \mathrm{Hom}_M({}^\circ\sigma_2, {}^\circ\sigma_1)$. It follows that the latter space is non-trivial, hence ${}^\circ\sigma_1 \sim {}^\circ\sigma_2$, since the representations involved are irreducible. □

Proposition 10.8 *For $j = 1, 2$, let $Q_j \in \mathbf{P}_\sigma$, $\omega_j \in \mathrm{X}^\wedge_{Q_j,*,ds}$, $\nu_j \in i\mathfrak{a}^{*\mathrm{reg}}_{Q_j\mathfrak{q}}$. Then the representations $\pi_j = \pi_{Q_j,\omega_j,\nu_j}$ are irreducible. Moreover, they are equivalent if and only if $Q_1 = Q_2$ and there exists $s \in W(\mathfrak{a}_{Q_1\mathfrak{q}})$ such that $\nu_2 = s\nu_1$ and $\omega_2 = s\omega_1$.*

Proof: Put $\xi_j = \xi_{\omega_j}$ and $Q = Q_1$. There exists $v \in {}^Q\mathcal{W}$, such that ω_1 belongs to the discrete series of $M_Q/M_Q \cap vHv^{-1}$. It follows from Lemma 10.6 that ω_1 has a real infinitesimal character for the center of $U(\mathfrak{m}_Q)$. A similar statement holds for Q_2, ω_2.

If α is a root of $\mathfrak{a}_Q$ in Q, then its restriction $\alpha_\mathfrak{q}$ to $\mathfrak{a}_{Q\mathfrak{q}}$ belongs to $\Sigma_r(Q)$. Moreover, $\langle \alpha\,,\ \nu_1 \rangle = \langle \alpha|_{\mathfrak{a}_{Q\mathfrak{q}}}\,,\ \nu_1 \rangle \neq 0$. Thus, it follows that ν_1 is regular with respect to the $\mathfrak{a}_Q$-roots in Q. A similar statement holds for ν_2.

Thus, Theorem 10.7 is applicable and we conclude that π_1 and π_2 are irreducible.

Assume that $\pi_1 \sim \pi_2$. Then by Theorem 10.7 (b) we conclude that there exists $k \in K$ such that $\mathrm{Ad}(k)\mathfrak{a}_1 = \mathfrak{a}_2$, $\nu_1 \circ \mathrm{Ad}(k)^{-1} = \nu_2$ and $\xi_1^k \sim \xi_2$. Since $\sigma\nu_j = -\nu_j$, for $j = 1, 2$, it follows by application of σ that also

$\nu_1 \circ \mathrm{Ad}(\sigma k)^{-1} = \nu_2$. We infer that $\mathrm{Ad}((\sigma k)^{-1}k)^{-1*}$ centralizes ν_1, hence belongs to the centralizer M_{1Q} of $\mathfrak{a}_{Q\mathrm{q}}$, by regularity of ν_1. The mentioned element therefore centralizes $\mathfrak{a}_Q$, from which we see that $\mathrm{Ad}(k) = \mathrm{Ad}(\sigma k)$ on $\mathfrak{a}_Q$. This implies that $\sigma \circ \mathrm{Ad}(k) = \mathrm{Ad}(k) \circ \sigma$ on $\mathfrak{a}_Q$, hence $\mathrm{Ad}(k)$ maps $\mathfrak{a}_{Q\mathrm{q}}$ onto $\mathfrak{a}_{Q2\mathrm{q}}$. We conclude that Q_1 and Q_2 are associated, hence equal. Put $Q = Q_1 = Q_2$.

It follows from the above that $\mathrm{Ad}(k)$ normalizes $\mathfrak{a}_{Q\mathrm{q}}$. Hence, $s := \mathrm{Ad}(k)|_{\mathfrak{a}_{Q\mathrm{q}}}$ belongs to $W(\mathfrak{a}_{Q\mathrm{q}})$, see [12], § 3. Finally, it follows that $s\nu_1 = \nu_2$ and $s[\xi_1] = [\xi_1^k] = [\xi_2]$. □

Theorem 10.9 *Let, for each $Q \in \mathbf{P}_\sigma$, an open and closed fundamental domain Ω_Q for the action of $W(\mathfrak{a}_{Q\mathrm{q}})$ on $i\mathfrak{a}_{Q\mathrm{q}}^{*\mathrm{reg}}$ be given. The Fourier transform $\mathfrak{F}$ induces the following Plancherel decomposition of the regular representation L of G in $L^2(\mathrm{X})$:*

$$L \simeq \oplus_{Q\in\mathbf{P}_\sigma} \widehat{\oplus}_{\omega\in \mathrm{X}^\wedge_{Q,*,ds}} \int_{\Omega_Q}^{\oplus} 1_{\bar{V}(Q,\omega)} \otimes \pi_{Q,\omega,\nu}\, |W||W_Q|^{-1}\, d\mu_Q(\nu), \quad (10.11)$$

and

$$\|f\|^2 = \sum_{Q\in\mathbf{P}_\sigma} \sum_{\omega\in \mathrm{X}^\wedge_{Q,*,ds}} \int_{\Omega_Q}^{\oplus} \|\mathfrak{F}_Q f(\omega : -\nu)\|^2\, |W||W_Q|^{-1}\, d\mu_Q(\nu), \quad (10.12)$$

for every $f \in L^2(\mathrm{X})$. In particular, for each $Q \in \mathbf{P}_\sigma$ and every $\omega \in \mathrm{X}^\wedge_{Q,*,ds}$, the induced representation $\pi_{Q,\omega,\nu}$ occurs with multiplicity $m_{Q,\omega} = \dim \bar{V}(Q,\omega)$, for almost every $\nu \in \Omega_Q$.

Proof: The fact that $\mathfrak{F}$ induces the isometric isomorphism of L with the direct integral as expressed by (10.11) and (10.12) follows from Lemma 10.5 applied to $-\Omega_Q$ in place of Ω, combined with Corollary 10.2 and (10.2). The occurring representations $\pi_{Q,\omega,\nu}$ are all irreducible, by Proposition 10.8. It remains to exclude double occurrences. For $j = 1, 2$, let $Q_j \in \mathbf{P}_\sigma$, $\omega_j \in \mathrm{X}^\wedge_{Q_j,*,ds}$, $\nu_j \in i\mathfrak{a}_{Q\mathrm{q}}^{*\mathrm{reg}}$, and assume that $\pi_{Q_1,\omega_1,\nu_1} \sim \pi_{Q_2,\omega_2,\nu_2}$. Then it follows from Proposition 10.8 that $Q_1 = Q_2$. Moreover, there exists $s \in W(\mathfrak{a}_{Q_1\mathrm{q}})$ such that $(s\nu_1, s\omega_1) = (\nu_2, \omega_2)$. Since $\Omega_{Q_1} = \Omega_{Q_2}$ is a fundamental domain for the $W(\mathfrak{a}_{Q_1\mathrm{q}})$-action, it follows that $s = 1$. □

We finish this section with a description of the image of the isometry $\mathfrak{F}: L^2(\mathrm{X}) \to \mathfrak{L}^2$.

