A comparison of Paley–Wiener theorems
for real reductive Lie groups

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Abstract. In this paper we make a detailed comparison between the Paley–Wiener theorems of J. Arthur and P. Delorme for a real reductive Lie group G. We prove that these theorems are equivalent from an a priori point of view. We also give an alternative formulation of the theorems in terms of the Hecke algebra of bi-K-finite distributions supported on K, a maximal compact subgroup of G. Our techniques involve derivatives of holomorphic families of continuous representations and Harish-Chandra modules.

Introduction

In this paper we make a detailed comparison between the Paley–Wiener theorem of J. Arthur [1], and the one recently established by P. Delorme [6].

Let G be a real reductive Lie group of the Harish-Chandra class and let K be a maximal compact subgroup. Let $C_c^\infty(G, K)$ denote the space of smooth compactly supported functions on G which behave finitely under both left and right translation by K. The Paley–Wiener theorem of each of the above mentioned authors describes the image of $C_c^\infty(G, K)$ under Fourier transformation, in terms of a so called Paley–Wiener space. In this paper we will show that the two Paley–Wiener spaces, denoted $\mathrm{PW}^A(G, K)$ and $\mathrm{PW}^D(G, K)$, respectively, are equal, without using the proof or the validity of any of the associated Paley–Wiener theorems. It thus follows that the two theorems are equivalent from an a priori point of view. Before we proceed with discussing the contents of our paper let us briefly recall that Arthur's proof of the Paley–Wiener theorem in [1] relied on a result of W. Casselman which has remained unpublished. In [3], the first named author and H. Schlichtkrull established a Paley–Wiener theorem for reductive symmetric spaces which implied Arthur's theorem in the special case of the group. In particular, the proof in [3] used an appropriate substitute for Casselman's result, derived in [2, §16].

Delorme's recent proof of a Paley–Wiener theorem for reductive groups in [6] is completely independent of the proofs just mentioned.

In order to be able to be more specific about the contents of this paper, we shall first give a more detailed description of the two Paley–Wiener theorems mentioned above.

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition associated with the maximal compact subgroup K. Here and in the following we use the convention to denote Lie groups by Roman capitals,

and their Lie algebras by the corresponding lower case German letters. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$, and let $A = \exp \mathfrak{a}$ be the associated vectorial closed subgroup of G. We denote by $\mathcal{P}(A)$ the (finite) set of cuspidal parabolic subgroups of G containing A. Each parabolic subgroup $P \in \mathcal{P}(A)$ has a Langlands decomposition of the form $P = M_P A_P N_P$, with $A_P \subset A$. Let $M_{P,\mathrm{ds}}^{\wedge}$ denote the set of (equivalence classes of) discrete series representations of M_P. For $(\xi, \mathcal{H}) \in M_{P,\mathrm{ds}}^{\wedge}$ and $\lambda \in \mathfrak{a}_{P\mathbb{C}}^*$, we denote by $\pi_{\xi,\lambda} = \pi_{P,\xi,\lambda}$ the representation of G induced from the representation $\xi \otimes (\lambda + \rho_P) \otimes 1$ of P. The associated module of smooth vectors for $\pi_{\xi,\lambda}$ has a realization on the space $C^\infty(K\!:\!\xi)$ consisting of smooth functions $\psi : K \to \mathcal{H}_\xi$, transforming according to the rule

$$\psi(mk) = \xi(m) f(k), \quad \text{for all } k \in K, m \in K \cap M_P.$$

Accordingly, each function $f \in C_c^\infty(G)$ has an operator valued Fourier transform

$$\hat{f}(P, \xi, \lambda) := \pi_{\xi,\lambda}(f) = \int_G f(x)\, \pi_{\xi,\lambda}(x)\, dx \in \mathrm{End}(C^\infty(K\!:\!\xi)).$$

Moreover, the endomorphism $\hat{f}(P, \xi, \lambda)$ depends holomorphically on the variable $\lambda \in \mathfrak{a}_{P\mathbb{C}}^*$. If f is bi-K-finite, then $\hat{f}(P, \xi, \lambda)$ belongs to the space $\mathcal{S}(P\!:\!\xi)$ of bi-K-finite elements of $\mathrm{End}(C^\infty(K\!:\!\xi))$. Moreover, the holomorphic maps $\lambda \mapsto \hat{f}(P, \xi, \lambda)$ are non-zero for only finitely many of the pairs (P, ξ). It follows that $\hat{f}(P)$ may be viewed as an element of the algebraic direct sum

$$(0.1) \qquad \bigoplus_{\xi \in M_{P,\mathrm{ds}}^{\wedge}} \mathcal{O}(\mathfrak{a}_{P\mathbb{C}}^*) \otimes \mathcal{S}(P\!:\!\xi).$$

We agree to define the pre-Paley–Wiener space $\mathrm{PW}_P^{\mathrm{pre}}(G, K)$ as the space of elements $(\varphi(P, \xi) \mid \xi \in M_{P,\mathrm{ds}}^{\wedge})$ in (0.1) for which there exists a constant $R > 0$ and for every $n > 0$ a constant $C_n > 0$ such that

$$(0.2) \qquad \|\varphi(P, \xi, \lambda)\| \le C_n (1 + |\lambda|)^{-n} e^{R|\mathrm{Re}\,\lambda|}$$

for all ξ, λ.

We can now describe the Paley–Wiener space involved in Arthur's theorem in [1]. Let P_0 be a fixed minimal parabolic subgroup in $\mathcal{P}(A)$. Its Langlands decomposition is of the form $P_0 = MAN_0$, where M is the centralizer of A in K. The Arthur Paley–Wiener space

$$(0.3) \qquad \mathrm{PW}^A(G, K)$$

is defined as the space of $\varphi \in \mathrm{PW}_{P_0}^{\mathrm{pre}}(G, K)$ satisfying all finite linear relations of the form

$$(0.4) \qquad \sum_i \langle \varphi(P_0, \xi_i, \lambda_i; u_i), \psi_i \rangle = 0,$$

with $\xi_i \in M^\wedge$, $\psi_i \in \mathcal{S}(P_0\!:\!\xi_i)_{K\times K}^*$, $\lambda_i \in \mathfrak{a}_{\mathbb{C}}^*$ and with $u_i \in S(\mathfrak{a}^*)$ acting as differential operators in the λ-variable (see Section 2.1 for notation), as soon as these relations are satisfied by all families of functions $(\lambda \mapsto \pi_{P_0,\xi,\lambda}(x) \mid \xi \in M^\wedge)$, for $x \in G$. These are the so-called Arthur–Campoli relations. In [1], Arthur defines a similar Paley–Wiener space involving all minimal parabolic subgroups from $\mathcal{P}(A)$. In [3] this space is shown to be isomorphic to the one defined in (0.3).

Next, let us describe the Paley–Wiener space introduced by Delorme [6]. The definition of this Paley–Wiener space involves the operation of taking successive derivatives of a family π_λ of representations, depending holomorphically on a parameter $\lambda \in \mathfrak{a}_{P\mathbb{C}}^*$, for some $P \in \mathscr{P}(A)$. Such an operation is encoded by a sequence $\eta = (\eta_1, \ldots, \eta_N)$ in $\mathfrak{a}_{P\mathbb{C}}^*$, listing the directions in which the derivatives should be taken successively. The associated family $\pi_\lambda^{(\eta)}$ is again a holomorphic family of representations. The operation of derivation also applies to a holomorphic family $\lambda \mapsto \varphi_\lambda$ of continuous endomorphisms of π_λ and then gives a holomorphic family $\varphi^{(\eta)}$ of endomorphisms of $\pi_\lambda^{(\eta)}$. Let $\mathscr{D}$ be the set of all 4-tuples (P, ξ, λ, η) with $P \in \mathscr{P}(A)$, $\xi \in M_{P,\mathrm{ds}}^\wedge$, $\lambda \in \mathfrak{a}_{P\mathbb{C}}^*$ and η a finite sequence of linear functionals from $\mathfrak{a}_{P\mathbb{C}}^*$ as above. Given a datum $\delta = (P, \xi, \lambda, \eta)$ we define $\pi_\delta := \pi_{P,\xi,\lambda}^{(\eta)}$. Moreover, given $\varphi \in \mathrm{PW}_P^{\mathrm{pre}}(G, K)$ we define φ_δ in a similar fashion. Finally, given a sequence $\delta = (\delta_1, \ldots, \delta_N)$ of data from $\mathscr{D}$, we write $\pi_\delta := \pi_{\delta_1} \oplus \cdots \oplus \pi_{\delta_N}$, and $\varphi_\delta := \varphi_{\delta_1} \oplus \cdots \oplus \varphi_{\delta_N}$.

Delorme's Paley–Wiener space is defined as the space $\mathrm{PW}^D(G, K)$ of functions $\varphi \in \bigoplus_{P \in \mathscr{P}(A)} \mathrm{PW}_P^{\mathrm{pre}}(G, K)$ such that

(a) for each finite sequence $\delta \in \mathscr{D}^N$ the function φ_δ preserves all invariant subspaces of π_δ;

(b) for any two finite sequences $\delta_1 \in \mathscr{D}^{N_1}$ and $\delta_2 \in \mathscr{D}^{N_2}$, and any two sequences of closed invariant subspaces $U_j \subset V_j$ for π_{δ_j} ($j = 1, 2$), the induced maps $\bar{\varphi}_{\delta_j} \in \mathrm{End}(V_j/U_j)$ are intertwined by all G-equivariant operators $T : V_1/U_1 \to V_2/U_2$.

There is a natural map $\mathrm{PW}^D(G, K) \to \mathrm{PW}_{P_0}^{\mathrm{pre}}(G, K)$, given by

$$(0.5) \qquad (\varphi(P, \xi) \mid P \in \mathscr{P}(A), \xi \in M_{P,\mathrm{ds}}^\wedge) \mapsto (\varphi(P_0, \xi) \mid \xi \in M^\wedge).$$

In this paper we show that the map (0.5) is a linear isomorphism from $\mathrm{PW}^D(G, K)$ onto $\mathrm{PW}^A(G, K)$, see Theorem 5.11.

To understand better the conditions involving the derivatives in the definition of the Paley–Wiener spaces, we start, in Section 2, with the study of holomorphic families and their derivatives. Instead of focusing on first order derivatives, we replace a holomorphic family by the associated holomorphic section in a suitable jet bundle. This idea also occurs in W. Casselman's paper [5, §9]. We reformulate it slightly, by using a suitable trivialization of the jet bundle. Our construction starts with fixing a finite dimensional module E for the ring $\mathscr{O}_0$ of germs of holomorphic functions (of λ) at zero. It then gives, for π_λ a holomorphic family of representations in a fixed complete space V, a new holomorphic family $\pi_\lambda^{(E)}$ in the space $E \otimes V$. The differentiation procedure of Delorme turns out to be a special case of this procedure, with E a suitable module of dimension 2. In the same Section 2 we study how the functor $\pi_\lambda \mapsto \pi_\lambda^{(E)}$ behaves with respect to analytic families of intertwining operators and with respect to induction.

In Section 3 we give an equivalent definition of Arthur's Paley–Wiener space by invoking the functors $\pi_\lambda \mapsto \pi_\lambda^{(E)}$, instead of the derivations given by elements of $S(\mathfrak{a}^*)$ in the Arthur–Campoli relations. These relations may then be reformulated as linear relations on differentiated families of representations.

In Section 4 we simplify the definition of $\mathrm{PW}^D(G, K)$. First of all, due to the intertwining relations in the definition, the space can be defined in terms of just the minimal parabolic subgroup P_0. Next, the intertwining conditions (b) turn out to be a consequence of the invariant subspace conditions (a).

In the final Section 5, we bring into play the Hecke algebra $\mathbb{H}(G, K)$ consisting of all bi-K-finite distributions on G supported by K. The importance of this algebra for representation theory is based on the fact that the category of Harish-Chandra modules is isomorphic to the category of finitely generated admissible modules for this algebra.

A key lemma in this section is the following. For (π, V) a Harish-Chandra module, let $\mathrm{End}(\pi)^{\#}$ denote the space of $K \times K$-finite endomorphisms φ of V with the property that for every positive integer n the Cartesian power $\varphi^{\times n}$ preserves all invariant subspaces of $V^{\times n}$. The mentioned key lemma asserts that

$$\mathrm{End}(\pi)^{\#} = \mathrm{image}(\mathbb{H}(G, K)) \subset \mathrm{End}(V).$$

It follows from this lemma that the Paley–Wiener space $\mathrm{PW}^D(G, K)$ allows the following description in terms of the Hecke algebra. For every finite dimensional $\mathcal{O}_0$-module E, and all finite sets $\Xi \subset M^{\wedge}$ and $\Lambda \subset \mathfrak{a}_{\mathbb{C}}^*$, we define the representation $\pi_{E,\Xi,\Lambda}$ to be the direct sum of the representations $\pi_{P_0,\xi,\lambda}^{(E)}$, for $(\xi, \lambda) \in \Xi \times \Lambda$. Moreover, for $\varphi \in \mathrm{PW}_{P_0}^{\mathrm{pre}}(G, K)$, we define the endomorphism $\varphi_{E,\Xi,\Lambda}$ of $\pi_{E,\Xi,\Lambda}$ by taking a similar direct sum. Then $\mathrm{PW}^D(G, K)$ maps isomorphically onto the space of $\varphi \in \mathrm{PW}_{P_0}^{\mathrm{pre}}(G, K)$ such that for all E, Ξ, Λ as above,

$$(0.6) \qquad \varphi_{E,\Xi,\Lambda} \in \pi_{E,\Xi,\Lambda}(\mathbb{H}(G, K)).$$

On the other hand, it follows from its definition that Arthur's Paley–Wiener space $\mathrm{PW}^A(G, K)$ is equal to the space of $\varphi \in \mathrm{PW}_{P_0}^{\mathrm{pre}}(G, K)$ such that for all E, Ξ, Λ as above, $\varphi_{E,\Xi,\lambda}$ is annihilated by the annihilator of $\pi_{E,\Xi,\lambda}(\mathbb{H}(G, K))$ in the contragredient module. Since this condition is equivalent to (0.6), it thus follows that the map (0.5) is a linear isomorphism onto $\mathrm{PW}^A(G, K)$ (Theorem 5.11).

