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ASYMPTOTIC EXPANSIONS ON SYMMETRIC SPACES

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Introduction

Let G/H be a semisimple symmetric space, where G is a connected semisimple real Lie group with an involution σ, and H is an open subgroup of the fix point group G^σ. Assume that G has finite center; then it is known that G has a σ-stable maximal compact subgroup K.

In harmonic analysis on the symmetric space G/H an important role is played by the K-finite functions f on G/H which are annihilated by a cofinite ideal of the algebra $\mathbf{D}(G/H)$ of invariant differential operators on G/H. The asymptotic behaviour of such functions is examined in [B87]. Using methods originally developed in [HC60] and [CM82] for the special case where H is compact, a converging series expansion of f at infinity is obtained. In fact, these expansions are obtained more generally for functions on G that are allowed to be H-finite on the right.

Let f be as above, but considered as a right H-invariant function on G. Then f can be written as an infinite sum of functions on G which are K-finite also on the right. Explicitly $f = \sum_{\delta \in K^\wedge} f^\delta$, where $f^\delta(x) = \dim \delta \int_K f(xk^{-1}) \chi_\delta(k)\, dk$ is right K-finite of type δ, χ_δ being the character of δ.

As a function which is K-finite on both sides, each f^δ has a converging series expansion at infinity, according to the above mentioned results of [HC60] and [CM82]. The purpose of the present note is to relate the coefficients in the expansion of f to the coefficients in the expansions of f^δ.

As an application of the relation between the coefficients, we prove the following result: the function f is bounded on G/H if and only if each of the functions f^δ is bounded on G. This result was obtained earlier by different methods in [FOS88], where it was used to prove (cf. Corollary 4 below): *if f generates (on the left) a unitarizable $(\mathfrak{g}, K)$-submodule of $C^\infty(G/H)$, then f is bounded.*

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1. Converging expansions of $K \times H$-finite, $\mathcal{Z}(\mathfrak{g})$-finite functions

Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be a σ-stable Cartan decomposition of the Lie algebra $\mathfrak{g}$ of G, and let $\mathfrak{g} = \mathfrak{h} + \mathfrak{q}$ be the decomposition in eigenspaces of σ. Choose a σ-stable maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$, and decompose it as the direct sum of $\mathfrak{a}_h = \mathfrak{a} \cap \mathfrak{h}$ and $\mathfrak{a}_q = \mathfrak{a} \cap \mathfrak{q}$.

Let $\Sigma \subset \mathfrak{a}^*$ be the restricted root system, Σ^+ a set of positive roots, Δ a set of simple roots, and $\mathfrak{a}^+$ the corresponding open chamber in $\mathfrak{a}$. We assume that $\mathfrak{a}$ and Σ^+ are chosen to be $\mathfrak{q}$-maximal and $\mathfrak{q}$-compatible, respectively (cf. [S84, p. 118-119]), and denote by Σ_q and Δ_q the sets of non-zero restrictions to $\mathfrak{a}_q$ of the elements in Σ and Δ, respectively. Let $\mathfrak{a}_q^+$ be the corresponding open chamber in $\mathfrak{a}_q$. It follows, in particular, that when P is the minimal parabolic subgroup of G associated with Σ^+, then the product PH is an open subset of G.

Let $\mathcal{Z}(\mathfrak{g})$ be the center of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of the complexification $\mathfrak{g}_c$, and let $f \in C^\infty(G)$ be a function which is $K \times H$-finite (that is, K-finite from the left and H-finite from the right) and $\mathcal{Z}(\mathfrak{g})$-finite (for example, f could be as in the introduction). Recall from [B87, Thm. 2.5] that there exists a finite set S of complex linear forms ν on $\mathfrak{a}_q$, a finite set of M of complex polynomials p on $\mathfrak{a}_q$, and, for each ν and p, a holomorphic function $F_{\nu,p}$ on D^{Δ_q} (where $D \subset \mathbb{C}$ is the unit disk) such that

$$f(a) = \sum_{\nu \in S, p \in M} F_{\nu,p}(\bar{\alpha}(a))\, p(\log a) a^\nu$$

for all $a \in A_q^+ = \exp \mathfrak{a}_q^+$. Here $\bar{\alpha}$ is the map from A_q^+ to $(0,1)^{\Delta_q}$ given by $\bar{\alpha}(a) = (a^{-\alpha})_{\alpha \in \Delta_q}$, where $a^{-\alpha} = e^{-\alpha(\log a)}$.