Lemma 10.10 *Let $Q \in \mathbf{P}_\sigma$ and $s \in W(\mathfrak{a}_{Q\mathrm{q}})$. Then for almost every $\nu \in i\mathfrak{a}_{Q\mathrm{q}}^*$, the unitary endomorphism $\mathfrak{C}_{Q,s}(\nu)$ of $\mathfrak{H}(Q)$ maps the subspace $\mathfrak{H}(Q,\omega)$ onto $\mathfrak{H}(Q, s\omega)$, for $\omega \in \mathrm{X}^\wedge_{Q,*,ds}$, intertwining the representations $\pi_{Q,\omega,\nu}$ and $\pi_{Q,s\omega,s\nu}$.*

Proof: Let $s \in W$. We will first show that, for almost every $\nu \in i\mathfrak{a}_{Q\mathrm{q}}^*$, the unitary endomorphism $\mathfrak{C}_{Q,s}(\nu)$ of $\mathfrak{H}(Q)$ intertwines the direct sum $\pi_{Q,\nu}$ of the representations $1 \otimes \pi_{Q,\omega,-\nu}$, for $\omega \in \mathrm{X}_{Q,*,ds}^{\wedge}$, with the similar direct sum $\pi_{Q,s\nu}$ of the representations $1 \otimes \pi_{Q,\omega,-s\nu}$. Let Ω be an open and closed fundamental domain for the $W(\mathfrak{a}_{Q\mathrm{q}})$-action on $i\mathfrak{a}_{Q\mathrm{q}}^{*\mathrm{reg}}$. Then the map $\mathfrak{F}_{Q\Omega}\colon f \mapsto \mathfrak{F}_Q(f)_\Omega$ is an equivariant isometry from $L_Q^2(\mathrm{X})$ onto $\mathfrak{L}^2(Q)_\Omega$, by Lemma 10.5. Similarly, the map $\mathfrak{F}_{Q\,s\Omega}$ is an intertwining isomorphism from $L_Q^2(\mathrm{X})$ onto $\mathfrak{L}^2(Q)_{s\Omega}$. Moreover, by (10.3), for every $f \in L_Q^2(\mathrm{X})$ we have

$$s^*(\mathfrak{F}_{Q\,s\Omega} f)(\nu) = \mathfrak{F}_{Q\,s\Omega} f(s\nu) = \mathfrak{C}_{Q,s}(\nu)\mathcal{F}_{Q\Omega} f(\nu),$$

for almost every $\nu \in \Omega$. It follows that the map $\varphi \mapsto s^{*-1}[\mathfrak{C}_{Q,s}(\,\cdot\,)\varphi]$ is an equivariant isometry from $\mathfrak{L}^2(Q)_\Omega$ onto $\mathfrak{L}^2(Q)_{s\Omega}$. Let $x \in G$. Then

$$s^*\,\pi_Q(x)s^{*-1}[\mathfrak{C}_{Q,s}(\,\cdot\,)\varphi] = \mathfrak{C}_{Q,s}(\,\cdot\,)\pi_Q(x)\varphi,$$

for every $\varphi \in \mathfrak{L}^2(Q)$. It follows that

$$\pi_{Q,s\nu}(x)\mathfrak{C}_{Q,s}(\nu) = \mathfrak{C}_{Q,s}(\nu)\pi_{Q,\nu}(x) \tag{10.13}$$

for almost every $\nu \in \Omega$. Since Ω was arbitrary, (10.13) holds for almost every $\nu \in i\mathfrak{a}_{Q\mathrm{q}}^*$. Select a countable dense subset G_0 of G. Then there exists a subset $\mathcal{A} \subset i\mathfrak{a}_{Q\mathrm{q}}^{*\mathrm{reg}}$ with complement of measure zero in $i\mathfrak{a}_{Q\mathrm{q}}^*$, such that $\mathfrak{C}_{Q,s}(\,\cdot\,)$ is represented by a function on $\mathcal{A}$ with values in $\mathrm{U}(\mathfrak{H}(Q))$, satisfying (10.13) for all $x \in G_0$ and $\nu \in \mathcal{A}$. By continuity of $\mathfrak{C}_{Q,s}(\nu)$ and of the representations $\pi_{Q,\nu}$ and $\pi_{Q,s\nu}$, it follows that (10.13) holds for all $\nu \in \mathcal{A}$ and all $x \in G$. In view of Theorem 10.7, the representation $1 \otimes \pi_{Q,\tilde{\omega},-s\nu}$, for $\tilde{\omega} \in \mathrm{X}_{Q,*,ds}^{\wedge}$ and $s \in \mathcal{A}$, is not disjoint from $1 \otimes \pi_{Q,\omega,-\nu}$, if and only if $\tilde{\omega} = s\omega$. All assertions now follow for all $\nu \in \mathcal{A}$. □

It follows from the above result that, for each $s \in W(\mathfrak{a}_{Q\mathrm{q}})$, we may define a unitary endomorphism $\Gamma_Q(s)$ of $\mathfrak{L}^2(Q)$ by

$$[\Gamma_Q(s)\varphi](\nu) = \mathfrak{C}_{Q,s}(s^{-1}\nu)\varphi(s^{-1}\nu),$$

for $\varphi \in \mathfrak{L}^2(Q)$ and almost every $\nu \in i\mathfrak{a}_{Q\mathrm{q}}^*$. Moreover, the map $\Gamma_Q(s)$ intertwines π_Q with itself. It follows from Proposition 10.3 that $s \mapsto \Gamma_Q(s)$ defines a unitary representation of $W(\mathfrak{a}_{Q\mathrm{q}})$ in $\mathfrak{L}^2(Q)$, commuting with the action of G. Accordingly, the associated space $\mathfrak{L}^2(Q)^{W(\mathfrak{a}_{Q\mathrm{q}})}$ of invariants is a closed G-invariant subspace of $\mathfrak{L}^2(Q)$.

Theorem 10.11

(a) *For each $Q \in \mathbf{P}_\sigma$, the image of $\mathfrak{F}_Q$ equals $\mathfrak{L}^2(Q)^{W(\mathfrak{a}_{Q\mathrm{q}})}$.*
(b) *The image of the Fourier transform $\mathfrak{F}$ is given by the following orthogonal direct sum*

$$\mathrm{im}\,(\mathfrak{F}) = \oplus_{Q\in\mathbf{P}_\sigma}\ \mathfrak{L}^2(Q)^{W(\mathfrak{a}_{Q\mathrm{q}})}.$$

Proof: From Proposition 10.3 it follows that $\mathfrak{F}_Q$ maps into $\mathcal{L}^2(Q)^{W(\mathfrak{a}_{Q\mathfrak{q}})}$. Thus, for (a) it remains to prove the surjectivity. Let Ω be an open and closed fundamental domain in $i\mathfrak{a}_{Q\mathfrak{q}}^{*\mathrm{reg}}$ for the action of $W(\mathfrak{a}_{Q\mathfrak{q}})$. Then the map $\varphi \mapsto \varphi|_\Omega$ is a bijection from $\mathcal{L}^2(Q)^{W(\mathfrak{a}_{Q\mathfrak{q}})}$ onto $\mathcal{L}^2(Q)_\Omega$. The surjectivity now follows by application of Lemma 10.5.