Returning to the original formulation of Arthur's Paley–Wiener theorem, we finally wish to mention that the condition (0.4) may be replaced by the condition that for all ξ_i, u_i, λ_i as in (0.4), there exists a $h \in \mathbb{H}(G, K)$ such that for all i,

$$\varphi(\xi_i, \lambda_i; u_i) = \pi_{P_0, \xi_i, \lambda_i; u_i}(h).$$

This characterization is given in Section 5.5 where it is used to derive, from Arthur's theorem, the Paley–Wiener theorem for bi-K-invariant functions, due to S. Helgason [8] and R. Gangolli [7].

1. Notation and preliminaries

Throughout this paper, G will be a real reductive Lie group in the Harish-Chandra class and K a maximal compact subgroup. Let $U(\mathfrak{g})$ be the universal enveloping algebra of the complexification $\mathfrak{g}_{\mathbb{C}}$ of $\mathfrak{g}$. We denote by $X \mapsto X^{\vee}$ the anti-automorphism of $U(\mathfrak{g})$ which on $\mathfrak{g}$ is given by $X \mapsto -X$.

In this paper, locally convex spaces will always be assumed to be Hausdorff and defined over $\mathbb{C}$.

For any continuous representation (τ, V) of K (in a quasi-complete locally convex space) and any class γ in the unitary dual $K^\wedge$ of K, the K-isotypic component of (τ, V) of type γ is denoted by V_γ. The associated K-equivariant projection onto V_γ is denoted by P_γ.

For every finite subset ϑ of $K^\wedge$, we put

$$V_\vartheta := \bigoplus_{\gamma \in \vartheta} V_\gamma \quad \text{and} \quad P_\vartheta := \bigoplus_{\gamma \in \vartheta} P_\gamma.$$

For any continuous representation (π, V) of G, with V a quasi-complete locally convex space, let V^∞ and V_K denote the vector subspaces of smooth and K-finite elements of V, respectively. The first one gives rise to a subrepresentation of π and the second one to its underlying $(\mathfrak{g}, K)$-module (π, V_K).

We say that a (continuous) G-representation or a $(\mathfrak{g}, K)$-module is admissible if all its K-isotypic components are finite dimensional. A Harish-Chandra module is an admissible $(\mathfrak{g}, K)$-module which is finitely generated as a $U(\mathfrak{g})$-module.

The space $C^\infty(G)$ of complex valued smooth functions on G is equipped with the left and right regular actions of G; the subspace $C_c^\infty(G)$ of compactly supported functions is invariant for these actions. The actions are continuous for the usual locally convex topologies on these spaces and may be dualized by taking contragredients. Let $\mathcal{E}'(G)$ denote the space of compactly supported distributions on G, i.e., the topological linear dual of $C^\infty(G)$. Fix a (bi-invariant) Haar measure dx on G. Then the linear map

$$C_c^\infty(G) \to \mathcal{E}'(G), \quad f \mapsto f\,dx$$

is an injective intertwining operator for both G-action. Accordingly, we will use this map to view $C_c^\infty(G)$ as a submodule of $\mathcal{E}'(G)$. For any continuous representation (π, V) and any $f \in C_c^\infty(G)$, let $\pi(f)$ denote the endomorphism of V defined by

$$\pi(f)v := \int_G f(x)\pi(x)v\,dx, \quad v \in V.$$

Then for all $v \in V^\infty$ and $\xi \in (V^\infty)^*$, the following equality holds:

$$\xi(\pi(f)v) = \langle f\,dx, \xi(\pi(\cdot)v)\rangle;$$

the bracket on the right-hand side of the equation indicates the natural pairing between $\mathcal{E}'(G)$ and $C^\infty(G)$. Let (π_i, V_i), $i = 1, 2$, be two $(\mathfrak{g}, K)$-modules. The space $\operatorname{Hom}(V_1, V_2)$ of (linear) homomorphisms from V_1 to V_2 is naturally endowed with a $(\mathfrak{g} \times \mathfrak{g}, K \times K)$-module structure. Indeed, for any $T \in \operatorname{Hom}(V_1, V_2)$,

$$(X_1, X_2)T = \pi_2(X_2) \circ T - T \circ \pi_1(X_1), \quad X_1, X_2 \in \mathfrak{g},$$
$$(k_1, k_2)T = \pi_2(k_2) \circ T \circ \pi_1(k_1^{-1}), \quad k_1, k_2 \in K.$$

Accordingly, the subspace $\operatorname{Hom}_{(\mathfrak{g}, K)}(V_1, V_2)$ of $(\mathfrak{g}, K)$-homomorphisms consists of the elements of $\operatorname{Hom}(V_1, V_2)$ which are invariant under the diagonal action.

Lemma 1.1. *Let (π, V) be an admissible representation of G and let ϑ_1 and ϑ_2 be finite subsets of $K^\wedge$. Then the linear map*

$$C_c^\infty(G) \to \mathrm{Hom}(V_{\vartheta_1}, V_{\vartheta_2}), \quad f \mapsto P_{\vartheta_2} \pi(f) P_{\vartheta_1}$$

uniquely extends to a continuous linear map from $\mathcal{E}'(G)$ to $\mathrm{Hom}(V_{\vartheta_1}, V_{\vartheta_2})$.

Proof. Uniqueness of the extension follows by density of $C_c^\infty(G)\,dx$ in $\mathcal{E}'(G)$. Let $v \in V_{\vartheta_1}$ and $\xi \in V_{\vartheta_2}^*$. Then by finite dimensionality of $\mathrm{Hom}(V_{\vartheta_1}, V_{\vartheta_2})$ it suffices to show that the linear map

$$\mathcal{L} : C_c^\infty(G) \to \mathbb{C}, \quad f \mapsto \xi(\pi(f)v)$$

extends continuously to $\mathcal{E}'(G)$. Define the function $m \in C^\infty(G)$ by $m(x) = \xi(\pi(x)v)$. Then $\mathcal{L}(f) = \langle f\,dx, m \rangle$, for $f \in C_c^\infty(G)$. Thus, $u \mapsto \langle u, m \rangle$ defines a continuous linear extension of $\mathcal{L}$ to $\mathcal{E}'(G)$. $\square$

We consider the convolution product $*$ on $C_c^\infty(G)$ given by

$$f * g(x) = \int_G f(y) g(y^{-1}x)\,dy,$$

for $f, g \in C_c^\infty(G)$ and $x \in G$. It defines an algebra structure on $C_c^\infty(G)$. The subspace $C_c^\infty(G, K)$ of left and right K-finite elements in $C_c^\infty(G)$ is closed under convolution, hence a subalgebra of $C_c^\infty(G)$. The convolution product has a unique extension to a separately continuous bilinear map $\mathcal{E}'(G) \times \mathcal{E}'(G) \to \mathcal{E}'(G)$, denoted $(u, v) \mapsto u * v$. This turns $\mathcal{E}'(G)$ into an algebra. It is readily seen that the subspace $\mathcal{E}'(G, K)$ of left and right K-finite elements in $\mathcal{E}'(G)$ is closed under convolution, hence a subalgebra. Likewise, the subspace $\mathcal{E}'_K(G)$ of distributions with support in K is a subalgebra, and so is the intersection

$$(1.1) \qquad\qquad \mathbb{H}(G, K) := \mathcal{E}'_K(G) \cap \mathcal{E}'(G, K).$$

The latter is also called the Hecke algebra of the pair (G, K) and is sometimes denoted by $\mathbb{H}$ for simplicity. From Lemma 1.1, we obtain the continuous linear map

$$(1.2) \qquad\qquad \mathcal{E}'(G, K) \to \mathrm{End}(V_K)_{K \times K}, \quad u \mapsto \pi(u)$$

which intertwines the $(\mathfrak{g} \times \mathfrak{g}, K \times K)$-actions. Here the space on the right is equipped with the weakest topology for which the $K \times K$-equivariant projections of finite rank are continuous. By application of Fubini's theorem we see that π is a morphism on the convolution algebra $C_c^\infty(G, K)$, which is a dense subalgebra of $\mathcal{E}'(G, K)$.

By separate continuity of $*$ and continuity of (1.2), it now follows that the map (1.2) is a homomorphism of algebras. Let dk denote the normalized Haar measure of K. Then each $\varphi \in C(K)$ defines a distribution $\varphi\,dk$ in $\mathcal{E}'_K(G)$, given by

$$\langle \varphi\,dk, f \rangle = \int_K \varphi(k) f(k)\,dk.$$

For a given representation $\gamma \in K^\wedge$, we define the distribution $\alpha_\gamma \in \mathbb{H}$ by

$$\alpha_\gamma = \dim(\gamma) \chi_{\gamma^\vee}\,dk,$$

where $\chi_{\gamma^\vee}$ denotes the character of the contragredient $\gamma^\vee$ of γ. Moreover, for $\vartheta \subset K^\wedge$ a finite subset, we define the element $\alpha_\vartheta \in \mathbb{H}$ by

$$\alpha_\vartheta := \sum_{\gamma \in \vartheta} \alpha_\gamma.$$

The functions α_ϑ, viewed as elements of $\mathbb{H}$, will later be seen to define an approximation of the identity in $\mathbb{H}$. Let $\vartheta_1, \vartheta_2 \subset K^\wedge$ be finite subsets. We agree to write $\mathcal{E}'(G, K)_{\vartheta_1 \vartheta_2}$ for the space of distributions $\varphi \in \mathcal{E}'(G, K)$ satisfying

$$\alpha_{\vartheta_1} * \varphi * \alpha_{\vartheta_2} = \varphi.$$

Then $\mathcal{E}'(G, K)_{\vartheta_1 \vartheta_2}$ consists of the distributions in $\mathcal{E}'(G, K)$ of left K-type in ϑ_1 and of right K-type in $\vartheta_2^\vee = \{\gamma^\vee \mid \gamma \in \vartheta_2\}$. Similarly, we write

$$C_c^\infty(G, K)_{\vartheta_1 \vartheta_2} := C_c^\infty(G, K) \cap \mathcal{E}'(G, K)_{\vartheta_1 \vartheta_2}$$

and

$$\mathbb{H}_{\vartheta_1 \vartheta_2} := \mathbb{H} \cap \mathcal{E}'(G, K)_{\vartheta_1 \vartheta_2}.$$

If U, V is a pair of admissible $(\mathfrak{g}, K)$-modules, we agree to write

$$\mathrm{Hom}(U, V)_{\vartheta_1 \vartheta_2} = \{A \in \mathrm{Hom}(U, V) \mid P_{\vartheta_1} A P_{\vartheta_2} = A\}$$

so that, for all admissible G-representations (π, V) and (τ, U),

$$\mathrm{Hom}(U, V)_{\vartheta_1 \vartheta_2} \simeq \mathrm{Hom}(U_{\vartheta_2}, V_{\vartheta_1}),$$

naturally. Viewing $\mathrm{Hom}(U, V)_{K \times K} \simeq V_K \otimes (U^*)_K$ as a $(\mathfrak{g} \times \mathfrak{g}, K \times K)$-module in a natural way, we see that

$$\mathrm{Hom}(U, V)_{\vartheta_1 \vartheta_2} = \mathrm{Hom}(U, V)_{\vartheta_1 \otimes \vartheta_2^\vee}.$$

In particular, it is readily seen that

$$\pi(\mathcal{E}'(G, K)_{\vartheta_1 \vartheta_2}) \subset \mathrm{End}(V)_{\vartheta_1 \vartheta_2} \simeq \mathrm{End}(V_K)_{\vartheta_1 \vartheta_2}.$$

Here we note that $\mathrm{End}(V_K)_{K \times K} \subset \mathrm{End}(V)$, naturally.

Proposition 1.2. *For any admissible representation (π, V) of G,*

$$\pi(C_c^\infty(G, K)) = \pi(\mathcal{E}'(G, K)) = \pi(\mathbb{H}).$$

Proof. We denote the three given subspaces of $\mathrm{End}(V_K)_{K \times K}$ by E_∞, $E_{\mathcal{E}'}$ and $E_\mathbb{H}$ respectively. Let β be the K-equivariant bilinear form on $\mathrm{End}(V_K)_{K \times K}$ given by

$$\beta(A, B) = \mathrm{tr}(A \circ B).$$

By admissibility of π it follows that β defines a non-degenerate pairing, which is perfect when restricted to $\mathrm{End}(V_K)_{\vartheta \vartheta}$, for ϑ any finite subset of $K^\wedge$. Therefore, it suffices to show that the β-orthocomplements $E_\infty^\perp$, $E_{\mathcal{E}'}^\perp$ and $\mathbb{H}^\perp$ are equal. Thus, let $T \in \mathrm{End}(V_K)_{K \times K}$. Then it suffices to show that the following assertions are equivalent:

(i) $\mathrm{tr}(T\pi(x)) = 0$, for all $x \in G$;

(ii) $\mathrm{tr}(T\pi(f)) = 0$, for all $f \in C_c^\infty(G, K)$;

(iii) $\mathrm{tr}(T\pi(f)) = 0$, for all $f \in \mathcal{E}'(G, K)$;

(iv) $\mathrm{tr}(T\pi(f)) = 0$, for all $f \in \mathbb{H}$.