Expanding each $F_{\nu,p}$ in its power series at 0 we obtain an expansion of f with polynomial coefficients:

$$f(a) = \sum_{\nu \in S - \mathbb{N}\Delta_q} P_\nu(f, \log a) a^\nu,$$

where $P_\nu(f) \in \operatorname{Span} M$ for each

$$\nu \in S - \mathbb{N}\Delta_q = \{s - \sum_{\alpha \in \Delta_q} n_\alpha \alpha \mid s \in S, n_\alpha = 0, 1, 2 \ldots\}.$$

For each $\epsilon > 0$, the sum converges absolutely and uniformly on the set $\{a \in A_q \mid \alpha(\log a) > \epsilon, \forall \alpha \in \Delta_q\}$. Each polynomial $P_\nu(f)$ is uniquely determined by f and ν. For convenience we put $P_\nu(f) = 0$ for $\nu \notin S - \mathbb{N}\Delta_q$.

In the special case where σ is the Cartan involution, so that $H = K$ and $\mathfrak{a}_q = \mathfrak{a}$, the above expansion on A^+ of a $K \times K$-finite $\mathcal{Z}(\mathfrak{g})$-finite function is the same as that given in [HC60] and [CM82].

Returning to the general case where H is noncompact, we write, as in the introduction, $f = \sum_{\delta \in K^\wedge} f^\delta$. Since the functions f^δ are $K \times K$-finite and $\mathcal{Z}(\mathfrak{g})$-finite, they admit converging expansions (according to the special case just mentioned)

$$f^\delta(a) = \sum_\xi P_\xi(f^\delta, \log a) a^\xi$$

for $a \in A^+$ with polynomials $P_\xi(f^\delta)$ on $\mathfrak{a}$, $\xi \in \mathfrak{a}_c^*$. Let T denote the set of weights μ of $\mathfrak{a}_h$ occurring in the finite dimensional representation of H generated by $R_h f, h \in H$.

Theorem 1. *Let $\nu \in (\mathfrak{a}_q)_c^*$. Then*

$$P_\nu(f) = \sum_{\delta \in K^\wedge} \sum_\xi P_\xi(f^\delta)|_{\mathfrak{a}_q},$$

where the inner sum extends over the finite set of $\xi \in \mathfrak{a}_c^*$ such that $\xi|_{\mathfrak{a}_q} = \nu$ and $\xi|_{\mathfrak{a}_h} \in T$. The degrees of the polynomials $P_\xi(f^\delta)$ in the sum are bounded by a constant independent of δ and ν, and the sum over δ converges locally uniformly on $\mathfrak{a}_q$.

The proof of the theorem will be given at the end of Section 3.

2. Asymptotic expansions of K-finite $\mathcal{Z}(\mathfrak{g})$-finite functions of at most exponential growth

Fix an ideal $I \subset \mathcal{Z}(\mathfrak{g})$ of finite codimension and a finite set $\mathcal{T} \subset K^\wedge$ of K-types, and put

$$E(I, \mathcal{T}) = \{f \in C^\infty(G; \mathcal{T}) \mid L_u f = 0, \forall u \in I\}.$$

Here $C^\infty(G; \mathcal{T})$ denotes the space of continuous functions on G whose left translates by elements of K span a finite dimensional space in which only K-types from $\mathcal{T}$ occur, and L denotes the left regular representation of G. Then G and $\mathfrak{g}$ act on $E(I, \mathcal{T})$ via the right regular representation R.

Let $J \subset \mathcal{U}(\mathfrak{g})$ be the left ideal generated by I and by the subspace $\cap_{\tau \in \mathcal{T}} \ker \tau$ in $\mathcal{U}(\mathfrak{k})$, and consider the $\mathcal{U}(\mathfrak{g})$-module $\mathcal{U}(\mathfrak{g})/J$. It is easily seen that every element in this module is $\mathfrak{k}$-finite, and since it is clearly finitely generated it follows from [W88, 3.4.7] that this is an admissible $(\mathfrak{g}, \mathfrak{k})$-module.