Finally, assertion (b) follows from (a) combined with Corollary 10.2. □

11 H-fixed generalized vectors, final remarks

In this section we compare our results with those obtained by P. Delorme in [21]. This comparison relies heavily on the automatic continuity theorem, due to W. Casselman and N.R. Wallach, see [18], Cor. 10.5 and [29], Thm. 11.6.7. We shall therefore first recall this result. The group decomposes as $G \simeq {}^\circ G \times \exp \mathfrak{c}_\mathfrak{p}$, where, as usual, ${}^\circ G$ denotes the intersection of all subgroups $\ker|\chi|$, with χ a continuous homomorphism $G \to \mathbb{R}^*$, and where $\mathfrak{c}_\mathfrak{p} = \mathrm{center}(\mathfrak{g}) \cap \mathfrak{p}$. Accordingly, we define the function $\|\cdot\|: G \to]0, \infty[$ by

$$\|x \exp H\| = \|\mathrm{Ad}(x)\|_{\mathrm{op}}\, e^{|H|}, \tag{11.1}$$

for $x \in {}^\circ G$ and $H \in \mathfrak{c}_\mathfrak{p}$; here $\|\cdot\|_{\mathrm{op}}$ denotes the operator norm on $\mathrm{End}(\mathfrak{g})$. Let $\mathfrak{a}_\mathfrak{p}$ be a maximal abelian subspace of $\mathfrak{p}$ containing $\mathfrak{a}_\mathfrak{q}$, and let $\Sigma(\mathfrak{a}_\mathfrak{p})$ be the root system of $\mathfrak{a}_\mathfrak{p}$ in $\mathfrak{g}$. Then one readily checks that

$$\|k_1 a k_2\| = \max_{\alpha \in \Sigma(\mathfrak{a}_\mathfrak{p})} a^\alpha, \tag{11.2}$$

for $k_1, k_2 \in K$ and $a \in A_\mathfrak{p} \cap {}^\circ G$. In particular, it follows that $\|\cdot\| \geq 1$ on G. Note that it follows from (11.1) and (11.2) that

$$\|x\| = \|x^{-1}\| \qquad (x \in G). \tag{11.3}$$

We recall from [29], 11.5.1, that a smooth representation π of G in a Fréchet space V is said to be of moderate growth if for each continuous seminorm s on V there exists a continuous seminorm p_s on V and a constant $d_s \in \mathbb{R}$ such that

$$s(\pi(x)v) \leq \|x\|^{d_s} p_s(v),$$

for all $v \in V$ and $x \in G$.

Theorem 11.1 (The automatic continuity theorem) *Let (π_j, V_j), for $j = 1, 2$, be smooth Fréchet representations of G of moderate growth, such that the associated $(\mathfrak{g}, K)$-modules $(V_j)_K$ are finitely generated. Then every $(\mathfrak{g}, K)$-equivariant linear map $(V_1)_K \to (V_2)_K$ extends (uniquely) to a continuous linear G-equivariant map $V_1 \to V_2$. Moreover, the image of the extension is a closed topological summand of V_2.*

Remark 11.2 A proof of this theorem, due to W. Casselman and N.R. Wallach, is given in [18], Cor. 10.5 and in [29], Thm. 11.6.7, but for a somewhat different class of real reductive groups. In [19], § 1, it is shown that the result is valid for groups of Harish-Chandra's class.

For any function $f: G \to \mathbb{C}$ and any non-negative real number $r \geq 0$ we define

$$\|f\|_r := \sup_{x \in G} \|x\|^{-r} |f(x)|. \tag{11.4}$$

Moreover, we define $C_r(G)$ to be the space of continuous functions $f: G \to \mathbb{C}$ with $\|f\|_r < \infty$. Then $C_r(G)$, equipped with the norm $\| \cdot \|_r$, is a Banach space.

Lemma 11.3 *For every $g \in G$, both the left regular action L_g and the right regular action R_g leave the space $C_r(G)$ invariant; their restrictions to the mentioned Banach space have operator norm at most $\|g\|^r$.*

Proof: For any function $f: G \to \mathbb{C}$, we define $f^\vee: G \to \mathbb{C}$ by $f^\vee(x) = f(x^{-1})$. It follows from (11.3) and (11.4) that $f \mapsto f^\vee$ is an isometry from the Banach space $C_r(G)$ onto itself, intertwining R_g with L_g. Therefore, it suffices to prove the assertions for the left action.

In view of (11.1) it follows from the multiplicative property of the operator norm that $\|gx\| \geq \|g^{-1}\|^{-1} \|x\|$ for all $x, g \in G$. Applying this inequality to the definition of $\|L_g f\|_r$, for $f \in C_r(G)$ and $g \in G$, we see that L_g acts on $C_r(G)$ with operator norm at most $\|g^{-1}\|^r$. The lemma now follows by application of (11.3). □

We note that the left regular representation L of G in $C_r(G)$ is not continuous if G is not compact. In fact, in that case there exists a function $f \in C_r(G)$ such that $L_g f$ has no limit for $g \to e$. However, L does induce a continuous representation in a subspace that we shall now introduce.

We define $C_r^\infty(G)$ to be the space of smooth functions $f: G \to \mathbb{C}$ with $L_u f \in C_r(G)$ for all $u \in U(\mathfrak{g})$. If $F \subset U(\mathfrak{g})$ is a finite subset, we define the seminorm $\nu_{F,r}$ on $C_r^\infty(G)$ by

$$\nu_{F,r}(f) := \max_{u \in F} \|L_u f\|_r.$$

We equip $C_r^\infty(G)$ with the locally convex topology induced by the collection of seminorms $\nu_{F,r}$, for $F \subset U(\mathfrak{g})$ finite. It is readily seen that the space $C_r^\infty(G)$, thus topologized, is a Fréchet space.

Remark 11.4 The space $C_r^\infty(G)$ has been introduced in [18], p. 424, where it was denoted by $A_{\mathrm{umg},r}(G)$. In the mentioned paper it is asserted that this space is a continuous G-module of moderate growth for the left action. The following result expresses that in fact this G-module is smooth.

Proposition 11.5 *Let $r \geq 0$. The space $C_r^\infty(G)$ is left G-invariant. Moreover, the left regular representation L of G in $C_r^\infty(G)$ is a smooth Fréchet representation of moderate growth.*

Proof: In [15], § 1, the space $\mathcal{A}_N(G/H)$, for $N \in \mathbb{N}$, is defined as the analogue of $C_r^\infty(G)$, with respect to the norm function $g \mapsto \|g\sigma(g)^{-1}\|$ on G/H, in place of the norm function $\|\cdot\|$ on G. Proposition 11.5 is the analogue of [15], Lemma 1. The proof of the mentioned lemma may be transferred to the present situation with obvious modifications. □

Corollary 11.6 *Let $r \geq 0$. Every closed G-submodule of $C_r^\infty(G)$ is a smooth Fréchet module of moderate growth.*

Proof: Immediate. See also [29], Lemma 11.5.2. □

Proposition 11.7 *Let (π, V) be a smooth Fréchet representation of G of moderate growth, such that V_K is finitely generated. Let $r \geq 0$ and let $T: V_K \to C_r^\infty(G)$ be a $(\mathfrak{g}, K)$-equivariant linear map. The map T has a unique extension to a continuous linear G-equivariant map $V \to C_r^\infty(G)$. The image of this extension is closed.*

Proof: Let W be the closure of the image of T in $C_r^\infty(G)$. Then W is a closed G-submodule of $C_r^\infty(G)$, hence a smooth Fréchet module of moderate growth. Moreover, $W_K = T(V_K)$ is finitely generated. By Theorem 11.1, T has a unique extension to a continuous linear G-equivariant map $\tilde{T}: V \to W$. The image of this extension is closed and contains a dense subspace of W, hence equals W. □

Lemma 11.8 *Let $r \geq 0$. Then the space $C_r^\infty(G)$ is right G-invariant. Moreover, if $y \in G$, then the right regular action R_y restricts to a continuous linear operator of $C_r^\infty(G)$.*

Proof: It follows from Lemma 11.3 that R_y is a continuous linear endomorphism of $C_r(G)$ with operator norm at most $\|y\|^r$. Since the action of R_y on $C^\infty(G)$ commutes with that of L_u, for every $u \in U(\mathfrak{g})$, it readily follows that R_y leaves the space $C_r^\infty(G)$ invariant and restricts to a continuous linear endomorphism of it. □

Let $r \geq 0$. We define

$$C_r^\infty(\mathrm{X}) := C_r^\infty(G) \cap C(G/H),$$

the space of right H-invariant functions in $C_r^\infty(G)$.