Obviously, (i) implies (ii). By density of $C_c^\infty(G, K)_{\vartheta\vartheta}$ in $\mathcal{E}'(G, K)_{\vartheta\vartheta}$, for every finite subset $\vartheta \subset K^\wedge$, if follows that (ii) implies (iii). Moreover, (iii) implies (iv). We will finish the proof by showing that (iv) implies (i). Assume (iv). For $x \in G$, we define $M(x) := \mathrm{tr}(T\pi(x))$. By admissibility and $K \times K$-finiteness of T it follows that M is an analytic function on G. From (iv) it follows that $\langle f, M \rangle = 0$ for any $f \in \mathbb{H}$. Fix a finite subset $\vartheta \subset K^\wedge$ such that $M \in C_c^\infty(G, K)_{\vartheta\vartheta}$. Then $M = \alpha_\vartheta * M * \alpha_\vartheta$. Let $u \in U(\mathfrak{g})$, and set

$$f := \alpha_\vartheta * (L_u \delta_e) * \alpha_\vartheta \in \mathbb{H},$$

where δ_e is the Dirac measure at the unit element e of G. Then

$$0 = \langle f, M \rangle = \langle L_u \delta_e, M \rangle = (L_{\check{u}} M)(e).$$

By analyticity this implies that M vanishes on the identity component G_0 of G. By K-stability of $\mathbb{H}$, we deduce that M vanishes on KG_0. Since G is of the Harish-Chandra class, this means that $M = 0$ on G. $\quad\square$

Corollary 1.3. *Let* (π, V) *be an admissible representation of G and assume that ϑ_1 and ϑ_2 are finite subsets of $K^\wedge$. Then*

$$\pi(C_c^\infty(G, K)_{\vartheta_1\vartheta_2}) = \pi(\mathcal{E}'(G, K)_{\vartheta_1\vartheta_2}) = \pi(\mathbb{H}_{\vartheta_1\vartheta_2}) = \pi(\mathbb{H}) \cap \mathrm{End}(V_K)_{\vartheta_1\vartheta_2}$$

Proof. This follows from Proposition 1.2 by using K-equivariant projections. $\quad\square$

2. Holomorphic families of representations and their derivatives

Let υ be a finite dimensional real linear space. For any open subset Ω of its complexification $\upsilon_\mathbb{C}$, we denote by $\mathcal{O}(\Omega)$ the space of holomorphic $\mathbb{C}$-valued functions on Ω, endowed with the topology of uniform convergence on compact subsets.

For $\mu \in \upsilon_\mathbb{C}$, we denote by $\mathcal{O}_\mu$ the algebra of germs at μ of holomorphic functions defined on a neighborhood of μ. For any Ω as above, $\mu \in \Omega$ and $f \in \mathcal{O}(\Omega)$, the germ of f at μ is denoted by $\gamma_\mu(f) \in \mathcal{O}_\mu$.

Let $\mathcal{P} = \mathcal{P}(\upsilon_\mathbb{C})$ denote the algebra of polynomial functions $\upsilon_\mathbb{C} \to \mathbb{C}$. Then the map $p \mapsto \gamma_0(p)$ is an embedding of algebras, $\mathcal{P} \hookrightarrow \mathcal{O}_0$. Accordingly, we shall view $\mathcal{P}$ as a subalgebra of $\mathcal{O}_0$.

The ring $\mathcal{O}_0$ is local; its unique maximal ideal $\mathcal{M}$ consists of the elements vanishing at 0. An ideal $\mathcal{I} \lhd \mathcal{O}_0$ is said to be cofinite if the quotient $\mathcal{O}_0/\mathcal{I}$ is finite dimensional as a vector space over $\mathbb{C}$. For $k \in \mathbb{N}$, let $\mathcal{P}_k$ denote the space of polynomial functions $\upsilon_\mathbb{C} \to \mathbb{C}$ of degree at most k. Then

$$(2.1) \qquad\qquad \mathcal{O}_0 = \mathcal{P}_k \oplus \mathcal{M}^{k+1}.$$

Therefore, the ideal $\mathcal{M}^{k+1}$ is cofinite in $\mathcal{O}_0$.

Lemma 2.1. *Let $\mathcal{I}$ be an ideal in $\mathcal{O}_0$. Then the following assertions are equivalent:*

(a) *The ideal $\mathcal{I}$ is cofinite.*

(b) *There exists a $k \in \mathbb{N}$ such that $\mathcal{M}^{k+1} \subset \mathcal{I}$.*

Proof. As $\mathcal{M}^{k+1}$ is cofinite, (b) implies (a). Conversely, assume (a).

The space $V = \mathcal{O}_0/\mathcal{I}$ is a finite dimensional vector space, and an $\mathcal{O}_0$-module for left multiplication. The associated algebra homomorphism $\mathcal{O}_0 \to \mathrm{End}(V)$ is denoted by Λ. As $\mathcal{O}_0$ is a commutative algebra and V is finite dimensional, there exists a positive integer p such that the module V decomposes as a finite direct sum of generalized weight spaces

$$V_\chi := \bigcap_{f \in \mathcal{O}_0} \ker(\Lambda(f) - \chi(f)\,\mathrm{id}_V)^p$$

with $\chi \in \widehat{\mathcal{O}}_0 := \mathrm{Hom}(\mathcal{O}_0, \mathbb{C})$. Since $\mathcal{O}_0$ is a local ring with maximal ideal $\mathcal{M}$, the set $\widehat{\mathcal{O}}_0$ of characters consists of the single element $\chi_0 : g \mapsto g(0)$. It follows that $V = V_{\chi_0}$, so that $(f - f(0))^p \in \mathcal{I}$ for all $f \in \mathcal{O}_0$. In particular, $f^p \in \mathcal{I}$ for all $f \in \mathcal{M}$. As an ideal, $\mathcal{M}$ is generated by n elements. Hence, $\mathcal{M}^{k+1} \subset \mathcal{I}$, for $k \geq np - 1$.   □

2.1. The derivation process. For each vector $X \in \mathfrak{v}$ we denote by ∂_X the first order differential operator given by $\partial_X \varphi(a) = \frac{d}{dz}\varphi(a + zX)|_{z=0}$ for $a \in \mathfrak{v}_\mathbb{C}$ and φ a holomorphic function defined on a neighborhood of a in $\mathfrak{v}_\mathbb{C}$. The map $X \mapsto \partial_X$ has a unique extension to an algebra isomorphism $u \mapsto \partial_u$ from the symmetric algebra $S(\mathfrak{v})$ of $\mathfrak{v}_\mathbb{C}$ onto the algebra of constant coefficient (holomorphic) differential operators on $\mathfrak{v}_\mathbb{C}$. We will follow Harish-Chandra's convention to write

$$(2.2) \qquad\qquad \varphi(a; u) := \partial_u \varphi(a),$$

for φ a holomorphic function defined on a neighborhood of a.

We define the pairing $\langle \cdot, \cdot \rangle$ between $\mathcal{O}_0$ and $S(\mathfrak{v})$ by

$$(2.3) \qquad\qquad \mathcal{O}_0 \times S(\mathfrak{v}) \to \mathbb{C}, \quad (\varphi, u) \mapsto \varphi(0; u).$$

For a given cofinite ideal $\mathcal{I} \lhd \mathcal{O}_0$, let $S_{\mathcal{I}}(\mathfrak{v})$ denote the annihilator of $\mathcal{I}$ in $S(\mathfrak{v})$ relative to this pairing. For $k \in \mathbb{N}$, let $S_k(\mathfrak{v})$ be the linear subspace of $S(\mathfrak{v})$ consisting of the elements of order at most k.

Lemma 2.2. *Let $k \in \mathbb{N}$. Then*

(a) $S_k(\mathfrak{v}) = S_{\mathcal{M}^{k+1}}(\mathfrak{v})$;

(b) *the pairing* (2.3) *induces a perfect pairing* $(\mathcal{O}_0/\mathcal{M}^{k+1}) \times S_k(\mathfrak{v}) \to \mathbb{C}$.

Proof. The pairing $\langle \cdot, \cdot \rangle$, defined in (2.3), vanishes on $\mathcal{M}^{k+1} \times S_k(\mathfrak{v})$. Thus, we have $S_k(\mathfrak{v}) \subset S_{\mathcal{M}^{k+1}}(\mathfrak{v})$.

From the decomposition $\mathcal{O}_0 = \mathcal{M}^{k+1} \oplus \mathcal{P}_k$, and non-degeneracy of the pairing, it follows that $S_{\mathcal{M}^{k+1}}(\mathfrak{v}) \hookrightarrow \mathcal{P}_k^*$. In particular, the dimension of $S_{\mathcal{M}^{k+1}}(\mathfrak{v})$ does not exceed the dimension of $\mathcal{P}_k$, which in turn equals the dimension of $S_k(\mathfrak{v})$. Assertion (a) now follows.

It also follows that the induced embedding $S_k(\mathfrak{v}) = S_{\mathcal{M}^{k+1}}(\mathfrak{v}) \hookrightarrow \mathcal{P}_k^* \simeq (\mathcal{O}_0/\mathcal{M}^{k+1})^*$ is an isomorphism onto. By finite dimensionality, this implies assertion (b).   □

Lemma 2.3. *Let $\mathcal{J}$ be a cofinite ideal of $\mathcal{O}_0$. Then the pairing (2.3) induces a linear isomorphism $S_{\mathcal{J}}(v) \simeq (\mathcal{O}_0/\mathcal{J})^*$.*

Proof. By Lemma 2.1 there exists a $k \in \mathbb{N}$ such that $\mathcal{M}^{k+1} \subset \mathcal{J}$. This inclusion induces an embedding of $(\mathcal{O}_0/\mathcal{J})^*$ into $(\mathcal{O}_0/\mathcal{M}^{k+1})^*$. In view of Lemma 2.2, it follows that the pairing induces an embedding $(\mathcal{O}_0/\mathcal{J})^* \hookrightarrow S(v)$. Its image is contained in the annihilator $S_{\mathcal{J}}(v)$, by definition of the latter. On the other hand, the pairing induces an inclusion $S_{\mathcal{J}}(v) \hookrightarrow \mathcal{O}_0^*$ and elements of $S_{\mathcal{J}}(v)$ vanish on $\mathcal{J}$, so that $S_{\mathcal{J}}(v) \hookrightarrow (\mathcal{O}_0/\mathcal{J})^*$. The result follows. $\qquad\square$

For $\mu \in v_{\mathbb{C}}$ we denote by T_μ the translation in $v_{\mathbb{C}}$ given by $v \mapsto v + \mu$. We note that the pull-back map $T_\mu^* : \varphi \mapsto \varphi \circ T_\mu$ induces a ring isomorphism from $\mathcal{O}_\mu$ onto $\mathcal{O}_0$.

Let $\mathcal{J} \lhd \mathcal{O}_0$ be a cofinite ideal and let Ω be an open subset of $v_{\mathbb{C}}$. Then for every $f \in \mathcal{O}(\Omega)$ and each $\mu \in \Omega$ we define

$$(2.4) \qquad J_{\mathcal{J}} f(\mu) := \mathrm{pr}_{\mathcal{J}}(\gamma_0(T_\mu^* f)) \in \mathcal{O}_0/\mathcal{J},$$

where $\mathrm{pr}_{\mathcal{J}}$ denotes the projection of $\mathcal{O}_0$ onto $\mathcal{O}_0/\mathcal{J}$.

In the following lemma, which is a straightforward consequence of the definitions, $\langle \cdot, \cdot \rangle$ denotes the pairing induced by (2.3), see Lemma 2.3.

Lemma 2.4. *Let $f \in \mathcal{O}(\Omega)$. Then for all $\mu \in \Omega$,*

$$\langle J_{\mathcal{J}} f(\mu), u \rangle = f(\mu; u), \quad u \in S_{\mathcal{J}}(v).$$

Corollary 2.5. *The map $f \mapsto J_{\mathcal{J}} f$ defines a continuous algebra homomorphism from $\mathcal{O}(\Omega)$ to $\mathcal{O}(\Omega, \mathcal{O}_0/\mathcal{J})$.*

Proof. Let $u \in S(v)$. For every $f \in \mathcal{O}(\Omega)$, the function $\partial_u f = f(\cdot; u)$ belongs to $\mathcal{O}(\Omega)$. Moreover, the map ∂_u is a continuous linear endomorphism of $\mathcal{O}(\Omega)$. In view of Lemma 2.4, it follows that for each $\xi \in (\mathcal{O}_0/I)^*$ the map $f \mapsto \xi \circ [J_{\mathcal{J}}(f)]$ is a continuous linear endomorphism of $\mathcal{O}(\Omega)$. By finite dimensionality of $\mathcal{O}_0/\mathcal{J}$, it follows that $J_{\mathcal{J}}$ is continuous.

The assertion that $J_{\mathcal{J}}$ is an algebra homomorphism follows from combining the observations that T_μ^*, γ_0, and $\mathrm{pr}_{\mathcal{J}}$ are algebra homomorphisms. $\qquad\square$

Example 2.6. Let $\xi \in v_{\mathbb{C}}^*$. Denote by e^ξ the holomorphic function on $v_{\mathbb{C}}$ given by

$$e^\xi(\mu) := e^{\xi(\mu)}, \quad \mu \in v_{\mathbb{C}}.$$

In terms of the canonical identification of the symmetric algebra $S(v)$ with the algebra $\mathcal{P}(v_{\mathbb{C}}^*)$ of polynomial functions on $v_{\mathbb{C}}^*$ we have $\partial_u e^\xi = u(\xi)e^\xi$, for $u \in S(v)$. Hence, if $\mathcal{J}$ is a cofinite ideal in $\mathcal{O}_0$, then for all $\mu \in v_{\mathbb{C}}$ and $u \in S_{\mathcal{J}}(v)$,

(i) $\langle J_{\mathcal{J}} e^\xi(\mu), u \rangle = u(\xi)e^{\xi(\mu)}$,

(ii) $J_{\mathcal{J}} e^\xi(\mu) = e^{\xi(\mu)} \mathrm{pr}_{\mathcal{J}} \circ \gamma_0(e^\xi)$.