Lemma 1. *The pairing*

$$\mathcal{U}(\mathfrak{g})/J \times E(I, \mathcal{T}) \to \mathbb{C},$$

defined by $(u, f) \to L_u f(e)$, is $\mathfrak{g}$-equivariant and nondegenerate in f.

Proof. The pairing is equivariant:

$$L_{Xu} f(e) = L_X(L_u f)(e) = -R_X(L_u f)(e) = -L_u(R_X f)(e).$$

It is nondegenerate in f because f is real analytic (cf. [HC60, p. 66]). □

Corollary 1. *The space of right K-finite functions in $E(I,\mathcal{T})$ is an admissible, finitely generated $(\mathfrak{g}, K)$-module for the right action of $\mathfrak{g}$ and K.*

Proof. It follows from Lemma 1 that the $\mathfrak{g}$-module $E(I,\mathcal{T})$ embeds into the linear dual of $\mathcal{U}(\mathfrak{g})/J$. Hence the K-finite functions embed into the $\mathfrak{k}$-finite dual of $\mathcal{U}(\mathfrak{g})/J$. Now apply [W88, 4.3.2]. (That all K-types occur in $E(I,\mathcal{T})$ with finite multiplicity could also be seen from [HC60, p. 65, Cor. 2]). □

For each $r \in \mathbb{R}$ we denote $C_r(G)$ the Banach space of continuous functions on G of at most exponential growth rate r, cf. [BS87, p. 113]. Then G acts continuously on $C_r(G)$ from both sides.

Fix r and let $\mathcal{E} = E(I,\mathcal{T}) \cap C_r(G)$ be equipped with the norm inherited from $C_r(G)$. With π equal to the right action R of G on $\mathcal{E}$ we then obtain an admissible Banach representation $(\pi, \mathcal{E})$ of G. Let $\mathcal{E}^\infty$ denote the space of C^∞-vectors of this representation, i.e., the space of functions $f \in E(I,\mathcal{T})$ for which $R_u f \in C_r(G)$ for all $u \in \mathcal{U}(\mathfrak{g})$, equipped with the natural Fréchet topology. Moreover, let $(\mathcal{E}^\infty)'_K$ be the space of K-finite vectors in the topological linear dual $(\mathcal{E}^\infty)'$ of $\mathcal{E}^\infty$; by [W88, 4.3.3] it can be identified with the space $V^\sim$ of K-finite vectors in the linear dual of the $(\mathfrak{g}, K)$-module V underlying $\mathcal{E}$.

Lemma 2. *Let $f \in \mathcal{E}^\infty$. There exist $v \in \mathcal{E}^\infty$ and $\sigma \in (\mathcal{E}^\infty)'_K$ such that, for all $x \in G$,*

$$f(x) = \sigma(\pi(x)v).$$

Proof. Let $v = f$ and let σ be the restriction to $\mathcal{E}^\infty$ of evaluation at the identity. Then $\sigma \in (\mathcal{E}^\infty)'$. Moreover, it follows from Lemma 1 that there is a surjection of the finite dimensional space $\mathcal{U}(\mathfrak{k})/\mathcal{U}(\mathfrak{k}) \cap J$ onto $\mathcal{U}(\mathfrak{k})\sigma$. Hence σ is K-finite (and the K-types occurring in the span of the K-translates of σ are contragredient to those in $\mathcal{T}$). □

Conversely, it follows from [W83, Lemma 5.1] that, for every admissible, finitely generated Banach representation $(\pi, \mathcal{H})$ and every $\sigma \in (\mathcal{H}^\infty)'_K$, there exists r, I, and $\mathcal{T}$ such that (1) the generalized matrix coefficient $\sigma(\pi(x)v)$ belongs to $\mathcal{E}^\infty$ for all $v \in \mathcal{H}^\infty$, and (2) the map taking $v \in \mathcal{H}^\infty$ to $\sigma(\pi(\cdot)v) \in \mathcal{E}^\infty$ is continuous.