Lemma 11.9 *Let $r \in \mathbb{R}$. The space $C_r^\infty(\mathrm{X})$ is a closed G-submodule of $C_r^\infty(G)$. In particular, it is a smooth Fréchet G-module of moderate growth.*

Proof: For every $h \in H$, the map $R_h - I$ restricts to a continuous linear operator of $C_r^\infty(G)$, by Lemma 11.8. The space $C_r^\infty(\mathrm{X})$ equals the intersection in $C_r^\infty(G)$ of the kernels of the operators $R_h - I$, for $h \in H$. Therefore, $C_r^\infty(\mathrm{X})$ is closed. The remaining assertion follows by application of Corollary 11.6. □

Remark 11.10 It follows from [4], Cor. 12.2, that the space $C_r^\infty(\mathrm{X})$ equals the space $\mathcal{A}_N(G/H)$, with $N = 2r$, defined in [15], § 1; the definition in the last mentioned paper is given for $N \in \mathbb{N}$, but makes sense for arbitrary real $N \geq 0$. Accordingly, Lemma 11.9 is due to [15]; see loc. cit. Lemma 1.

If V is a locally convex space, we denote its continuous linear dual by V'. Unless otherwise specified, we equip it with the strong dual topology.

Corollary 11.11 *Let (π, V) be a smooth Fréchet representation of G of moderate growth, such that V_K is finitely generated. Let $r \in \mathbb{R}$ and let $T: V_K \to C_r^\infty(\mathrm{X})$ be a $(\mathfrak{g}, K)$-equivariant linear map.*

(a) *The map T has a unique extension to a continuous linear G-equivariant map $\tilde{T}: V \to C_r^\infty(\mathrm{X})$.*
(b) *The linear functional $\mathrm{ev}_e \circ T: v \mapsto Tv(e)$ has a unique extension to a continuous linear functional $\eta_T \in V'$.*
(c) *The functional η_T is H-invariant and $\tilde{T}$ may be represented as the generalized matrix coefficient map given by*

$$T(v)(x) = \eta_T(\pi(x)^{-1}v), \qquad (x \in G/H).$$

Proof: From Proposition 11.7 it follows that T has a unique extension to a continuous linear G-equivariant map $\tilde{T}: V \to C_r^\infty(G)$. The image of $\tilde{T}$ is a closed subspace W of $C_r^\infty(G)$ which contains the image of T as a dense subspace. In view of Lemma 11.9 it follows that $W \subset C_r^\infty(\mathrm{X})$. The extended functional is given by $\eta_T = \mathrm{ev}_e \circ \tilde{T}: v \mapsto \tilde{T}v(e)$. The assertions of (c) readily follow by G-equivariance. □

Using the above result we shall be able to express our Eisenstein integrals as matrix coefficients of principal series representations. As a preparation we need to relate the function $\| \cdot \|$, defined in (11.1), to the $G = KA_{\mathrm{q}}H$-decomposition. Following [12], Eqn. (10.1), we define the distance function $l_{\mathrm{X}}: G \to [0, \infty[$ by

$$l_{\mathrm{X}}(kah) = |\log a|,$$

for $k \in K$, $a \in A_{\mathrm{q}}$ and $h \in H$.

Lemma 11.12 *There exists a constant $s > 0$ such that*

$$e^{l_{\mathrm{X}}(x)} \leq \|x\|^s, \qquad (x \in G).$$

Proof: One readily sees that it suffices to prove this in case $G = {}^\circ G$. Moreover, since the functions of x on both sides of the equality are left K-invariant, we may reduce to the case that G is connected and semisimple, with finite center. From [6], Lemma 14.4, we deduce, using the equality $\|x\| = \|x^{-1}\|$, that $\|a\| \leq \|ah\|$ for all $a \in A_{\mathrm{q}}$ and $h \in H$. Hence, by the $G = KA_{\mathrm{q}}H$ decomposition, it suffices to prove the estimate

$$e^{|\log a|} \leq \|a\|^s, \qquad (a \in A_{\mathrm{q}}), \tag{11.5}$$

for some $s > 0$ independent of a. Let m be the minimal value of the continuous function $\max\{\alpha \mid \alpha \in \Sigma(\mathfrak{a}_\mathfrak{p})\}$ on the unit sphere in $\mathfrak{a}_\mathfrak{q}$. Then $m > 0$. Using (11.2) we see that the estimate (11.5) holds for $s \geq m^{-1}$. □

Let $Q \in \mathcal{P}_\sigma$ and $\xi \in \mathrm{X}^\wedge_{Q,*,ds}$ be fixed throughout the rest of this section.

Lemma 11.13 *Let $\vartheta \subset \widehat{K}$ be a finite subset, let $\psi \in \mathcal{A}_{2,Q}(\tau_\vartheta)_\xi$ and let $\nu_0 \in \mathfrak{a}^*_{Q\mathfrak{q}\mathbb{C}}$ be a regular point for the Eisenstein integral $E^\circ(Q : \psi : \nu)$. There exist an open neighborhood U of ν_0 and a constant $r > 0$ such that $\nu \mapsto E^\circ(Q : \psi : \nu)$ is a bounded function on U with values in $C^\infty_r(\mathrm{X}) \otimes \mathbf{V}_\vartheta$.*

Proof: It follows from [12], Prop. 13.14, combined with Lemma 11.12, that there exist an open neighborhood Ω of ν_0 and a polynomial function $p \in \Pi_{\Sigma_r(Q)}(\mathfrak{a}^*_{Q\mathfrak{q}})$, such that the function $\nu \mapsto p(\nu)E^\circ(Q : \nu)$ is holomorphic on Ω as a function with values in $C^\infty_r(\mathrm{X}) \otimes \mathrm{Hom}(\mathcal{A}_{2,Q}(\tau_\vartheta), \mathbf{V}_\vartheta)$ and such that for every continuous seminorm μ on the latter tensor product space, the function $\nu \mapsto \mu(p(\nu)E^\circ(Q : \nu))$ is bounded on Ω.

Select an open neighborhood U of ν_0 with compact closure contained in Ω such that $E^\circ(Q : \psi : \cdot)$ is holomorphic on an open neighborhood of $\overline{U}$. Then it follows by a straightforward application of Cauchy's integral formula in the variable ν, see, e.g., [4], proof of Lemma 6.1, that for every continuous seminorm μ' on $C^\infty_r(\mathrm{X}) \otimes \mathbf{V}_\vartheta$ the function $\nu \mapsto \mu'(E^\circ(Q : \psi : \nu))$ is bounded on U. □

Lemma 11.14 *Let $\nu \in \mathfrak{a}^*_{Q\mathfrak{q}\mathbb{C}}$. The representation $\pi_{Q,\xi,\nu}$ of G in $C^\infty(K : \xi)$ is a smooth Fréchet representation of moderate growth. Moreover, the associated $(\mathfrak{g}, K)$-module $C^\infty(K : \xi)_K$ is finitely generated.*

Proof: It follows from Remark 4.2 that $V := C^\infty(K : \xi)$, equipped with $\pi_{Q,\xi,\nu}$, is the space of C^∞-vectors for the Hilbert representation $\mathrm{Ind}_Q^G(\xi \otimes \nu \otimes 1)$. It now follows from [29], Lemma 11.5.1, that V is a smooth Fréchet G-module of moderate growth. The last assertion is well known, see also Proposition 5.1 for a stronger assertion. □