Definition 2.7. Let E be a finite dimensional $\mathcal{O}_0$-module. For every $f \in \mathcal{O}(\Omega)$ and all $\mu \in \Omega$, we define $f^{(E)}(\mu) \in \mathrm{End}(E)$ by

$$f^{(E)}(\mu)e := \gamma_0(T_\mu^* f) \cdot e, \quad e \in E.$$

Example 2.8. If $E = \mathcal{O}_0/\mathcal{I}$ for some cofinite ideal $\mathcal{I}$ of $\mathcal{O}_0$, then, for any $f \in \mathcal{O}(\Omega)$,

$$J_{\mathcal{I}} f(\mu) = f^{(E)}(\mu)(1 + \mathcal{I}), \quad \mu \in \Omega.$$

Let A be an algebra and E an A-module. We denote by $\mathrm{ann}_A(E)$ the annihilator of E in A, i.e., the kernel of the natural algebra homomorphism $A \to \mathrm{End}(E)$. If E is finite dimensional, $\mathrm{ann}_A(E)$ is a cofinite ideal of A.

Lemma 2.9. *Let $\Omega \subset \mathfrak{v}_{\mathbb{C}}$ be open, and let E be a finite dimensional $\mathcal{O}_0$-module. Then for every $f \in \mathcal{O}(\Omega)$ and all $\mu \in \Omega$,*

(a) $f^{(E)}(\mu)e = J_{\mathrm{ann}_{\mathcal{O}_0}(E)} f(\mu) \cdot e$, *for all $e \in E$;*

(b) $(T_\mu^* f)^{(E)} = T_\mu^* f^{(E)}$.

Proof. These formulas follow by straightforward computation. □

Lemma 2.10. *Let E be a finite dimensional $\mathcal{O}_0$-module and let $\eta \in \mathrm{End}(E)^*$. Then there exists an element $u = u_\eta \in S(\mathfrak{v})$ such that*

$$\eta \circ f^{(E)} = \partial_u f,$$

for every open $\Omega \subset \mathfrak{v}_{\mathbb{C}}$ and all $f \in \mathcal{O}(\Omega)$.

Proof. It suffices to prove this for $\eta = e^* \otimes e$, with $e^* \in E^*$ and $e \in E$. Then, for $f \in \mathcal{O}(\Omega)$ and $\lambda \in \Omega$,

(2.5) $$\eta \circ f^{(E)}(\lambda) = e^*(\gamma_0(T_\lambda^* f) \cdot e).$$

Let $\mathcal{I}$ be the (cofinite) annihilator of e in $\mathcal{O}_0$. Then the linear functional $L : \varphi \mapsto e^*(\varphi \cdot e)$ on $\mathcal{O}_0$ factors through a linear map $\mathcal{O}_0/\mathcal{I} \to \mathbb{C}$. Hence, in view of Lemma 2.3, there exists an element $u \in S_{\mathcal{I}}(\mathfrak{v})$ such that $L(\varphi) = \partial_u \varphi(0)$ for all $\varphi \in \mathcal{O}_0$. It follows that the expression on the right-hand side of (2.5) equals $\partial_u f(\lambda)$. □

We shall also need a kind of converse to the above lemma.

Lemma 2.11. *Let $F \subset S(\mathfrak{v})$ be a finite subset. Then there exists a finite dimensional $\mathcal{O}_0$-module E, and linear functionals $\eta_u \in \mathrm{End}(E)^*$, for $u \in F$, such that*

$$\eta_u \circ f^{(E)} = \partial_u f$$

for every $u \in F$, every open $\Omega \subset \mathfrak{v}_{\mathbb{C}}$ and all $f \in \mathcal{O}(\Omega)$.

Proof. By taking direct sums of finite dimensional $\mathcal{O}_0$-modules we may reduce to the case that F consists of a single element $u \in S(\mathfrak{v})$. Let k be the order of u. Then $\varphi \mapsto \varphi(0; u)$ defines a linear functional e^* on $\mathcal{O}_0/\mathcal{I}$, for $\mathcal{I} = \mathcal{M}^{k+1}$. We put $E = \mathcal{O}_0/\mathcal{I}$ and let e denote the image of $1 \in \mathcal{O}_0$ in E. Let $\eta = e^* \otimes e$ be the linear functional on $\mathrm{End}(E)$ defined by $T \mapsto e^*(Te)$. Then for all $f \in \mathcal{O}(\Omega)$ and all $\lambda \in \Omega$ we have

$$\eta \circ f^{(E)}(\lambda) = \eta(\gamma_0(T_\lambda^* f)) = e^*(\gamma_0(T_\lambda^* f) \cdot e) = \partial_u(\gamma_0(T_\lambda^* f))(0) = \partial_u f(\lambda). \quad □$$

We retain the assumption that Ω is an open subset of $\mathfrak{v}_{\mathbb{C}}$.

Corollary 2.12. *Let E be a finite dimensional $\mathcal{O}_0$-module. Then $f \mapsto f^{(E)}$ is a continuous algebra homomorphism from $\mathcal{O}(\Omega)$ to $\mathcal{O}(\Omega, \operatorname{End}(E))$.*

Proof. The map is an algebra homomorphism by Lemma 2.9 (a) and Corollary 2.5. The continuity is an immediate consequence of Lemma 2.10. □

We agree to use the following notation for the map of Corollary 2.12,

$$(2.6) \qquad J^{(E)} : f \mapsto f^{(E)}, \quad \mathcal{O}(\Omega) \to \mathcal{O}(\Omega, \operatorname{End}(E)).$$

The following property is an immediate consequence of the definitions; here we keep in mind that $\operatorname{End}(E) \oplus \operatorname{End}(F) \hookrightarrow \operatorname{End}(E \oplus F)$, naturally.

Property 2.13. *Let E, F be two finite dimensional $\mathcal{O}_0$-modules. Then for every $f \in \mathcal{O}(\Omega)$,*

$$f^{(E \oplus F)} = f^{(E)} \oplus f^{(F)}.$$

To prepare for deriving more properties of the map $J^{(E)}$, we formulate a few results on finite dimensional $\mathcal{O}_0$-modules.

Lemma 2.14. *Let E be a finite dimensional $\mathcal{O}_0$-module. Then E is cyclic if and only if there exists a cofinite ideal $\mathcal{I}$ of $\mathcal{O}_0$ such that $E \simeq \mathcal{O}_0/\mathcal{I}$.*

Proof. Straightforward. □

Corollary 2.15. *Let E be a finite dimensional $\mathcal{O}_0$-module. Then there exist finitely many cofinite ideals $\mathcal{I}_1, \ldots, \mathcal{I}_n$ of $\mathcal{O}_0$ such that E is a quotient of the direct sum $\mathcal{O}_0/\mathcal{I}_1 \oplus \cdots \oplus \mathcal{O}_0/\mathcal{I}_n$ of $\mathcal{O}_0$-modules. In particular, there exist $k, N \in \mathbb{N}$ such that E is a quotient of the $\mathcal{O}_0$-module $(\mathcal{O}_0/\mathcal{M}^{k+1})^N$ for some $k, N \in \mathbb{N}$.*

Proof. The first assertion results from the previous lemma. The second follows from Lemma 2.1. □

Besides the decomposition (2.1), we have the following decomposition of $\mathcal{P} = \mathcal{P}(v_{\mathbb{C}})$, for $k \in \mathbb{N}$,

$$\mathcal{P} = \mathcal{P}_k \oplus (\mathcal{M} \cap \mathcal{P})^{k+1}.$$

Hence, the embedding $\mathcal{P} \hookrightarrow \mathcal{O}_0$ induces, for each $k \in \mathbb{N}$, an isomorphism of algebras

$$(2.7) \qquad \iota_k : \mathcal{P}/(\mathcal{M} \cap \mathcal{P})^{k+1} \xrightarrow{\sim} \mathcal{O}_0/\mathcal{M}^{k+1}.$$

It follows that $\mathcal{P}/(\mathcal{M} \cap \mathcal{P})^{k+1}$ is a local ring, with unique maximal ideal equal to $(\mathcal{M} \cap \mathcal{P})/(\mathcal{M} \cap \mathcal{P})^{k+1}$. Thus, if $\tilde{\mathcal{I}} \subset \mathcal{P}$ is an ideal with $(\mathcal{M} \cap \mathcal{P})^{k+1} \subset \tilde{\mathcal{I}}$, then $\tilde{\mathcal{I}} \subset \mathcal{M}$.

Lemma 2.16. *Let $\tilde{\mathcal{I}}$ be an ideal of $\mathcal{P}$. Then the following assertions are equivalent:*

(a) *There exists a $k \in \mathbb{N}$ such that $(\mathcal{M} \cap \mathcal{P})^{k+1} \subset \tilde{\mathcal{I}}$.*

(b) *There exists an ideal $\mathcal{I} \lhd \mathcal{O}_0$ of finite codimension, such that $\tilde{\mathcal{I}} = \mathcal{I} \cap \mathcal{P}$.*

If any of these conditions is fulfilled, then the ideal $\mathcal{I}$ in (b) is unique and the embedding of $\mathcal{P}$ into $\mathcal{O}_0$ induces an isomorphism of algebras $\mathcal{P}/\tilde{\mathcal{I}} \to \mathcal{O}_0/\mathcal{I}$.

Proof. Assume (a). The image $\tilde{\jmath}'$ of $\tilde{\jmath}$ in $\mathcal{P}/(\mathcal{M} \cap \mathcal{P})^{k+1}$ is an ideal. Its image $\iota_k(\tilde{\jmath}')$ is an ideal of $\mathcal{O}_0/\mathcal{M}^{k+1}$. Let $\jmath$ be the preimage of $\iota_k(\tilde{\jmath}')$ in $\mathcal{O}_0$. Then the following diagram commutes:

$$
\begin{array}{ccc}
\mathcal{P}/(\mathcal{M} \cap \mathcal{P})^{k+1} & \xrightarrow{\ \iota_k\ } & \mathcal{O}_0/\mathcal{M}^{k+1} \\
\ \downarrow{\scriptstyle p} & & \ \downarrow{\scriptstyle q} \\
\mathcal{P}/\tilde{\jmath} & \xrightarrow{\ \ \mathbf{i}\ \ } & \mathcal{O}_0/\jmath.
\end{array}
$$

Here $\mathbf{i}$ is induced by the inclusion map $\mathcal{P} \hookrightarrow \mathcal{O}_0$. The kernel of p equals $\tilde{\jmath}'$ and the kernel of q equals $\iota_k(\tilde{\jmath}')$. As ι_k is an isomorphism of algebras, it follows that $\mathbf{i}$ is an isomorphism of algebras, and (b) is immediate.

Conversely, assume (b). Let $\jmath_j \lhd \mathcal{O}_0$ be ideals such that $\tilde{\jmath} = \jmath_j \cap \mathcal{P}$, for $j = 1, 2$. We will complete the proof by showing that (a) holds and that $\jmath_1 = \jmath_2$.

Since $\jmath_1$ and $\jmath_2$ are cofinite, there exists a constant $k \in \mathbb{N}$ such that $\mathcal{M}^{k+1} \subset \jmath_j$, for both $j = 1, 2$. Therefore,

$$(\mathcal{M} \cap \mathcal{P})^{k+1} \subset (\mathcal{M}^{k+1} \cap \mathcal{P}) \subset \tilde{\jmath}$$

and (a) follows. Moreover, for each $j = 1, 2$ we have the following commutative diagram:

$$
\begin{array}{ccc}
\mathcal{P}/(\mathcal{M} \cap \mathcal{P})^{k+1} & \xrightarrow{\ \iota_k\ } & \mathcal{O}_0/\mathcal{M}^{k+1} \\
\ \downarrow{\scriptstyle p} & & \ \downarrow{\scriptstyle q_j} \\
\mathcal{P}/\tilde{\jmath} & \xrightarrow{\ \ \mathbf{i}_j\ \ } & \mathcal{O}_0/\jmath_j.
\end{array}
$$

From the assumption on $\jmath_j$ it follows that the map $\mathbf{i}_j$ is injective. Moreover, since ι_k and q_j are surjective, it follows that $\mathbf{i}_j$ is an isomorphism of algebras, for $j = 1, 2$. This implies that $\iota_k(\ker p) = \ker(q_j)$. Since $\jmath_j$ equals the preimage of $\ker(q_j)$ in $\mathcal{O}_0$, for $j = 1, 2$, it follows that $\jmath_1 = \jmath_2$. $\qquad\square$

Let $\mathrm{FM}_{\mathcal{O}_0}$ denote the category of finite dimensional $\mathcal{O}_0$-modules and $\mathrm{FM}_{\mathcal{P}}$ the category of finite dimensional $\mathcal{P}$-modules E for which there exists a $k \in \mathbb{N}$ such that

$$(\mathcal{M} \cap \mathcal{P})^{k+1} \subset \mathrm{ann}_{\mathcal{P}}(E).$$

If E, F belong to $\mathrm{FM}_{\mathcal{P}}$ then the $\mathrm{FM}_{\mathcal{P}}$-morphisms are defined to be the $\mathcal{P}$-module homomorphisms $E \to F$. We observe that their kernels and images belong to the category $\mathrm{FM}_{\mathcal{P}}$ as well. If E is a finite dimensional $\mathcal{O}_0$-module, then its annihilator $\jmath$ is cofinite. Since $\mathcal{P} \hookrightarrow \mathcal{O}_0$, the space E is a finite dimensional $\mathcal{P}$-module as well and its annihilator $\tilde{\jmath} := \mathrm{ann}_{\mathcal{P}}(E)$ in $\mathcal{P}$ is given by $\tilde{\jmath} = \jmath \cap \mathcal{P}$. Furthermore, by Lemma 2.1 there exists a $k \in \mathbb{N}$ such that

$$(\mathcal{M} \cap \mathcal{P})^{k+1} \subset \mathrm{ann}_{\mathcal{P}}(E).$$

We conclude that there is a well-defined forgetful functor $\mathcal{F} : \mathrm{FM}_{\mathcal{O}_0} \to \mathrm{FM}_{\mathcal{P}}$.