The following theorem is now a direct consequence of [W88, Thm. 4.4.3, cf. also BS87] for the case of K-fixed functions. (Wallach only states the theorem for Hilbert representations, but it holds as well for Banach representations, cf. [W83, Thm. 5.8]).

Theorem 2 (Wallach). *Fix r, I, and $\mathcal{T}$ as above. Then there exists a finite set $E^\circ \subset \mathfrak{a}_c^*$ with the following properties. For every $f \in \mathcal{E}^\infty$ and every $\xi \in E^\circ - \mathbb{N}\Delta$ there exists a polynomial $p_\xi(f)$ on $\mathfrak{a}$ such that*

$$f(\exp tH) \underset{t\to\infty}{\sim} \sum_\xi p_\xi(f, tH) e^{t\xi(H)},$$

for every $H \in \mathfrak{a}^+$. Here the asymptotic relation $\underset{t\to\infty}{\sim}$ means that, for all $N \in \mathbb{R}$, there exist positive numbers C and ϵ such that

$$|f(\exp tH) - \sum_{\operatorname{Re}\xi\geq 0} p_\xi(f,tH)e^{t\xi(H)}| \leq Ce^{(N-\epsilon)t} \text{ for all } t\geq 0,$$

and that this inequality is locally uniform in $H \in \mathfrak{a}^+$.

Moreover, there exists $d \in \mathbb{N}$ such that, for each ξ, the map $f \to p_\xi(f)$ is continuous and linear from $\mathcal{E}^\infty$ to the space P_d of all polynomials on $\mathfrak{a}$ of degree $\leq d$.

Remark 1. Let $j(V^\sim)$ be the Jacquet module (cf. [W88, 4.1.5]) of the Harish-Chandra module $V^\sim = (\mathcal{E}^\infty)'_K$. It follows from [W88, Lemma 4.1.4] that $j(V^\sim)$ is generated as a $\mathcal{U}(\mathfrak{g})$-module by a finite dimensional $\mathfrak{a}$-stable subspace, say of dimension d_0. Since the adjoint action of $\mathfrak{a}$ on $\mathcal{U}(\mathfrak{g})$ is semisimple, it then follows that the representation of $\mathfrak{a}_c$ in $j(V^\sim)$ admits a simultaneous Jordan decomposition whose nilpotent part has nilpotent order at most d_0. This implies that, for each $k \in \mathbb{N}$ and $\xi \in \mathfrak{a}_c^*$, the generalized weight space $(V^\sim/\mathfrak{n}^k V^\sim)_\xi$ for the weight ξ is annihilated by $(H-\xi(H))^{d_0}$ for all $H \in \mathfrak{a}$. For the constant d in the final statement of the theorem one may take d_0.

Remark 2. The theorem stated here deals with the asymptotic behavior of f in the direction of the open chamber A^+. In fact a more general result, describing also the asymptotics 'along the walls,' is contained in [W88]. In [BS89] we study these expansions (for the K-fixed case) and prove a relation between coefficients in the expansions along the walls and coefficients p_ξ in Theorem 2 (cf. [BS89, Thm. 3.1]). However, these results are not needed here.

Let $\mathcal{S}$ be Wallach's space of rapidly decreasing functions on G (cf. [W88, 7.1.2]; $\mathcal{S} = \cap_{p>0}\mathcal{C}^p(G)$, where $\mathcal{C}^p(G)$ is Harish-Chandra's L^p-Schwartz space), and let $\mathcal{S}'$ be the dual space. Fix a Haar measure dx on G. Following [BS87, part II] (to obtain congruence with [BS87], replace $f(x)$ by $f(x^{-1})$), we obtain:

Corollary 2. *For every $f \in \mathcal{E}$ and $\xi \in E^\circ - \mathbb{N}\Delta$ there exists a polynomial $p_\xi(f)$ on $\mathfrak{a}$ with coefficients in $\mathcal{S}'$ such that*

$$f(\exp(tH)x) \sim \sum_\xi p_\xi(f,tH)(x)e^{t\xi(H)}$$

as $t \to +\infty$, for every $H \in \mathfrak{a}^+$. Here the relation $\sim$ means that the following asymptotic relation holds for all $\phi \in \mathcal{S}$ (in the sense described in Thm. 2):

$$\int_G f(\exp(tH)x)\phi(x)dx \underset{t\to\infty}{\sim} \sum_\xi p_\xi(f,tH)(\phi)e^{t\xi(H)}.$$

Moreover, for each ξ, the map $f \to p_\xi(f)$ is continuous and G-equivariant (for the right actions) from $\mathcal{E}$ to $P_d \otimes \mathcal{S}'$.

3. Application to $K \times H$-finite, $\mathcal{Z}(\mathfrak{g})$-finite functions

Here is the relation between the expansions in the previous two sections:

Theorem 3. *Let $f \in E(I, \mathcal{T})$ and assume that f is right H-finite. Then there exists $r \in \mathbb{R}$ such that $f \in C_r(G)$, and hence Corollary 2 applies to f. The restrictions to the open set PH of the distribution coefficients of $p_\xi(f)$ are, via the Haar measure dx, given by real analytic functions, and hence they can be evaluated at the identity. They satisfy the following relation with the polynomials P_ν of Section 1: for all $\nu \in (\mathfrak{a}_q)^*_c$*

$$P_\nu(f) = \sum_\xi p_\xi(f, e)|_{\mathfrak{a}_q},$$

where the sum extends over the finite set of $\xi \in \mathfrak{a}_q^*$ such that $\xi|_{\mathfrak{a}_q} = \nu$ and $\xi|_{\mathfrak{a}_h} \in \mathcal{T}$.

Proof. For the existence of r, see Remark 14.5 in [BS87]. In the special case where $\mathcal{T}$ consists only of the trivial K-type (so that f is left K-invariant), the theorem can be derived from [BS87, Sections 14-16] as follows.* That $p_\xi(f)$ is real analytic on PH is stated in Corollary 16.2. For the τ-spherical function F associated to f (cf. [BS87, p. 148]) it follows from (14.8), (16.4) and (16.9) that $p_\xi(F, e)|_{\mathfrak{a}_q}$ can be obtained from $P_\nu(F)$, where $\nu = \xi|_{\mathfrak{a}_q}$, by projecting it onto the generalized weight space in E_τ of $\mathfrak{a}_h$-weight $\mu = \xi|_{\mathfrak{a}_h}$. Hence the summation over all ξ such that $\xi|_{\mathfrak{a}_q} = \nu$ and $\xi|_{\mathfrak{a}_h} \in \mathcal{T}$ yields $\sum_\xi p_\xi(F, e)|_{\mathfrak{a}_q} = P_\nu(F)$, from which the stated result for f follows. The only difficulty in extending this proof to the general $\mathcal{T}$ is contained in the following lemma, which generalizes [BS87, Lemma 15.1]. □

Define $a_t \in A^+$ for $t \in (0,1)^\Delta$ by $(a_t)^{-\alpha} = t_\alpha$ for $\alpha \in \Delta$. Then $t \to 0$ is equivalent to $\alpha(\log a_t) \to +\infty$ for all $\alpha \in \Delta$.

Lemma 3. *There exist an open neighborhood Ω_0 of $(e, 0)$ in $G \times \mathbb{R}^\Delta$ and real analytic maps $h, a, k : \Omega_0 \to H, A, K$, respectively, such that:*

(i) *For all $(g, t) \in \Omega_0$, with $t \in (0,1)^\Delta$,*

$$ga_t = h(g,t)a(g,t)a_t k(g,t).$$

(ii) *If $(g, 0) \in \Omega_0$ and $x = man \in P$, then $(gx, 0) \in \Omega_0$, $h(gx, 0) = h(g, 0)$, $a(gx, 0) = a(g, 0)a$, and $k(gx, 0) = k(g, 0)m$.*

(iii) *For $t \in \mathbb{R}^\Delta$ near 0 we have $h(e,t) = a(e,t) = k(e,t) = e$.*

Proof. The existence of h and a is given in [BS87, Lemma 15.1]. To prove the existence of k we need the following lemma. Let $\bar{P} = M A \bar{N}$ be the minimal parabolic opposite to P.