Proposition 11.15 *Let $\nu \in \mathfrak{a}^*_{Q\mathfrak{q}\mathbb{C}} \setminus \cup\mathcal{H}(Q, \xi)$.*

(a) *There exists a constant $r \in \mathbb{R}$ such that $J_{Q,\xi,\nu}$ maps $\bar{V}(Q, \xi) \otimes C^\infty(K : \xi)_K$ into the space $C^\infty_r(\mathrm{X})$.*
(b) *Let $r \in \mathbb{R}$ be a constant as in (a). The map $J_{Q,\xi,\nu}$ has a unique extension to a continuous linear map from $\bar{V}(Q, \xi) \otimes C^\infty(K : \xi)$ into $C^\infty_r(\mathrm{X})$. The extension intertwines the G-representations $I \otimes \pi_{Q,\xi,-\nu}$ and L.*

Proof: Fix ν as above. By Lemma 11.14, there exists a finite subset $\vartheta \subset \widehat{K}$ such that $C^\infty(K : \xi)_\vartheta$ generates $C^\infty(K : \xi)_K$ as the $(\mathfrak{g}, K)$-module associated with $\pi_{Q,\xi,-\nu}$. Let $r \in \mathbb{R}$ be associated with ϑ as in Lemma 11.13. Then it follows from (4.20) that $J_{Q,\xi,\nu}$ maps $\bar{V}(Q, \xi) \otimes C^\infty(K : \xi)_\vartheta$ into $C^\infty_r(\mathrm{X})$. The map $J_{Q,\xi,\nu}$ is $(\mathfrak{g}, K)$-equivariant, by Theorem 4.6. Since $C^\infty(K : \xi)_\vartheta$

generates $C^\infty(K:\xi)_K$, whereas $C^\infty_r(\mathrm{X})$ is $(\mathfrak{g},K)$-invariant, assertion (a) follows.

Assume that r is a constant as in (a). Then it follows from Theorem 4.6 that the map $J_{Q,\xi,\nu}$ is $(\mathfrak{g},K)$-equivariant. In view of Lemma 11.14 and assertion (a), we may apply Corollary 11.11 with $T = J_{Q,\xi,\nu}$. Assertion (b) follows. □

If $\nu \in \mathfrak{a}^*_{Q\mathbb{C}} \setminus \cup\mathcal{H}(Q,\xi)$, we denote the continuous linear extension of $J_{Q,\xi,\nu}$ by the same symbol. We denote the conjugate of the topological linear dual of $C^\infty(K:\xi)$ by $C^{-\infty}(K:\xi)$. The G-representation on the latter space, induced by dualization of $\pi_{\xi,\nu} = \pi_{Q,\xi,\nu}$, is denoted by $\pi^{-\infty}_{\xi,\nu} = \pi^{-\infty}_{Q,\xi,\nu}$.

The sesquilinear pairing $C^\infty(K:\xi)\times C^\infty(K:\xi)\to\mathbb{C}$, given by (4.5) induces a continuous linear embedding $C^\infty(K:\xi)\hookrightarrow C^{-\infty}(K:\xi)$, intertwining the representations $\pi_{\xi,-\bar\nu}$ and $\pi^{-\infty}_{\xi,\nu}$. The latter may therefore be viewed as the continuous linear extension of $\pi_{\xi,-\bar\nu}$. Accordingly, we shall sometimes use the notation $\pi_{\xi,-\bar\nu}$ for the representation $\pi^{-\infty}_{\xi,\nu}$.

We denote by $V(Q,\xi)$ the conjugate space of $\bar V(Q,\xi)$, and define the linear map $j^\circ(Q:\xi:\bar\nu)\colon V(Q,\xi)\to C^{-\infty}(K:\xi)$ by

$$\langle\varphi\,|\,j^\circ(Q:\xi:\bar\nu)(\eta)\rangle = J_{Q,\xi,\nu}(\eta\otimes\varphi)(e). \tag{11.6}$$

Then by Proposition 11.15 and Corollary 11.11, the image of $j^\circ(Q:\xi:\bar\nu)$ is contained in the subspace of $C^{-\infty}(K:\xi)$ consisting of H-invariants for the representation $\pi_{\xi,\bar\nu}$; we agree to denote this subspace by $C^{-\infty}(Q:\xi:\bar\nu)^H$.

We may now represent the Eisenstein integral as a matrix coefficient. The following formula generalizes the similar formula for Q minimal, see [7], Eqn. (53).

Lemma 11.16 *Let $\nu\in\mathfrak{a}^*_{Q\mathbb{C}}\setminus\cup\mathcal{H}(Q,\xi)$, let $\vartheta\subset\widehat{K}$ be a finite subset and let $T=\eta\otimes\varphi\in\bar V(Q,\xi)\otimes C^\infty(K:\xi)_\vartheta$. Then*

$$E^\circ_\vartheta(Q:\psi_T:\nu)(x)(k) = \langle\varphi\,|\,\pi_{Q,\xi,\bar\nu}(kx)j^\circ(Q:\xi:\bar\nu)\eta\rangle \quad (x\in\mathrm{X},\ k\in K).$$

Proof: This follows from (4.20) and (11.6), by application of Corollary 11.11 (c). □

To identify our Eisenstein integral with the one introduced by P. Delorme in [19], we recall some results from [16], § 2.4.

For each $v\in {}^Q\mathcal{W}$, we denote by $\mathcal{V}(Q,\xi,v)$ the space of $M_Q\cap vHv^{-1}$-fixed elements in $\mathcal{H}^{-\infty}_\xi$, the conjugate of the topological linear dual of $\mathcal{H}^\infty_\xi$. The space $\mathcal{V}(Q,\xi,v)$ is finite dimensional by [1], Lemma 3.3. We introduce the formal direct sum

$$\mathcal{V}(Q,\xi) := \oplus_{v\in{}^Q\mathcal{W}}\ \mathcal{V}(Q,\xi,v).$$

If $u\in C^{-\infty}(Q:\xi:\nu)^H$, then on an open neighborhood of any $v\in{}^Q\mathcal{W}$ in K, the functional u may be represented by a unique continuous function with values in $\mathcal{H}^{-\infty}_\xi$, via the sesquilinear pairing (4.5). Its value $\mathrm{ev}_v(u)$ in v

is therefore a well defined element of $\mathcal{V}(Q,\xi,v)$. See [16], § 3.3, for details. The direct sum of the maps ev_v, for $v \in {}^Q\mathcal{W}$, is denoted by

$$\mathrm{ev} = \oplus_{v\in {}^Q\mathcal{W}}\ \mathrm{ev}_v\colon\quad C^{-\infty}(Q:\xi:\nu)^H \to \mathcal{V}(Q,\xi).$$

We have the following result, due to [3] for minimal Q and to [16] in general.

Theorem 11.17 *There exists a unique meromorphic function $j(Q,\xi,\cdot)$ on $\mathfrak{a}^*_{Q\mathbb{C}}$ with values in* $\mathrm{Hom}(\mathcal{V}(Q,\xi), C^{-\infty}(K:\xi))$ *such that the following two conditions are fulfilled.*

(a) *For regular values of ν, the image of $j(Q:\xi:\nu)$ is contained in $C^{-\infty}(Q:\xi:\nu)^H$.*

(b) *For regular values of ν, we have* $\mathrm{ev}\circ j(Q:\xi:\nu) = I_{\mathcal{V}(Q,\xi)}$.