Lemma 2.17. *The forgetful functor $\mathcal{F} : \mathrm{FM}_{\mathcal{O}_0} \to \mathrm{FM}_{\mathcal{P}}$ is an isomorphism of categories.*

Proof. Let E be a non-trivial $\mathcal{P}$-module from the category $\mathrm{FM}_{\mathcal{P}}$ and let $\varphi : \mathcal{P} \to \mathrm{End}(E)$ be the associated algebra homomorphism. As the ideal $\tilde{\mathcal{I}} = \ker \varphi$ satisfies condition (a) of Lemma 2.1 it follows that there exists a (unique) cofinite ideal $\mathcal{I} \lhd \mathcal{O}_0$ such that $\mathcal{I} \cap \mathcal{P} = \tilde{\mathcal{I}}$. Via the isomorphism $\mathcal{P}/\tilde{\mathcal{I}} \simeq \mathcal{O}_0/\mathcal{I}$ we equip E with a compatible structure of $\mathcal{O}_0$-module.

A compatible $\mathcal{O}_0$-module structure on E corresponds to an algebra homomorphism $\psi : \mathcal{O}_0 \to \mathrm{End}(E)$ which restricts to φ on $\mathcal{P}$. For such a ψ we have $\ker \varphi = \tilde{\mathcal{I}} \cap \mathcal{P}$, hence $\ker \psi = \mathcal{I}$. In view of the isomorphism $\mathcal{P}/\tilde{\mathcal{I}} \simeq \mathcal{O}_0/\mathcal{I}$, it follows that ψ is uniquely determined by φ. We conclude that E has a unique compatible structure of $\mathcal{O}_0$-module.

Let now F be a second $\mathcal{P}$-module from the category $\mathrm{FM}_{\mathcal{P}}$. If $f : E \to F$ is a linear map, then it readily follows from the above that f is a $\mathcal{P}$-module morphism if and only if f is a morphism for the compatible structures of $\mathcal{O}_0$-modules. This implies that $\mathcal{F}$ is an isomorphism of categories. $\qquad \square$

Corollary 2.18. *For every pair E, F of finite dimensional $\mathcal{O}_0$-modules,*

$$\mathrm{Hom}_{\mathcal{O}_0}(E, F) = \mathrm{Hom}_{\mathcal{P}}(E, F).$$

Our next objective is to consider tensor products in the categories $\mathrm{FM}_{\mathcal{P}}$ and $\mathrm{FM}_{\mathcal{O}_0}$.

Let $n \in \mathbb{N}^*$ and consider the n-fold Cartesian product $v_{\mathbb{C}}^n$ of $v_{\mathbb{C}}$. Projection onto the j-th coordinate is denoted by pr_j. Pull-back by pr_j defines an embedding of algebras $\mathrm{pr}_j^* : p \mapsto p \circ \mathrm{pr}_j, \mathcal{P} \to \mathcal{P}(v_{\mathbb{C}}^n)$. The multi-linear map

$$(p_1, \ldots, p_n) \mapsto \prod_{j=1}^n \mathrm{pr}_j^*(p_j)$$

induces an isomorphism of algebras $\mathcal{P}^{\otimes n} \to \mathcal{P}(v_{\mathbb{C}}^n)$, via which we shall identify the elements of these spaces. Accordingly,

$$(p_1 \otimes \cdots \otimes p_n)(\mu_1, \ldots, \mu_n) = p_1(\mu_1) \cdots p_n(\mu_n),$$

for $p_j \in \mathcal{P}$ and $\mu_j \in v_{\mathbb{C}}$. The maximal ideal $\mathcal{M}_{\mathcal{P}^{\otimes n}}$ in $\mathcal{P}(v_{\mathbb{C}}^n)$ consisting of the polynomials vanishing at 0 is now given by $\mathcal{M}_{\mathcal{P}^{\otimes n}} = \sum_{i=1}^n \mathcal{M}_{\mathcal{P},n,i}$, where

$$\mathcal{M}_{\mathcal{P},n,i} := \mathcal{P} \otimes \cdots \otimes \mathcal{P} \otimes \overbrace{(\mathcal{M} \cap \mathcal{P})}^{i} \otimes \mathcal{P} \otimes \cdots \otimes \mathcal{P}.$$

Lemma 2.19. *Let $E_1, \ldots, E_n$ be finite dimensional $\mathcal{P}$-modules from the category $\mathrm{FM}_{\mathcal{P}}$. Then $E_1 \otimes \cdots \otimes E_n$ is a $\mathcal{P}^{\otimes n}$-module from the category $\mathrm{FM}_{\mathcal{P}^{\otimes n}}$.*

Proof. There exists a $k \in \mathbb{N}$ such that $(\mathcal{M} \cap \mathcal{P})^{k+1}$ annihilates each of the modules E_i, for $1 \leq i \leq n$. It is now readily checked that

$$(\mathcal{M}_{\mathcal{P}^{\otimes n}})^{n(k+1)} \subset \mathrm{ann}_{\mathcal{P}^{\otimes n}}(E_1 \otimes \cdots \otimes E_n). \qquad \square$$

We consider the map

$$\alpha_n : v_{\mathbb{C}}^n \to v_{\mathbb{C}}, \quad (\mu_1, \ldots, \mu_n) \mapsto \mu_1 + \cdots + \mu_n.$$

Pull-back by α_n induces an algebra homomorphism

(2.8)
$$\alpha_n^* : \mathcal{P} \to \mathcal{P}^{\otimes n}.$$

Lemma 2.20. *The homomorphism α_n^* maps the maximal ideal $\mathcal{M} \cap \mathcal{P}$ of $\mathcal{P}$ into the maximal ideal $\mathcal{M}_{\mathcal{P}^{\otimes n}}$ of $\mathcal{P}^{\otimes n}$.*

Proof. The ideal $\mathcal{M} \cap \mathcal{P}$ is generated by the first order polynomials $\xi \in \mathfrak{v}_{\mathbb{C}}^*$. Now

$$(2.9) \qquad \alpha_n^*(\xi) = \sum_{i=1}^{n} 1 \otimes \cdots \otimes 1 \otimes \overset{i}{\xi} \otimes 1 \otimes \cdots \otimes 1,$$

and the result follows. □

Corollary 2.21. *Let $n \in \mathbb{N}^*$. Via the homomorphism (2.8), every finite dimensional $\mathcal{P}^{\otimes n}$-module from the category $\mathrm{FM}_{\mathcal{P}^{\otimes n}}$ becomes a $\mathcal{P}$-module from the category $\mathrm{FM}_{\mathcal{P}}$.*

Proof. Straightforward. □

Remark 2.22. If E is a finite dimensional $\mathcal{P}$-module, we define

$$m_E : \mathcal{P} \to \mathrm{End}(E)$$

by $m_E(p)e := p \cdot e$, for $p \in \mathcal{P}$ and $e \in E$.

If $E_1, \ldots, E_n$ are finite dimensional modules from the category $\mathrm{FM}_{\mathcal{P}}$, then by combining Lemma 2.19 and Corollary 2.21 we may equip the tensor product $E_1 \otimes \cdots \otimes E_n$ with the structure of a module from the same category $\mathrm{FM}_{\mathcal{P}}$. The module structure is given by the rule

$$m_{E_1 \otimes \cdots \otimes E_n} = (m_{E_1} \otimes \cdots \otimes m_{E_n}) \circ \alpha_n^*.$$

In view of (2.9) this module structure is completely determined by the rule

$$(2.10) \quad m_{E_1 \otimes \cdots \otimes E_n}(\xi) = \sum_{i=1}^{n} \mathrm{id}_{E_1} \otimes \cdots \otimes \mathrm{id}_{E_{i-1}} \otimes m_{E_i}(\xi) \otimes \mathrm{id}_{E_{i+1}} \otimes \cdots \otimes \mathrm{id}_{E_n}$$

for $\xi \in \mathfrak{v}_{\mathbb{C}}^* \subset \mathcal{P}$.

Accordingly, if $E_1, \ldots, E_n$ are finite dimensional $\mathcal{O}_0$-modules, then in particular they are $\mathcal{P}$-modules from the category $\mathrm{FM}_{\mathcal{P}}$. We equip the module $E_1 \otimes \cdots \otimes E_n$ from $\mathrm{FM}_{\mathcal{P}}$ with the unique compatible structure of $\mathcal{O}_0$-module.

Lemma 2.23. *Let $E_1, \ldots, E_n$ be finite dimensional $\mathcal{O}_0$-modules. Then, as $\mathcal{O}_0$-modules:*

$$E_1 \otimes \cdots \otimes E_n \simeq E_1 \otimes (E_2 \otimes \cdots \otimes E_n).$$

Proof. In view of Remark 2.22, it suffices to establish the identity as an identity of $\mathcal{P}$-modules. Thus, it suffices to show that

$$m_{E_1 \otimes \cdots \otimes E_n} = m_{E_1 \otimes (E_2 \otimes \cdots \otimes E_n)}.$$

Since $\mathfrak{v}_{\mathbb{C}}^*$ generates $\mathcal{P}$, it suffices to check this identity on any element $\xi \in \mathfrak{v}_{\mathbb{C}}^*$. This is easily done by using the identity (2.10). □

We return to the setting of an open subset $\Omega \subset \mathfrak{v}_{\mathbb{C}}$ and resume our study of the map $J^{(E)} : f \mapsto f^{(E)}$, $\mathcal{O}(\Omega) \to \mathcal{O}(\Omega, \mathrm{End}(E))$, introduced in (2.6), for E a finite dimensional

$\mathcal{O}_0$-module. In the following we shall use the natural identification

$$\mathcal{O}(\Omega, \mathrm{End}(E)) \simeq \mathcal{O}(\Omega) \otimes \mathrm{End}(E).$$

Lemma 2.24. *The restriction to $\mathcal{P}$ of the map $J^{(E)}$ is given by*

$$J^{(E)}|_{\mathcal{P}} = (\mathrm{id}_{\mathcal{P}} \otimes m_E) \circ \alpha_2^*.$$

In particular, this restriction maps $\mathcal{P}$ into $\mathcal{P} \otimes \mathrm{End}(E)$.

Proof. Let $p \in \mathcal{P}$. For any $\mu \in \mathfrak{v}_{\mathbb{C}}$, we have $T_\mu^* p \in \mathcal{P}$, and for any $e \in E$,

$$p^{(E)}(\mu)e = (T_\mu^* p) \cdot e = m_E(T_\mu^* p)e.$$

Hence,

$$(\mathrm{ev}_\mu \otimes \mathrm{id}_{\mathrm{End}(E)}) \circ J^{(E)}|_{\mathcal{P}} = m_E \circ T_\mu^*|_{\mathcal{P}}.$$

Viewing $\mathrm{ev}_\mu \otimes \mathrm{id}_{\mathcal{P}}$ as a $\mathcal{P}$-valued function on $\mathcal{P} \otimes \mathcal{P}$, we may identify the map T_μ^* with $(\mathrm{ev}_\mu \otimes \mathrm{id}_{\mathcal{P}}) \circ \alpha_2^*$ on $\mathcal{P}$. Using the relation

$$m_E \circ (\mathrm{ev}_\mu \otimes \mathrm{id}_{\mathcal{P}}) = (\mathrm{ev}_\mu \otimes \mathrm{id}_{\mathrm{End}(E)}) \circ (\mathrm{id}_{\mathcal{P}} \otimes m_E),$$

we obtain the assertion of the lemma. $\qquad\square$

Proposition 2.25. *Let E_1 and E_2 be two finite dimensional $\mathcal{O}_0$-modules. Then, for every $f \in \mathcal{O}(\Omega)$,*

$$(J^{(E_1)} \otimes \mathrm{id}_{\mathrm{End}(E_2)}) f^{(E_2)} = f^{(E_1 \otimes E_2)}.$$

Proof. By density of the subspace $\mathcal{P}$ in $\mathcal{O}(\Omega)$ and continuity of the maps $J^{(E_1 \otimes E_2)}$ and $(J^{(E_1)} \otimes \mathrm{id}_{\mathrm{End}(E_2)}) \circ J^{(E_2)}$, see Corollary 2.12, it suffices to establish the validity of the identity on a fixed element $p \in \mathcal{P}$. By definition of $J^{(E_1)}$, we have $\mathrm{ev}_0 \circ J^{(E_1)}q = m_{E_1}(q)$ for each $q \in \mathcal{P}$. Using Lemma 2.24 we obtain

$$
\begin{aligned}
(J^{(E_1)} \otimes \mathrm{id}_{\mathrm{End}(E_2)}) p^{(E_2)}(0) &= (m_{E_1} \otimes \mathrm{id}_{\mathrm{End}(E_2)}) p^{(E_2)} \\
&= (m_{E_1} \otimes \mathrm{id}_{\mathrm{End}(E_2)}) \circ (\mathrm{id}_{\mathcal{P}} \otimes m_{E_2})(\alpha_2^* p) \\
&= (m_{E_1} \otimes m_{E_2}) \circ \alpha_2^*(p) \\
&= m_{E_1 \otimes E_2}(p) = p^{(E_1 \otimes E_2)}(0).
\end{aligned}
$$

The result now follows by translation invariance; see Lemma 2.9 (b). $\qquad\square$

Proposition 2.26. *Let $E_1, \dots, E_n$ be finite dimensional $\mathcal{O}_0$-modules. Then, for every $f \in \mathcal{O}(\Omega)$,*

$$(J^{(E_1)} \circ \cdots \circ J^{(E_n)})(f) = f^{(E_1 \otimes \cdots \otimes E_n)}.$$

In the formulation of this proposition we have slightly abused notation, by using the abbreviation $J^{(E_k)}$ for $J^{(E_k)} \otimes \mathrm{id}_{\mathrm{End}(E_{k+1})} \otimes \cdots \otimes \mathrm{id}_{\mathrm{End}(E_n)}$.