*Notice that in [BS87] the sides from which K and H act are reversed.

Lemma 4. *There exist an open neighborhood U_1 of $(e,0)$ in $G \times \mathbb{R}^\Delta$ and unique real analytic maps $z_1, k_1 : U_1 \to \bar{N}A, K$, respectively, such that*

$$ga_t = z_1(g,t)a_t k_1(g,t)$$

for $(g,t) \in U_1$ with $t \in (0,1)^\Delta$. Moreover, $z_1(e,t) = k_1(e,t) = e$, and if $x = man \in P$ then $z_1(gx,0) = z_1(g,0)a$ and $k_1(gx,0) = k_1(g,0)m$.

Proof. The uniqueness is clear from the uniqueness in the Iwasawa decomposition. By the $\bar{N}AMN$ decomposition it suffices to prove the existence of such maps on a neighbourhood of $(e,0)$ in $N \times \mathbb{R}^\Delta$. Now use [BS87, Lemma 8.6] and its proof. □

Proof of Lemma 3. Let Ω_0, h, and a be as in [BS87, Lemma 15.1]. Then, for (z,t) an element of $(\bar{N}A \times (0,1)^\Delta) \cap \Omega_0$, we have

$$za_t k = h(z,t)a(z,t)a_t$$

for some $k \in K$. From the uniqueness in Lemma 4 we infer that

$$z = z_1(h(z,t)a(z,t),t) \text{ and } k = k_1(h(z,t)a(z,t),t)$$

for (z,t) near $(e,0)$ in $(\bar{N}A \times (0,1)^\Delta) \cap \Omega_0$. Hence

$$za_t = h(z,t)a(z,t)a_t k(z,t)$$

where $k(z,t) = k_1(h(z,t)a(z,t),t)^{-1}$ is defined and real analytic on a neighborhood of $(e,0)$ in $\bar{N}A \times \mathbb{R}^\Delta$.

For $(g,t) \in G \times (0,1)^\Delta$ near $(e,0)$, Lemma 4 gives

$$\begin{aligned} ga_t &= z_1(g,t)a_t k_1(g,t) \\ &= h(z_1(g,t),t)a(z_1(g,t),t)a_t k(z_1(g,t),t)k_1(g,t), \end{aligned}$$

and Lemma 3 follows. □

Proof of Theorem 1. From Theorem 3 we have

$$P_\nu(f) = \sum_{\xi|_{\mathfrak{a}_q}=\nu,\, \xi|_{\mathfrak{a}_h} \in T} p_\xi(f,e)|_{\mathfrak{a}_q}$$

for all $\nu \in (\mathfrak{a}_q)^*_c$, and, when applied to the case $H = K$,

$$P_\xi(f^\delta) = p_\xi(f^\delta, e)$$

for all $\xi \in \mathfrak{a}^*_c$. In particular, $\deg P_\xi(f^\delta) = \deg p_\xi(f^\delta, e) \leq d$. From the continuity and linearity of the map $f \to p_\xi(f)$ it follows that

$$p_\xi(f) = \sum_\delta p_\xi(f^\delta)$$

for all ξ. Combining these equations we get

$$P_\nu(f) = \sum_\xi p_\xi(f,e)|_{\mathfrak{a}_q} = \sum_{\xi,\delta} p_\xi(f^\delta,e)|_{\mathfrak{a}_q} = \sum_{\xi,\delta} P_\xi(f^\delta)|_{\mathfrak{a}_q},$$

and the theorem follows. □

4. Application to boundedness

For simplicity we consider in this section only functions on G that are right H-fixed. Notice that in this case the set $T \subset (\mathfrak{a}_h)^*_c$ in Theorem 1 consists only of the element 0. In [B87, Thm. 6.4] (and in [CM82, Thm. 7.5] for the case of $H = K$) a criterion for $f \in L^p(G/H)$ is given in terms of the coefficients $P_\nu(f)$, where $1 \leq p < \infty$. The following theorem supplements this (at $p = \infty$), and the proof is essentially the same (it is in fact slightly easier). It is convenient to rewrite the series expansion of f as follows