There exists a locally finite collection $\mathcal{H} = \mathcal{H}(j,Q,\xi)$ of hyperplanes in $\mathfrak{a}^*_{Q\mathbb{C}}$ such that each $\nu \in \mathfrak{a}^*_{Q\mathbb{C}} \setminus \cup\mathcal{H}$ is a regular value for $j(Q:\xi:\cdot)$ and the associated map $j(Q:\xi:\nu)$ is surjective from $\mathcal{V}(Q,\xi)$ onto $C^{-\infty}(Q:\xi:\nu)^H$.

Finally, each $\nu \in \mathfrak{a}^*_{Q\mathbb{C}}$ with $\mathrm{Re}\,\nu + \rho_Q$ strictly $\Sigma_r(Q)$-anti-dominant is a regular value for $j(Q:\xi:\cdot)$. Moreover, for such ν and every $\eta \in \mathcal{V}(Q,\xi)$, the element $j(Q:\xi:\nu)\eta \in C^{-\infty}(K:\xi)$ is representable by a continuous function $u\colon K \to \mathcal{H}^{-\infty}_\xi$, in the sense that

$$\langle \varphi \mid j(Q:\xi:\nu)\eta\rangle = \int_K \langle \varphi(k) \mid u(k)\rangle\, dk, \qquad (\varphi \in C^\infty(K:\xi)).$$

Proof: This follows from [16], Prop. 2, Thm. 1 and Thm. 3. □

The Eisenstein integrals of Delorme are built in terms of matrix coefficients coming from a subspace $\mathcal{V}_{ds}(Q,\xi)$ of $\mathcal{V}(Q,\xi)$, which is defined as follows, see [19], § 8.3. Let $v \in {}^Q\mathcal{W}$. An element $\eta \in \mathcal{V}(Q,\xi,v)$ naturally determines the M_Q-equivariant embedding $\iota_\eta\colon \mathcal{H}^\infty_\xi \to C^\infty(M_Q/M_Q\cap vHv^{-1})$, given by

$$\iota_\eta(u)(m) = \langle \xi(m)^{-1}u \mid \eta\rangle, \qquad (m \in M_Q).$$

We denote by $\mathcal{V}_{ds}(Q,\xi,v)$ the subspace of $\eta \in \mathcal{V}(Q,\xi)$ with the property that ι_η maps into $L^2(M_Q/M_Q\cap vHv^{-1})^\infty$. Note that for such η the map ι_η extends to a continuous linear map $\mathcal{H}_\xi \to L^2(\mathrm{X}_{Q,v})$; see [17], Lemma 1. Moreover, the map $\eta \mapsto \iota_\eta$ defines a linear isomorphism from $\mathcal{V}_{ds}(Q,\xi,v)$ onto $V(Q,\xi,v)$, via which we shall identify.

We define the subspace $\mathcal{V}_{ds}(Q,\xi)$ of $\mathcal{V}(Q,\xi)$ as the direct sum of the spaces $\mathcal{V}_{ds}(Q,\xi,v)$, for $v \in {}^Q\mathcal{W}$. Via the direct sum of the above isomorphisms, we obtain the natural isomorphism

$$\mathcal{V}_{ds}(Q,\xi) \simeq V(Q,\xi).$$

Accordingly, the map j° introduced in (11.6) may be viewed as a linear map

$$j^\circ(Q:\xi:\nu)\colon\ \mathcal{V}_{ds}(Q,\xi) \to C^{-\infty}(Q:\xi:\nu)^H,$$

defined for $\nu \in \mathfrak{a}^*_{Q\mathbb{C}} \setminus \cup\mathcal{H}(Q,\xi)$.

To relate this map with the map $j(Q:\xi:\nu)$ of Theorem 11.17 we need standard intertwining operators. We recall from [29] and [16] that for a parabolic subgroup $P \in \mathcal{P}_\sigma$ with split component equal to A_Q, the standard intertwining operator $A(Q:P:\xi:\nu)$ between the representations $\pi_{P,\xi,\nu}$ and $\pi_{Q,\xi,\nu}$ on $C^\infty(K:\xi)$ is given by an absolutely convergent integral for $\nu \in \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$ with $\langle \mathrm{Re}\,\nu - \rho_Q\,,\,\alpha\rangle > 0$ for every $\alpha \in \Sigma_r(P) \cap \Sigma_r(\bar{Q})$, and allows a meromorphic continuation in ν. Its adjoint is a continuous linear endomorphism of $C^{-\infty}(K:\xi)$, intertwining the representations $\pi^{-\infty}_{Q,\xi,\nu}$ and $\pi^{-\infty}_{P,\xi,\nu}$. It extends the standard intertwining operator $A(P:Q:\xi:-\bar{\nu})$, and is therefore denoted by the same symbol. Thus,

$$A(Q:P:\xi:\nu)^* = A(P:Q:\xi:-\bar{\nu}). \tag{11.7}$$

We also recall that

$$A(P:Q:\xi:\nu)\circ A(Q:P:\xi:\nu) = \eta(Q:P:\xi:\nu)\, I_{C^\infty(K:\xi)}$$

with $\eta(Q:P:\xi:\cdot)$ a non-zero scalar meromorphic function on $\mathfrak{a}^*_{P\mathrm{q}\mathbb{C}} = \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$. In particular, it follows that the standard intertwining operator is invertible for ν in an open dense subset of $\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$.

Lemma 11.18 *Let $\nu \in \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$ be such that $\mathrm{Re}\,\nu - \rho_Q$ is strictly $\Sigma_r(Q)$-dominant. Then, for every $\eta \in \mathcal{V}(Q,\xi)$ and $\varphi \in C^\infty(K:\xi)_K$, and for each $v \in {}^Q\mathcal{W}$, all $m \in M_Q$ and all $X \in \mathfrak{a}^+_{Q\mathrm{q}}$,*

$$\lim_{t\to\infty} a_t^{-\nu+\rho_Q}\ \langle \varphi \,|\, \pi_{\bar{Q},\xi,\bar{\nu}}(ma_t\, v)\, j(\bar{Q}:\xi:\bar{\nu})\eta\rangle$$
$$= \langle A(Q:\bar{Q}:\xi:-\nu)\varphi(e) \,|\, \xi(m)\eta_v\rangle,$$

where $a_t = \exp tX$.

Proof: The result is equivalent to Lemma 16 in [19]. We refer to the proof given there. □

Theorem 11.19 *Let $\eta \in \mathcal{V}_{ds}(Q,\xi)$. Then $j^\circ(Q:\xi:\cdot)\eta$ is holomorphic as a function on $\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}} \setminus \cup\mathcal{H}(Q,\xi)$ with values in $C^{-\infty}(K:\xi)$. Moreover,*

$$j^\circ(Q:\xi:\nu)\eta = A(\bar{Q}:Q:\xi:\nu)^{-1} j(\bar{Q}:\xi:\nu)\eta,$$

as an identity of meromorphic $C^{-\infty}(K:\xi)$-valued functions in $\nu \in \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}} \setminus \cup\mathcal{H}(Q,\xi)$. In particular, $j^\circ(Q:\xi:\cdot)\eta$ extends to a meromorphic $C^{-\infty}(K:\xi)$-valued function on $\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$.

For the proof of this result we need the following lemma.

Lemma 11.20 *Let $\vartheta \subset \widehat{K}$ be a finite subset. There exists an open dense subset Ω of the set of points $\nu \in \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$ with $\mathrm{Re}\,\nu$ strictly $\Sigma_r(Q)$-dominant, such that the following holds. Let $\psi \in \mathcal{A}_{2,Q}(\tau_\vartheta)$, $v \in {}^Q\mathcal{W}$, $m \in \mathrm{X}_{Q,v,+}$, $X \in \mathfrak{a}^+_{Q\mathrm{q}}$ and put $a_t = \exp tX$, $(t \in \mathbb{R})$. Then, for every $\nu \in \Omega$,*

$$\lim_{t\to\infty} a_t^{-\nu+\rho_Q}\, E^\circ(Q:\psi:\nu)(ma_t v) = (\psi)_v(m).$$

Proof: Let ω be the set of regular points for $E^\circ(Q:\cdot)$, and ω_+ the subset of $\nu \in \omega$ with $\mathrm{Re}\,\nu$ strictly $\Sigma_r(Q)$-dominant.