Proof. In view of Lemma 2.23, the result follows by repeated application of Proposition 2.25. $\qquad\square$

Let $\lambda \in \upsilon_{\mathbb{C}}$. By $\mathcal{I}_\lambda$ we denote the subset of $\mathcal{O}_0$ consisting of all elements $\varphi \in \mathcal{M}$ satisfying

$$(2.11) \qquad\qquad \partial_\lambda \varphi \in \mathcal{M}.$$

By the Leibniz rule, $\mathcal{I}_\lambda$ is an ideal of $\mathcal{O}_0$ containing $\mathcal{M}^2$.

Lemma 2.27. *Assume $\lambda \neq 0$. Then $\mathcal{I}_\lambda$ is an ideal in $\mathcal{O}_0$ of codimension 2. If $\xi \in \upsilon_{\mathbb{C}}^*$ is such that $\xi(\lambda) \neq 0$, then $\mathcal{O}_0 = \mathbb{C}1 \oplus \mathbb{C}\xi \oplus \mathcal{I}_\lambda$.*

Proof. Consider the linear map $L : \mathcal{O}_0 \to \mathbb{C}^2$ defined by $L(\varphi) = (\varphi(0), \partial_\lambda \varphi(0))$. Then $\mathcal{I}_\lambda$ equals the kernel of L, hence is of codimension at most 2. On the other hand, L is readily seen to be injective on $\mathbb{C}1 \oplus \mathbb{C}\xi$, and the result follows. $\qquad\square$

Let $\tilde{\mathcal{I}}_1, \ldots, \tilde{\mathcal{I}}_n$ be a collection of ideals from $\mathcal{P}$ containing $(\mathcal{M} \cap \mathcal{P})^{k+1}$ for some $k \in \mathbb{N}$. Then the algebra homomorphism $\alpha_n^* : \mathcal{P} \to \mathcal{P}^{\otimes n}$ induces an algebra homomorphism

$$\bar{\alpha}_n^* : \mathcal{P} \to (\mathcal{P}/\tilde{\mathcal{I}}_1) \otimes \cdots \otimes (\mathcal{P}/\tilde{\mathcal{I}}_n)$$

by composition with the natural projection from $\mathcal{P}^{\otimes n}$ onto the quotient algebra on the right.

Lemma 2.28. *Let $\lambda \in \upsilon_{\mathbb{C}} \setminus \{0\}$, and let $\tilde{\mathcal{I}} \lhd \mathcal{P}$ be an ideal with $(\mathcal{M} \cap \mathcal{P})^{k+1} \subset \tilde{\mathcal{I}}$ for some $k \in \mathbb{N}$. Then the kernel of*

$$\bar{\alpha}_2^* : \mathcal{P} \to \mathcal{P}/\tilde{\mathcal{I}} \otimes (\mathcal{P}/\mathcal{I}_\lambda \cap \mathcal{P})$$

is equal to $\{p \in \tilde{\mathcal{I}} \mid \partial_\lambda p \in \tilde{\mathcal{I}}\}$. In particular, this kernel contains $(\mathcal{M} \cap \mathcal{P})^{k+2}$.

Proof. The last assertion follows from the first one by application of the Leibniz rule. We turn to the proof of the first assertion.

We fix $\xi \in \upsilon_{\mathbb{C}}^*$ such that $\xi(\lambda) = 1$. By Lemma 2.27 we have $\mathcal{P} = (\mathbb{C}1 \oplus \mathbb{C}\xi) \oplus (\mathcal{P} \cap \mathcal{I}_\lambda)$ as a linear space. Let $\pi : \mathcal{P} \to \mathbb{C}1 \oplus \mathbb{C}\xi$ denote the associated projection operator. Then it is readily checked that

$$\pi(p) = p(0) + \partial_\lambda p(0)\xi, \quad p \in \mathcal{P}.$$

This implies that for all $p \in \mathcal{P}$ we have

$$(\mathrm{id}_{\mathcal{P}} \otimes \pi) \circ \alpha_2^*(p) = p \otimes 1 + \partial_\lambda p \otimes \xi.$$

It follows that $\bar{\alpha}_2^*(p) = 0$ if and only if $p \in \tilde{\mathcal{I}}$ and $\partial_\lambda p \in \tilde{\mathcal{I}}$. $\qquad\square$

Lemma 2.29. *Let $\lambda_1, \ldots, \lambda_n \in \upsilon_{\mathbb{C}} \setminus \{0\}$. If $n > 1$, the kernel of*

$$\bar{\alpha}_n^* : \mathcal{P} \to (\mathcal{P}/\mathcal{I}_{\lambda_1} \cap \mathcal{P}) \otimes \cdots \otimes (\mathcal{P}/\mathcal{I}_{\lambda_n} \cap \mathcal{P})$$

equals the kernel of

$$\bar{\alpha}_2^* : \mathcal{P} \to (\mathcal{P}/\ker \bar{\alpha}_{n-1}^*) \otimes (\mathcal{P}/\mathcal{I}_{\lambda_n} \cap \mathcal{P}),$$

where $\bar{\alpha}_{n-1}^*$ denotes the composition on the left of the algebra homomorphism α_{n-1}^* with the projection $\mathcal{P}^{\otimes(n-1)} \to (\mathcal{P}/\mathcal{I}_{\lambda_1} \cap \mathcal{P}) \otimes \cdots \otimes (\mathcal{P}/\mathcal{I}_{\lambda_{n-1}} \cap \mathcal{P})$.

Proof. The map $\bar{\alpha}^*_{n-1}$ induces an embedding (denoted by the same symbol)

$$\bar{\alpha}^*_{n-1} : \mathcal{P} / \ker \bar{\alpha}^*_{n-1} \hookrightarrow (\mathcal{P} / \mathcal{I}_{\lambda_1} \cap \mathcal{P}) \otimes \cdots \otimes (\mathcal{P} / \mathcal{I}_{\lambda_{n-1}} \cap \mathcal{P}).$$

Moreover, the following diagram commutes:

$$
\begin{array}{ccc}
\mathcal{P} & \xrightarrow{\quad \bar{\alpha}^*_n \quad} & (\mathcal{P} / \mathcal{I}_{\lambda_1} \cap \mathcal{P}) \otimes \cdots \otimes (\mathcal{P} / \mathcal{I}_{\lambda_n} \cap \mathcal{P}) \\
{\scriptstyle \bar{\alpha}^*_2} \downarrow & \nearrow {\scriptstyle \bar{\alpha}^*_{n-1} \otimes \mathrm{id}_{\mathcal{P}/(\mathcal{I}_{\lambda_n} \cap \mathcal{P})}} & \\
(\mathcal{P} / \ker \bar{\alpha}^*_{n-1}) \otimes (\mathcal{P} / \mathcal{I}_{\lambda_n} \cap \mathcal{P}) & &
\end{array}
$$

Thus, $\ker \bar{\alpha}^*_n = \ker \bar{\alpha}^*_2$. □

Lemma 2.30. *Let $\lambda_1, \ldots, \lambda_n \in v_{\mathbb{C}} \setminus \{0\}$. Then the kernel of*

$$\bar{\alpha}^*_n : \mathcal{P} \to (\mathcal{P} / \mathcal{I}_{\lambda_1} \cap \mathcal{P}) \otimes \cdots \otimes (\mathcal{P} / \mathcal{I}_{\lambda_n} \cap \mathcal{P})$$

consists of the elements p of $\mathcal{M} \cap \mathcal{P}$ satisfying $p(\cdot; \lambda_{j_1} \cdots \lambda_{j_l}) \in \mathcal{M}$, for all $1 \leq l \leq n$, $1 \leq j_1 < \cdots < j_l \leq n$. In particular, this kernel contains $(\mathcal{M} \cap \mathcal{P})^{n+1}$.

Proof. The final statement follows from the first by repeated application of the Leibniz rule. The first statement follows by induction on n, by application of Lemmas 2.28 and 2.29. □

Proposition 2.31. (a) *Let $E_1, \ldots, E_n$ be finite dimensional $\mathcal{O}_0$-modules. There exist $N, k \in \mathbb{N}$ such that $E_1 \otimes \cdots \otimes E_n$ is a quotient of the $\mathcal{O}_0$-module $\mathbb{C}^N \otimes \mathcal{O}_0 / \mathcal{M}^{k+1}$ (action on the second tensor component).*

(b) *Let $\mathcal{I}$ be a cofinite ideal of $\mathcal{O}_0$. There exist $\lambda_1, \ldots, \lambda_n \in v_{\mathbb{C}}$ such that the algebra homomorphism*

$$\bar{\alpha}^*_n : \mathcal{P} \to (\mathcal{P} / \mathcal{I}_{\lambda_1} \cap \mathcal{P}) \otimes \cdots \otimes (\mathcal{P} / \mathcal{I}_{\lambda_n} \cap \mathcal{P})$$

satisfies $\ker \bar{\alpha}^*_n \subset \mathcal{I}$. In particular, the $\mathcal{O}_0$-module $\mathcal{O}_0 / \mathcal{I}$ is a subquotient of

$$(\mathcal{O}_0 / \mathcal{I}_{\lambda_1}) \otimes \cdots \otimes (\mathcal{O}_0 / \mathcal{I}_{\lambda_n}).$$

Proof. Assertion (a) follows from Corollary 2.15 and (2.7). For the second assertion, we note that by Lemma 2.17, there exists a number $k \in \mathbb{N}$ such that $(\mathcal{M} \cap \mathcal{P})^{k+1} \subset \mathcal{I} \cap \mathcal{P} \subset \mathcal{M}$. Fix a basis $\{X_1, \ldots, X_N\}$ of v. Put $n := kN$ and

$$\lambda_{kj+i} := X_j \quad \text{for } 0 \leq j \leq N - 1 \text{ and } 1 \leq i \leq k.$$

By Lemma 2.30, the ideal $\ker \bar{\alpha}^*_n$ is equal to

$$\{p \in \mathcal{M} \cap \mathcal{P} \mid p(\cdot; \lambda_{j_1} \cdots \lambda_{j_l}) \in \mathcal{M}, 1 \leq l \leq n, 1 \leq j_1 < \cdots < j_j \leq n\}.$$

Hence, if $p \in \ker \bar{\alpha}^*_n$, then

$$p(0; X^\beta) = 0, \quad |\beta| \leq k.$$

This implies that $p \in (\mathcal{M} \cap \mathcal{P})^{k+1}$. Thus, $\ker \bar{\alpha}^*_n \subset \mathcal{I}$. In particular, it follows that the $\mathcal{P}$-module $\mathcal{P} / (\mathcal{I} \cap \mathcal{P})$ is a subquotient of $(\mathcal{P} / \mathcal{I}_{\lambda_1} \cap \mathcal{P}) \otimes \cdots \otimes (\mathcal{P} / \mathcal{I}_{\lambda_n} \cap \mathcal{P})$. The final assertion now follows by application of Lemmas 2.16 and 2.17. □

2.2. Holomorphic families of continuous representations. We retain our assumption that Ω is an open subset of the finite dimensional complex linear space $\mathfrak{v}_{\mathbb{C}}$.

If X is a locally compact Hausdorff space, and V a locally convex (Hausdorff) space, then by $C(X, V)$ we denote the space of V-valued continuous functions on X, equipped with the topology of uniform convergence on compact subsets. If V is quasi-complete, then $C(X, V)$ is quasi-complete as well.

Let V be a quasi-complete locally convex space. The subspace $\mathcal{O}(\Omega, V)$ of V-valued holomorphic functions on Ω is closed in $C(\Omega, V)$ hence quasi-complete of its own right. For our further considerations, it is important to note that the algebraic tensor product $\mathcal{P}(\mathfrak{v}_{\mathbb{C}}) \otimes V$ is dense in $\mathcal{O}(\Omega, V)$, see Lemma A.1.

If V, W are locally convex spaces, we write $\mathrm{Hom}(V, W)$ for the space of continuous complex linear maps $V \to W$. This space, equipped with the strong operator topology, is locally convex again. Moreover, if V is barrelled and W quasi-complete, then $\mathrm{Hom}(V, W)$ is quasi-complete as well, see the paragraph before Lemma A.4. As usual, we write $\mathrm{End}(V)$ for $\mathrm{Hom}(V, V)$. Let U be a third locally convex space. Then the composition map

$$\beta : \mathrm{Hom}(U, V) \times \mathrm{Hom}(V, W) \to \mathrm{Hom}(U, W), \quad (A, B) \mapsto B \circ A$$

is bilinear and separately continuous. Moreover, if V is barrelled, then by the principle of uniform boundedness, β is continuous relative to (i.e., when restricted to) sets of the form $\mathrm{Hom}(U, V) \times C$, with $C \subset \mathrm{Hom}(V, W)$ compact; see Lemma A.4. Now assume that U, V and W are quasi-complete locally convex spaces. If U and V are barrelled, then it follows from the material in the appendix, see Lemma A.6, that the natural pointwise composition defines a bilinear map

$$\beta_* : \mathcal{O}(\Omega, \mathrm{Hom}(U, V)) \times \mathcal{O}(\Omega, \mathrm{Hom}(V, W)) \to \mathcal{O}(\Omega, \mathrm{Hom}(U, W)).$$

Definition 2.32. A holomorphic family of continuous representations of G over Ω is a pair (π, V) such that the following conditions are fulfilled:

(a) V is a Fréchet space.

(b) π is a continuous map from G to $\mathcal{O}(\Omega, \mathrm{End}(V))$ satisfying:

   (1) $\pi(g_1 g_2) = \pi(g_1)\pi(g_2)$, for $g_1, g_2 \in G$,
      $\pi(e_G) = 1_\pi$, where $1_\pi(\mu) = \mathrm{id}_V$ for all $\mu \in \Omega$;

   (2) for every $k \in K$, the $\mathrm{End}(V)$-valued function $\pi(k)$ is constant on Ω.

A holomorphic family of smooth representations of G over Ω is a family (π, V) as above such that in addition

$$(2.12) \qquad\qquad (g, \lambda) \mapsto \pi(g)(\lambda)v, \quad G \times \Omega \to V$$

is smooth for every $v \in V$.