$$f(a) = \sum_{\nu \in S - \mathbb{N}\Delta_q, n \in \mathbb{N}^{\Delta_q}} c_{\nu,n} (\log a)^n a^\nu \qquad (a \in A_q^+),$$

where $c_{\nu,n} \in \mathbb{C}$, and where $(\log a)^n$ is defined as $\prod_{\beta \in \Delta_q} (\beta(\log a))^{n_\beta}$. The $c_{\nu,n}$ are uniquely determined by f and by the choice of basis Δ_q defining the open chamber A_q^+.

Theorem 4. *Let f be a K-finite, $\mathbf{D}(G/H)$-finite function on G/H. Then the following three statements are equivalent:*

(i) *f is bounded on G/H.*
(ii) *For every choice Δ_q of basis for Σ_q, and for every ν, the function $a \to P_\nu(\log a) a^\nu$ is bounded on A_q^+.*
(iii) *For every choice Δ_q of basis for Σ_q, for every $\nu = \sum_{\beta \in \Delta_q} \nu_\beta \beta$ and $n \in \mathbb{N}^{\Delta_q}$ with $c_{\nu,n} \neq 0$, and for every $\beta \in \Delta_q$, we have:*

$$\operatorname{Re} \nu_\beta \leq 0; \text{ in fact, if } n_\beta \neq 0, \text{ then } \operatorname{Re} \nu_\beta < 0.$$

We can now derive the following result, which was first obtained in joint work of Flensted-Jensen, Oshima and the second author ([FOS88, Lemma 3.2]).

Corollary 3. *Let f be as above, and let $f = \sum_{\delta \in K^\wedge} f^\delta$. Then f is bounded if and only if each f^δ is bounded.*

Proof. If f is bounded, then obviously each f^δ is bounded. Conversely, assume that every f^δ is bounded, and fix Δ_q and $\nu \in S - \mathbb{N}\Delta_q$.

From Theorem 4 (applied to the special case $H = K$) it follows that $P_\xi(f^\delta, \log a) a^\xi$ is bounded on A^+ for all $\xi \in \mathfrak{a}_c^*$. Hence it is also bounded on A_q^+ because A_q^+ is contained in the closure of A^+. In particular, this holds with ξ given by $\xi|_{\mathfrak{a}_q} = \nu$ and $\xi|_{\mathfrak{a}_h} = 0$. For this ξ we write

$$P_\xi(f^\delta, \log a) = \sum_{n \in \mathbb{N}^{\Delta_q}} c_n^\delta (\log a)^n$$

for $a \in A_q$. If $P_\xi(f^\delta) \neq 0$, then the boundedness of $P_\xi(f^\delta, \log a) a^\nu$ on A_q^+ implies that, for each $\beta \in \Delta_q$, we have $\operatorname{Re} \nu_\beta \leq 0$. Moreover, if $n \in \mathbb{N}^{\Delta_q}$,

$c_n^\delta \neq 0$, and $n_\beta \neq 0$, then $\operatorname{Re} \nu_\beta < 0$. However, Theorem 1 gives that

$$c_{\nu,n} = \sum_\delta c_n^\delta,$$

and hence, if $c_{\nu,n} \neq 0$, then $c_n^\delta \neq 0$ for some δ. Hence $\operatorname{Re} \nu_\beta \leq 0$, and if $n_\beta \neq 0$, then $\operatorname{Re} \nu_\beta < 0$. Now Theorem 4 can be applied once more. □

Finally, we notice the following corollary, also from [FOS88]:

Corollary 4. *Let f be as above, and assume that the $(\mathfrak{g}, K)$-module V_f generated by f (on the left) is unitarizable. Then f is bounded.*

Proof. Since projection onto the space of functions of right K-type δ is a left homomorphism, V_{f^δ} is a unitarizable representation. Since f^δ is a $K \times K$-finite matrix coefficient of V_{f^δ} (cf. Lemma 2), it is thus in fact a matrix coefficient of a unitary representation, and hence it is bounded. Now apply Corollary 3. □

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