Fix a minimal parabolic group P from $\mathcal{P}_\sigma$, contained in Q. Then, by [12], Prop. 13.15, the family $f: (\nu, x) \mapsto E^\circ(Q:\nu:x)\psi$ belongs to $\mathcal{E}^{\mathrm{hyp}}_Q(\mathrm{X}:\tau)$. Moreover, for each $u \in N_K(\mathfrak{a}_\mathrm{q})$, the set of exponents $\mathrm{Exp}\,(P, u \mid f_\nu)$ is contained in the collection $W^{P|Q}(\nu + \Lambda(P|Q)) - \rho_P - \mathbb{N}\Delta(P)$, for $\nu \in \omega$.

Fix $\nu \in \omega_+$ and let ξ be an exponent in $\mathrm{Exp}\,(Q, v \mid f_\nu)$. Then it follows by application of [11], Thm. 3.5, that $\xi = w(\nu+\Lambda)|_{\mathfrak{a}_{Q\mathrm{q}}} - \rho_Q - \mu$, for certain $w \in W^{P|Q}$, $\Lambda \in \Lambda(P|Q)$ and $\mu \in \mathbb{N}\Delta_r(Q)$. It follows from the definitions preceding [12], Prop. 13.15, that $w\Lambda \in -\mathbb{R}^+\Delta(P)$. Hence $\mathrm{Re}\,\xi(X) + \rho_Q(X) \le w\mathrm{Re}\,\nu(X)$, with equality if and only if $w\Lambda|_{\mathfrak{a}_{Q\mathrm{q}}} = 0$ and $\mu = 0$. Now $\mathrm{Re}\,\nu$ is strictly $\Sigma_r(Q)$-dominant and $X \in \mathfrak{a}^+_{Q\mathrm{q}}$. Hence, by a well known result on root systems, $\mathrm{Re}\,\nu(X) \ge \mathrm{Re}\,s\nu(X)$, for each $s \in W$, with equality if and only if s centralizes $\mathfrak{a}_{Q\mathrm{q}}$. Since $W^{P|Q} \cap W_Q = \{e\}$, we conclude that

$$\mathrm{Re}\,\xi(X) < (\mathrm{Re}\,\nu - \rho_Q)(X)$$

for every exponent $\xi \in \mathrm{Exp}\,(Q, v \mid f_\nu)$, different from $\nu - \rho_Q$. It follows that

$$\lim_{t\to\infty} a_t^{-\nu+\rho_Q} f_\nu(ma_t v) = \lim_{t\to\infty} q_{\nu-\rho_Q}(Q, v \mid f_\nu, tX)(m), \tag{11.8}$$

for every $\nu \in \omega_+$ for which the limit on the right-hand side exists. It follows from [12], Def. 13.7 and Prop. 13.6, that there exists a non-empty open subset Ω_0 of $\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$ such that

$$q_{\nu-\rho_Q}(Q, v \mid f_\nu, X)(m) = \psi_v(m) \tag{11.9}$$

for all $\nu \in \Omega_0$, $v \in {}^Q\mathcal{W}$, $m \in \mathrm{X}_{Q,v,+}$ and $X \in \mathfrak{a}_{Q\mathrm{q}}$. On the other hand, by [11], Thm. 7.7, there exists an open dense subset Ω_1 of $\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$ such that the expression on the left-hand side of (11.9) depends holomorphically on $\nu \in \Omega_1$, for all v, m, X as above. By analytic continuation it follows that (11.9) is valid for $\nu \in \Omega_1$. This implies that the limit on the right-hand side of (11.8) has the value $\psi_v(m)$, for every $\nu \in \omega_+ \cap \Omega_0$. □

Proof of Theorem 11.19: Fix $\varphi \in C^\infty(K:\xi)_K$ with $\varphi(e) \neq 0$ and put $T = \eta \otimes \varphi$. Select a finite subset $\vartheta \subset \widehat{K}$ such that $\varphi \in C^\infty(K:\xi)_\vartheta$. Then, in the notation of (4.13), $\psi_T \in \mathcal{A}_{2,Q}(\tau_\vartheta)$. Fix $v \in {}^Q\mathcal{W}$. Then the preimage $M_{Q,v,+}$ of $\mathrm{X}_{Q,v,+}$ under the canonical map $M_Q \to \mathrm{X}_{Q,v}$ is open dense in M_Q. Fix $m \in M_Q$ and $X \in \mathfrak{a}^+_{Q\mathrm{q}}$. We agree to write $a_t = \exp tX$. Let $\mathcal{A}$ be the open dense subset of $\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}} \setminus \cup\mathcal{H}(Q,\xi)$ consisting of points ν where both intertwining operators $\nu \mapsto A(Q:\bar{Q}:\xi:-\nu)^\pm$ are regular. By Lemma 11.16 we may write, for $\nu \in \mathcal{A}$,

$$\begin{aligned} &E^\circ_\vartheta(Q:\psi_T:\nu)(ma_t v)(e) \\ &\quad = \langle \varphi \mid \pi_{Q,\xi,\bar{v}}(ma_t v) j^\circ(Q:\xi:\bar{v})\rangle \\ &\quad = \langle A(Q:\bar{Q}:\xi:-\nu)^{-1}\varphi \mid \pi_{\bar{Q},\xi,\bar{v}}(ma_t v) A(\bar{Q}:Q:\xi:\bar{v}) j^\circ(Q:\xi:\bar{v})\eta\rangle. \end{aligned}$$

Replacing $\mathcal{A}$ if necessary, we may in addition assume that the conjugate $\bar{\mathcal{A}}$ of $\mathcal{A}$ has empty intersection with the set $\cup\mathcal{H}$, where $\mathcal{H} = \mathcal{H}(j, \bar{Q}, \xi)$ is as in Theorem 11.17, with $\bar{Q}$ in place of Q. By the mentioned theorem it then follows, for $\nu \in \mathcal{A}$, that

$$A(\bar{Q} : Q : \xi : \bar{\nu}) j^\circ(Q : \xi : \bar{\nu})\eta = j(\bar{Q} : \xi : \bar{\nu})\eta(\bar{\nu}), \tag{11.10}$$

for $\eta(\bar{\nu}) \in \mathcal{V}(\bar{Q}, \xi)$ given by $\eta(\bar{\nu}) = \mathrm{ev} \circ A(\bar{Q} : Q : \xi : \bar{\nu}) j^\circ(Q : \xi : \bar{\nu})\eta$. Using Lemma 11.18 we now conclude that, for $\nu \in \mathcal{A}$ with $\mathrm{Re}\,\nu - \rho_Q$ strictly $\Sigma_r(Q)$-dominant,

$$\begin{aligned} \lim_{t\to\infty} a_t^{-\nu+\rho_Q} E^\circ_\vartheta(Q : \psi_T : \nu)(ma_t v)(e) \\ = \langle \varphi(e) \,|\, \xi(m)\mathrm{ev}_v \circ A(\bar{Q} : Q : \xi : \bar{\nu}) j^\circ(Q : \xi : \bar{\nu})\eta \rangle \\ = \langle \varphi(e) \,|\, \xi(m)\eta(\bar{\nu})_v \rangle. \end{aligned} \tag{11.11}$$