Remark 2.33. Given $g \in G$ and $\lambda \in \Omega$ we agree to write $\pi_\lambda(g) := \pi(g)(\lambda)$. The condition that V is Fréchet ensures that the principle of uniform boundedness is valid. By application of this principle it follows that the map

$$G \times \Omega \times V \to V, \quad (g, \lambda, v) \mapsto \pi_\lambda(g)v$$

is continuous and holomorphic in λ. More generally we could have given the definition of a holomorphic family of continuous representations under the weaker assumption that V be quasi-complete and barrelled. All results of the present section and the next are in fact valid under this weaker assumption. However, in Section 2.4 the assumption that V is Fréchet will really be needed.

Definition 2.34. Let (π, V_π) and (ρ, V_ρ) be two holomorphic families of continuous representations of G over Ω. A holomorphic family of intertwining operators, T, between (π, V_π) and (ρ, V_ρ) is an element of $\mathcal{O}(\Omega, \operatorname{Hom}(V_\pi, V_\rho))$ satisfying

$$T\pi(g) = \rho(g)T, \quad g \in G.$$

Definition 2.35. The category HF_G is defined as follows.

(a) The objects are the holomorphic families of continuous representations of G over Ω.

(b) The morphisms are the holomorphic families of intertwining operators between two objects.

For any $(\pi, V_\pi) \in \operatorname{HF}_G$, the identity morphism is the holomorphic family 1_π of intertwining operators defined by

$$1_\pi(\lambda) := \operatorname{id}_{V_\pi}, \quad \lambda \in \Omega.$$

The composition $T' \circ T$ of two (composable) morphisms is given by pointwise composition: $(T' \circ T)_\lambda = T'_\lambda \circ T_\lambda$. It is again a holomorphic family of intertwining operators, by virtue of Lemma A.6.

If V and W are locally convex spaces, and E a finite dimensional complex linear space, then the map $(A_1, A_2) \mapsto A_1 \otimes A_2$ induces a linear isomorphism

$$\operatorname{End}(E) \otimes \operatorname{Hom}(V, W) \simeq \operatorname{Hom}(E \otimes V, E \otimes W),$$

which we shall use for identifying these spaces. Accordingly, if V and W are quasi-complete, and V barrelled, then

$$J^{(E)} : T \mapsto T^{(E)}$$

defines a continuous linear map

$$\mathcal{O}(\Omega, \operatorname{Hom}(V, W)) \to \mathcal{O}(\Omega, \operatorname{Hom}(E \otimes V, E \otimes W))$$

(see Remark A.7).

Definition 2.36. Let E be a finite dimensional $\mathcal{O}_0$-module.

(a) For $(\pi, V) \in \operatorname{HF}_G$, we define $\pi^{(E)}$ to be the continuous map

$$\pi^{(E)} = J^{(E)} \circ \pi : g \mapsto \pi(g)^{(E)}, \quad G \to \mathcal{O}(\Omega, \operatorname{End}(E \otimes V)).$$

(b) For any morphism $T : (\pi, V_\pi) \to (\rho, V_\rho)$ of HF_G, we define

$$T^{(E)} := J^{(E)}(T) \in \mathcal{O}(\Omega, \operatorname{Hom}(E \otimes V_\pi, E \otimes V_\rho)).$$

Proposition 2.37. *Let E be a finite dimensional $\mathcal{O}_0$-module. Then $J^{(E)}$ defines a functor from HF_G to itself.*

Proof. Let $(\pi, V) \in \mathrm{HF}_G$. We will first show that $(\pi^{(E)}, E \otimes V) \in \mathrm{HF}_G$. Let β denote the composition map in $\mathrm{End}(V)$. It is a bilinear map, which preserves holomorphy on Ω by Lemma A.6. Moreover, by Corollary A.12 we have, for all $g_1, g_2 \in G$,

$$\pi^{(E)}(g_1 g_2) = \pi(g_1 g_2)^{(E)} = (\pi(g_1)\pi(g_2))^{(E)}$$
$$= \pi(g_1)^{(E)}\pi(g_2)^{(E)} = \pi^{(E)}(g_1)\pi^{(E)}(g_2).$$

Also, $\pi^{(E)}(e_G) = 1_\pi^{(E)} = 1_{\pi^{(E)}}$. For $k \in K$, the $\mathrm{End}(V)$-valued function $\pi(k)$ is a constant on Ω, and therefore, so is the $\mathrm{End}(E \otimes V)$-valued function $\pi^{(E)}(k) = \pi(k)^{(E)}$. Hence (1) and (2) of Definition 2.32 (b) are fulfilled. It remains to show that, for any morphism $T : (\pi, V_\pi) \to (\rho, V_\rho)$ of HF_G, $T^{(E)}$ is a morphism of HF_G from $(\pi^{(E)}, E \otimes V_\pi)$ to $(\rho^{(E)}, E \otimes V_\rho)$. For this it suffices to show that, for $g \in G$,

$$T^{(E)}\pi^{(E)}(g) = \rho^{(E)}(g)T^{(E)}.$$

This follows from Corollary A.12, see also the first part of the present proof. □

The category HF_G has a null object, $(0, \{0\})$, and one can define a biproduct in HF_G as follows. Let $(\pi, V_\pi), (\rho, V_\rho) \in \mathrm{HF}_G$. Set, for any $g \in G$, $\lambda \in \Omega$ and $(v, w) \in V_\pi \oplus V_\rho$,

$$(\pi \oplus \rho)_\lambda(g)(v, w) := (\pi_\lambda(g)v, \rho_\lambda(g)w).$$

Then $(\pi \oplus \rho, V_\pi \oplus V_\rho)$ defines an object of HF_G. We define the full subcategory HF_G^∞ of HF_G by stipulating that the objects are the holomorphic families of smooth representations over Ω (the set of morphisms between objects in HF_G^∞ coincides with the set of morphisms between the objects viewed as objects for the bigger category HF_G. Likewise, the full subcategory $\mathrm{HF}_G^{\infty,\mathrm{adm}}$ of HF_G^∞ consists of the objects (π, V) of HF_G^∞ with V_K admissible.

If $(\pi, V) \in \mathrm{HF}_G$, then the identity morphism 1_π is constant as an $\mathrm{End}(V)$-valued function on Ω. This enables us to define a particular subcategory.

Definition 2.38. The subcategory $^\circ\mathrm{HF}_G$ of HF_G is defined as follows.

(a) The objects are the objects of HF_G.

(b) If (π, V) and (ρ, W) are objects of HF_G, then the associated collection of $^\circ\mathrm{HF}_G$-morphisms consists of all HF_G-morphisms T in $\mathcal{O}(\Omega, \mathrm{Hom}(V, W))$ which are constant as a function on Ω.

In a similar fashion, we define the subcategories $^\circ\mathrm{HF}_G^\infty$ of HF_G^∞ and $^\circ\mathrm{HF}_G^{\infty,\mathrm{adm}}$ of $\mathrm{HF}_G^{\infty,\mathrm{adm}}$.

Note that $^\circ\mathrm{HF}_G^{\infty,\mathrm{adm}}$ is a full subcategory of $^\circ\mathrm{HF}_G^\infty$, which in turn is a full subcategory of $^\circ\mathrm{HF}_G$.

Remark 2.39. For every $E \in \mathrm{FM}_{\mathcal{O}_0}$, the functor $J^{(E)} : \mathrm{HF}_G \to \mathrm{HF}_G$ leaves all subcategories $\mathrm{HF}_G^\infty, \mathrm{HF}_G^{\infty,\mathrm{adm}}, {}^\circ\mathrm{HF}_G, {}^\circ\mathrm{HF}_G^\infty$ and $^\circ\mathrm{HF}_G^{\infty,\mathrm{adm}}$ invariant.

Lemma 2.40. *Let $\psi : E \to E'$ be a morphism in $\mathrm{FM}_{\mathcal{O}_0}$ and let (π, V_π) be a holomorphic family of representations. Then $\psi \otimes 1_\pi$ intertwines the families $\pi^{(E)}$ and $\pi^{(E')}$.*

Proof. If $f \in \mathcal{O}(\Omega)$, then from the definitions it readily follows that

$$\psi \circ f^{(E)}(\mu) = f^{(E')}(\mu) \circ \psi, \quad \mu \in \Omega.$$

From this and the identification $\mathrm{End}(E) \otimes \mathrm{End}(V_\pi) \simeq \mathrm{End}(E \otimes V_\pi)$, it follows that for all $g \in G$ the map $\psi \otimes 1_\pi : E \otimes V_\pi \to E' \otimes V_\pi$ intertwines $\pi_\mu^{(E)}(g) = \pi(g)^{(E)}(\mu)$ with $\pi_\mu^{(E')}(g) = \pi(g)^{(E')}(\mu)$.   □

The above lemma justifies the following definition.

Definition 2.41. Given any $(\pi, V_\pi) \in \mathrm{obj}(\mathrm{HF}_G)$, we define the functor X_π from $\mathrm{FM}_{\mathcal{O}_0}$ to $^\circ\mathrm{HF}_G$ as follows.

(a) For an object $E \in \mathrm{FM}_{\mathcal{O}_0}$, the associated object of $^\circ\mathrm{HF}_G$ is given by

$$X_\pi(E) := (\pi^{(E)}, E \otimes V_\pi).$$

(b) For a morphism $\psi : E \to E'$ of $\mathrm{FM}_{\mathcal{O}_0}$, the associated morphism of $^\circ\mathrm{HF}_G$ is given by

$$X_\pi(\psi) := \psi \otimes 1_\pi : (\pi^{(E)}, E \otimes V_\pi) \to (\pi^{(E')}, E' \otimes V_\pi).$$

Remark 2.42. It is readily checked that X_π is a functor. Indeed, X_π respects composition of morphisms, and $X_\pi(\mathrm{id}_E) = \mathrm{id}_{E \otimes V_\pi} = 1_{\pi^{(E)}}$.

As $\mathrm{FM}_{\mathcal{O}_0}$ is an abelian category, we have the usual notion of finite direct sums and exact sequences in $\mathrm{FM}_{\mathcal{O}_0}$.

The category Vect of complex vector spaces is abelian. If $T : (\pi, V) \to (\rho, W)$ is a morphism in $^\circ\mathrm{HF}_G$, then there exists a unique linear map $T_0 : V \to W$ such that $T(\lambda) = T_0$ for all $\lambda \in \Omega$. By abuse of notation we will write T for T_0. We thus have a forgetful functor $^\circ\mathrm{HF}_G \to$ Vect.

The category $^\circ\mathrm{HF}_G$ is not abelian. Nevertheless, we may use the forgetful functor to define exact sequences.

Definition 2.43. A sequence $((\pi_k, V_k), T_k)$, $p \le k \le q$, in the category $^\circ\mathrm{HF}_G$, where $p, q \in \mathbb{Z}$, $p < q$, will be called exact if its image under the forgetful functor $^\circ\mathrm{HF}_G \to$ Vect is exact, i.e., the image of T_{k-1} equals the kernel of T_k for all $p < k \le q$.

Lemma 2.44. *Let $(\pi, V_\pi) \in \mathrm{obj}(\mathrm{HF}_G)$. Then the functor $X_\pi : \mathrm{FM}_{\mathcal{O}_0} \to {}^\circ\mathrm{HF}_G$ has the following properties.*

(a) *It sends every short exact sequence $0 \to E \to E' \to E'' \to 0$ to a similar short exact sequence in $^\circ\mathrm{HF}_G$.*

(b) *It sends every exact sequence of the form $0 \to E \to E'$ in $\mathrm{FM}_{\mathcal{O}_0}$ to an exact sequence of similar form in $^\circ\mathrm{HF}_G$.*

(c) *It sends every exact sequence of the form $E' \to E'' \to 0$ in $\mathrm{FM}_{\mathcal{O}_0}$ to an exact sequence of similar form in $^\circ\mathrm{HF}_G$.*

(d) *It sends a direct sum of the form $E = E_1 \oplus E_2$ in $\mathrm{FM}_{\mathcal{O}_0}$ to a similar direct sum in $^\circ\mathrm{HF}_G$.*

Proof. Let FDVect denote the (abelian) category of finite dimensional complex linear spaces. Then we have a forgetful functor $\mathcal{F}$ from $\mathrm{FM}_{\mathcal{O}_0}$ to FDVect. A sequence in $\mathrm{FM}_{\mathcal{O}_0}$ is exact if and only if its image under $\mathcal{F}$ is exact in FDVect. According to the definition above, the forgetful functor $\mathcal{F}' : {}^\circ\mathrm{HF}_G \to \mathrm{Vect}$ has a similar property. Given $U \in \mathrm{Vect}$ we define the functor $X_U : \mathrm{FDVect} \to \mathrm{Vect}$ by $X_U(E) = E \otimes U$ for an object E of FDVect. A morphism $f : E \to E'$ in $\mathrm{FM}_{\mathcal{O}_0}$ is mapped to $X_U(f) := f \otimes \mathrm{id}_U : E \otimes U \to E' \otimes U$. It is readily seen that the functor X_U is exact, and has the obvious properties analogous to (a)–(d). Since $\mathcal{F}' \circ X_\pi = X_{\mathcal{F}'(\pi,V)} \circ \mathcal{F}$, assertions (a), (b) and (c) of the lemma follow. For assertion (d) it remains to be shown that each natural embedding $\mathrm{i}_j : E_j \to E$, for $j = 1, 2$, is mapped to an embedding $X_\pi(\mathrm{i}_j)$ from $X_\pi(E_j)$ onto a closed subspace of $X_\pi(E)$. Let $p_j : E \to E_j$ be the natural projection. Then by exactness of the sequence $0 \to E_1 \to E \to E_2 \to 0$ it follows from the established assertion (a) that $0 \to X_\pi(E_1) \to X_\pi(E) \to X_\pi(E_2) \to 0$ is exact. This implies that $X_\pi(\mathrm{i}_1)$ has closed image in $X_\pi(E)$. Likewise, $X_\pi(\mathrm{i}_2)$ is seen to have close image. $\qquad\square$

Remark 2.45. In view of Remark 2.39, Lemma 2.44 has an obvious generalization to objects from HF_G^∞ and from $\mathrm{HF}_G^{\infty,\mathrm{adm}}$.