In particular, this holds for ν contained in the non-empty open set $\mathcal{A} \cap \Omega$, with Ω as in Lemma 11.20. For such ν it follows by the mentioned lemma that the limit in (11.11) also equals $(\psi_T)_v(m) = \langle \varphi(e) \,|\, \xi(m)\eta_v \rangle$. We deduce that, for $\nu \in \bar{\mathcal{A}} \cap \bar{\Omega}$, where the bar denotes conjugation,

$$\langle \varphi(e) \,|\, \xi(m)\, \eta(\nu)_v \rangle = \langle \varphi(e) \,|\, \xi(m)\eta_v \rangle, \tag{11.12}$$

for all $m \in M_{Q,v,+}$. By continuity and density it follows that the identity (11.12) holds for all $m \in M_Q$. Since $\varphi(e) \in \mathcal{H}_\xi^\infty \setminus \{0\}$, it follows by irreducibility of the G-module $\mathcal{H}_\xi^\infty$ that $\eta(\nu)_v = \eta_v$, for all $\nu \in \bar{\mathcal{A}} \cap \bar{\Omega}$. This identity holds for every $v \in {}^Q\mathcal{W}$, since the sets $\mathcal{A}$ and Ω are independent of the element $v \in {}^Q\mathcal{W}$. Combining this with (11.10) we deduce that, for every $\nu \in \bar{\mathcal{A}} \cap \bar{\Omega}$,

$$j^\circ(Q : \xi : \nu)\eta = A(\bar{Q} : Q : \xi : \nu)^{-1} j(\bar{Q} : \xi : \nu)\eta. \tag{11.13}$$

Let $f(\nu)$ denote the expression on the left-hand side and $g(\nu)$ that on the right-hand side of the above equation. Then g is a meromorphic $C^{-\infty}(K : \xi)$-valued function on $\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$, by Theorem 11.17 and meromorphy of the intertwining operator. If $\varphi \in C^\infty(K : \xi)_K$, then $\nu \mapsto \langle f(\nu) \,|\, \varphi \rangle$ is a holomorphic function of $\nu \in \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}} \setminus \cup\mathcal{H}(Q, \xi)$, by Lemma 11.16. On the other hand, $\nu \mapsto \langle g(\nu) \,|\, \varphi \rangle$ is a meromorphic function on $\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$. By analytic continuation we deduce that

$$\langle f(\nu) \,|\, \varphi \rangle = \langle g(\nu) \,|\, \varphi \rangle, \tag{11.14}$$

as an identity of meromorphic functions in the variable $\nu \in \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}} \setminus \cup\mathcal{H}(Q, \xi)$. From the holomorphy of the function on the left-hand side it follows that the function on the right-hand side is actually regular on $\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}} \setminus \cup\mathcal{H}(Q, \xi)$,

for every $\varphi \in C^{-\infty}(K:\xi)_K$. The latter space is dense in $C^\infty(K:\xi)$ and $\nu \mapsto g(\nu)$ is a meromorphic $C^{-\infty}(K:\xi)$-valued function. It follows that g is regular on $\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}} \setminus \cup\mathcal{H}(Q,\xi)$. It now follows from (11.14) that the element $f(\nu) \in C^{-\infty}(K:\xi)$ equals $g(\nu)$, for every $\nu \in \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}} \setminus \cup\mathcal{H}(Q,\xi)$. This implies all assertions of the theorem. □

It follows from the above result that the distribution vector $j^\circ(Q:\xi:\nu)\eta$, defined for $\eta \in \mathcal{V}_{ds}(Q,\xi)$, coincides with the similarly denoted distribution vector defined in [7], (3.13).

Corollary 11.21 *Let (τ, V_τ) be a finite dimensional unitary representation of K, let $P \in \mathcal{P}_\sigma$ and $\psi \in \mathcal{A}_{2,P}(\tau)$. Then the Eisenstein integral $E^\circ(P:\psi:\lambda)$, for $\lambda \in \mathfrak{a}^*_{P\mathrm{q}\mathbb{C}}$, coincides with the normalized Eisenstein integral $E^\circ(P,\psi,-\lambda)$ defined in [17], § 5.1.*

Proof: By the functorial property of Lemma 4.5, which is satisfied by both Eisenstein integrals, it suffices to prove the result for $\tau = \tau_\vartheta$, with $\vartheta \subset \widehat{K}$ a finite subset. By linearity it suffices to prove the assertion for $\psi = \psi_{\eta\otimes f}$, where $\xi \in \mathrm{X}^\wedge_{P,*,ds}$, $\eta \in \mathcal{V}_{ds}(P,\xi)$ and $f \in C^\infty(K:\xi)_\vartheta$. The associated normalized Eisenstein integral is denoted $E^\circ(P,\psi,\nu)$ in [17], § 5.1. It is represented as a matrix coefficient in [17], Prop. 4. This representation coincides with the one given in Lemma 11.16. □

It follows from the equality of the normalized Eisenstein integrals stated above, that the Plancherel theorems formulated in [12], § 23, and Sect. 10, coincide with the ones of P. Delorme formulated in [21], Sects. 3.3 and 3.4. However, the chosen normalizations of measures are different, resulting in different constants. We shall finish this section by relating the various constants. The normalization of measures for the present paper is described in [12], § 5. The normalization given in [21], § 0, follows essentially the same conventions of interdependence, with one crucial difference. A choice of invariant measure dx for X determines the same choice of Haar measure da for A_q in both papers. In our paper we fix the Lebesgue measure $d\lambda$ on $i\mathfrak{a}^*_\mathrm{q}$ that makes the Euclidean Fourier transform an isometry from $L^2(A_\mathrm{q}, da)$ onto $L^2(i\mathfrak{a}^*_\mathrm{q}, |W|\,d\lambda)$. On the other hand, in [21], § 0, the convention is to fix the measure $d\lambda' := |W|\,d\lambda$ instead.

If $Q \in \mathcal{P}_\sigma$, the same convention applies to the normalizations of invariant measures $dx_{Q,v}$ on $\mathrm{X}_{Q,v}$, for $v \in {}^Q\mathcal{W}$, versus a choice of normalization of da_Q on the group ${}^*A_{Q\mathrm{q}}$, which is 'the A_q of $(M_Q, M_Q \cap vHv^{-1})$.' This determines a normalization $d\lambda_Q$ of Lebesgue measure on $i\,{}^*\mathfrak{a}^*_{Q\mathrm{q}}$. The corresponding measure of [21] is given by $d\lambda'_Q = |W_Q|\,d\lambda_Q$. In both papers, one chooses the measure on $i\mathfrak{a}^*_{Q\mathrm{q}}$ to be the quotient of the chosen measures on $i\mathfrak{a}^*_\mathrm{q}$ and $i^*\mathfrak{a}^*_{Q\mathrm{q}}$. This results in a choice of Lebesgue measure $d\mu_Q$ on $i\mathfrak{a}^*_{Q\mathrm{q}}$ in the present paper. The similar measure $d\mu'_Q$ in [21] is then given by

$d\mu'_Q = [W\colon W_Q]\, d\mu_Q$. For the constants in the Plancherel formula, see e.g. [12], Thm. 23.1 (d), this means that $[W\colon W_Q^*]$ should be replaced by

$$[W\colon W_Q]^{-1}\big[W\colon W_Q^*\big] = \big[W_Q^*\colon W_Q\big]^{-1} = |W(\mathfrak{a}_{Q\mathfrak{q}})|^{-1}.$$

The latter is indeed the constant occurring in, e.g., [21], Thm. 3 (iii).

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