Proposition 2.46. (a) *Let $E_1, \ldots, E_n \in \mathrm{FM}_{\mathcal{O}_0}$ and set $E := E_1 \otimes \cdots \otimes E_n$. Then there exist $N, k \in \mathbb{N}$ such that, for any object (π, V_π) in HF_G (resp. $\mathrm{HF}_G^\infty, \mathrm{HF}_G^{\infty,\mathrm{adm}}$), the family $(\pi^{(E)}, E \otimes V_\pi)$ is a quotient of*

$$\left(\mathrm{id}_{\mathbb{C}^N} \otimes \pi^{(\mathcal{O}_0/\mathcal{M}^{k+1})}, \mathbb{C}^N \otimes (\mathcal{O}_0/\mathcal{M}^{k+1} \otimes V_\pi) \right)$$

in the category $^\circ\mathrm{HF}_G$ (resp. $^\circ\mathrm{HF}_G^\infty, {}^\circ\mathrm{HF}_G^{\infty,\mathrm{adm}}$).

(b) *Let $E \in \mathrm{FM}_{\mathcal{O}_0}$. Then there exist $\lambda_1, \ldots, \lambda_n \in \mathfrak{v}_{\mathbb{C}}$ such that, for any object (π, V_π) in HF_G (resp. $\mathrm{HF}_G^\infty, \mathrm{HF}_G^{\infty,\mathrm{adm}}$), the family $(\pi^{(E)}, E \otimes V_\pi)$ is a subquotient of*

$$\left(\pi^{((\mathcal{O}_0/\mathcal{I}_{\lambda_1}) \otimes \cdots \otimes (\mathcal{O}_0/\mathcal{I}_{\lambda_n}))}, (\mathcal{O}_0/\mathcal{I}_{\lambda_1}) \otimes \cdots \otimes (\mathcal{O}_0/\mathcal{I}_{\lambda_n}) \otimes V_\pi \right)$$

in the category $^\circ\mathrm{HF}_G$ (resp. $^\circ\mathrm{HF}_G^\infty, {}^\circ\mathrm{HF}_G^{\infty,\mathrm{adm}}$).

Proof. This follows from Proposition 2.31 combined with Lemma 2.44. $\qquad\square$

2.3. Holomorphic families of admissible $(\mathfrak{g}, K)$-modules. For the purpose of this paper, it is convenient to introduce the following notion of holomorphic families of admissible $(\mathfrak{g}, K)$-modules. Recall that Ω is an open subset of the finite dimensional complex linear space $\mathfrak{v}_{\mathbb{C}}$.

Definition 2.47. A holomorphic family of admissible $(\mathfrak{g}, K)$-modules over Ω is a triple (π_1, π_2, V) satisfying the following conditions.

(a) V is a complex vector space.

(b) π_1 is a map from $U(\mathfrak{g}) \times \Omega$ to $\mathrm{End}(V)$ such that

  (1) for each $\lambda \in \Omega$, the map $\pi_1(\cdot, \lambda)$ is a Lie algebra homomorphism;

  (2) for each $u \in U(\mathfrak{g})$ and every $v \in V$, the vector subspace of V generated by the $\pi_1(u, \lambda)v$, $\lambda \in \Omega$, is finite dimensional and the map $\lambda \mapsto \pi_1(u, \lambda)v$ is holomorphic from Ω into this subspace.

(c) π_2 is a Lie group homomorphism from K to $\mathrm{GL}(V)$ such that

  (1) for each $v \in V$, the vector subspace of V generated by the $\pi_2(k)v$, $k \in K$, is finite dimensional;

  (2) for all $v \in V$, $\lambda \in \Omega$, $u \in U(\mathfrak{g})$, $k \in K$ and $X \in \mathfrak{k}$,

$$\pi_1(\mathrm{Ad}(k)u, \lambda)v = \pi_2(k)\pi_1(u, \lambda)v,$$

$$\frac{d}{dt}\big[\pi_2(\exp(tX))v\big]\big|_{t=0} = \pi_1(X, \lambda)v.$$

(d) For every $\delta \in K^\wedge$, the K-isotypic component $\pi_{2,\delta}$ of π_2 of type δ is finite dimensional.

Remark 2.48. Let (π_1, π_2, V) be as in the above definition. Given $\lambda \in \Omega$ we agree to write $\pi_{1\lambda}$ for the map $\pi_1(\cdot, \lambda) : U(\mathfrak{g}) \to \mathrm{End}(V)$. Then $(\pi_{1\lambda}, \pi_2, V)$ is an admissible $(\mathfrak{g}, K)$-module.

We also need the following notion of holomorphic family of intertwining operators.

Definition 2.49. Let (π_1, π_2, V) and (ρ_1, ρ_2, W) be two holomorphic families of admissible $(\mathfrak{g}, K)$-modules over Ω. A holomorphic family of intertwining operators between (π_1, π_2, V) and (ρ_1, ρ_2, W) is a function $T : \Omega \to \mathrm{Hom}(V, W)$ satisfying the following conditions.

(a) $T(\lambda)\pi_1(u, \lambda) = \rho_1(u, \lambda)T(\lambda)$, for all $\lambda \in \Omega$ and $u \in U(\mathfrak{g})$.

(b) $T(\lambda)\pi_2(k) = \rho_2(k)T(\lambda)$, for all $\lambda \in \Omega$ and $k \in K$.

(c) For any finite dimensional subspace $\tilde{V}$ of V, there exists a finite dimensional subspace $\tilde{W}$ of W such that $T(\lambda)(\tilde{V}) \subset \tilde{W}$ for all $\lambda \in \Omega$, and the associated function $\lambda \mapsto T(\lambda)|_{\tilde{V}}$ belongs to $\mathcal{O}(\Omega, \mathrm{Hom}(\tilde{V}, \tilde{W}))$.

Let $\mathrm{HF}^{\mathrm{adm}}_{(\mathfrak{g},K)}$ denote the corresponding category of holomorphic families of admissible $(\mathfrak{g}, K)$-modules.

Let (π, V) be an object in the category $\mathrm{HF}^{\infty,\mathrm{adm}}_G$. Then for each $\lambda \in \Omega$, $u \in U(\mathfrak{g})$ and $v \in V$ we may define

$$\pi(u)(\lambda)v := L_{u^\vee}(g \mapsto \pi(g)(\lambda)v)|_{g=e}.$$

We put $\pi_1(u, \lambda) = \pi(u)(\lambda)|_{V_K}$ and $\pi_2(k) = \pi(k)|_{V_K}$.

Lemma 2.50. *Let $(\pi, V) \in \mathrm{HF}^{\infty,\mathrm{adm}}_G$ and let π_1, π_2 be defined as above. Then*

$$(\pi, V)_K := (\pi_1, \pi_2, V_K)$$

is a holomorphic family in $\mathrm{HF}^{\mathrm{adm}}_{(\mathfrak{g},K)}$. Moreover, $(\cdot)_K : (\pi, V) \mapsto (\pi, V)_K$ defines a functor $\mathrm{HF}^{\infty,\mathrm{adm}}_G \to \mathrm{HF}^{\mathrm{adm}}_{(\mathfrak{g},K)}$.

Proof. It follows from the smoothness of the map (2.12), the continuity of the map $L_{u^\vee} : C^\infty(G) \to C^\infty(G)$ and the holomorphy with respect to λ, that $\varphi : \lambda \mapsto \pi(u)(\lambda)v$ defines a holomorphic function $\Omega \to V$. Let now $v \in V_K$ and let $\vartheta_2 \subset \widehat{K}$ denote the set of K-types appearing in the $\pi(K)$-span of v. Moreover, let ϑ_1 be the set of K-types which appear in the $\mathrm{Ad}(K)$-span of u. Finally, let ϑ be the union of the sets of K-types of $\delta_1 \otimes \delta_2$, for $\delta_j \in \vartheta_j$, $j = 1, 2$. Then φ has its image contained in the finite dimensional subspace $V_\vartheta \subset V_K$. It follows that φ is holomorphic as a function $\Omega \to V_\vartheta$. This shows that (π_1, π_2, V_K) satisfies condition (2) of Definition 2.47 (b). The other conditions of that definition are pointwise in λ, and therefore consequences of the standard theory of assigning the $(\mathfrak{g}, K)$-module of K-finite vectors to an admissible smooth Fréchet representation, see for instance [10, Lemma 3.3.5]. The latter assignment is a functor from the category of admissible smooth Fréchet representations to the category of Harish-Chandra modules. This implies that $(\pi, V) \mapsto (\pi, V)_K$ has functorial properties which are pointwise in λ. This in turn is readily seen to imply that $(\,\cdot\,)_K$ is a functor as stated. $\qquad\square$

We will now discuss the functor $J^{(E)}$ on the level of holomorphic families of admissible $(\mathfrak{g}, K)$-modules.

Definition 2.51. Let $E \in \mathrm{FM}_{\mathcal{O}_0}$. For any $(\pi_1, \pi_2, V) \in \mathrm{HF}^{\mathrm{adm}}_{(\mathfrak{g},K)}$ the triple

$$\left(J^{(E)}\pi_1, J^{(E)}\pi_2, J^{(E)}V \right),$$

also denoted $(\pi_1^{(E)}, \pi_2^{(E)}, V^{(E)})$, is defined as follows.

(a) $J^{(E)}V := E \otimes V$.

(b) $J^{(E)}\pi_1$ is the map from $U(\mathfrak{g}) \times \Omega$ to $\mathrm{End}(E \otimes V)$ given by

$$J^{(E)}\pi_1(u, \lambda)e \otimes v := (\pi_1(u, \,\cdot\,)v)^{(E)}(\lambda)e,$$

for $u \in U(\mathfrak{g}), \lambda \in \Omega, e \in E$, and $v \in V$.

(c) $J^{(E)}\pi_2 := \mathrm{id}_E \otimes \pi_2$.

We now have functors $J^{(E)}$ on $\mathrm{HF}^{\infty,\mathrm{adm}}_G$ and $\mathrm{HF}^{\mathrm{adm}}_{(\mathfrak{g},K)}$. They are linked as follows.

Lemma 2.52. (a) *For any $E \in \mathrm{FM}_{\mathcal{O}_0}$, the assignment $J^{(E)}$ defines a functor from* $\mathrm{HF}^{\mathrm{adm}}_{(\mathfrak{g},K)}$ *to itself.*

(b) *The following diagram commutes:*

$$
\begin{array}{ccc}
\mathrm{HF}^{\infty,\mathrm{adm}}_G & \xrightarrow{\ (\cdot)_K\ } & \mathrm{HF}^{\mathrm{adm}}_{(\mathfrak{g},K)} \\
{\scriptstyle J^{(E)}}\big\downarrow & & \big\downarrow{\scriptstyle J^{(E)}} \\
\mathrm{HF}^{\infty,\mathrm{adm}}_G & \xrightarrow[\ (\cdot)_K\ ]{} & \mathrm{HF}^{\mathrm{adm}}_{(\mathfrak{g},K)} \,.
\end{array}
$$

Proof. Assertion (a) follows by similar arguments as in the proof of Proposition 2.37. For (b), assume that (π, V) is a family in $\mathrm{HF}^{\infty,\mathrm{adm}}_G$. Put $(\pi, V)_K = (\pi_1, \pi_2, V_K)$, then

$$J^{(E)}((\pi, V)_K) = \left(\pi_1^{(E)}, \pi_2^{(E)}, (V_K)^{(E)} \right).$$

On the other hand, $J^{(E)}(\pi, V) = (\pi^{(E)}, V^{(E)})$ and

$$\left(J^{(E)}(\pi, V)\right)_K = \left((\pi^{(E)})_1, (\pi^{(E)})_2, (V^{(E)})_K\right)$$

Now $V^{(E)} = E \otimes V$ is equipped with the K-action on the second component, so that $(V^{(E)})_K = E \otimes V_K = (V_K)^{(E)}$. Moreover,

$$\pi_2^{(E)}(k) = 1_E \otimes \pi_2(k) = 1_E \otimes \pi(k, \lambda)|_{V_K} = \pi^{(E)}(k, \lambda)|_{E \otimes V_K} = (\pi^{(E)})_2(k).$$

It remains to establish the identity

$$(2.13) \qquad \pi_1^{(E)} = (\pi^{(E)})_1.$$

Since both are representations of $U(\mathfrak{g})$ in $E \times V_K$, it suffices to check the identity on a fixed element $X \in \mathfrak{g}$. Fix $e \in E$ and $v \in V$. We first observe that

$$\pi_1^{(E)}(X)v = (\pi_1(X, \cdot)v)^{(E)} = \left[\frac{d}{dt}\pi(\exp tX)(\cdot)v\right]^{(E)}\Big|_{t=0}.$$

In view of the natural identification $\mathrm{End}(E) \otimes V \simeq \mathrm{Hom}(E, E \otimes V)$, we note that

$$\begin{aligned}
(\pi^{(E)})_1(X, \cdot)v &= \frac{d}{dt}\left[\pi^{(E)}(\exp tX)(\cdot)v\right]\Big|_{t=0} \\
&= \frac{d}{dt}\left[(\pi(\exp tX)(\cdot)v)^{(E)}\right]\Big|_{t=0}.
\end{aligned}$$

The identity (2.13) now follows by application of the lemma below. □

Lemma 2.53. *Let V be a quasi-complete locally convex space, and let $\varphi : \mathbb{R} \times \Omega \to V$ be a C^1-map which is holomorphic in the second variable. Then*

$$(2.14) \qquad \frac{d}{dt}\left[\varphi(t, \cdot)^{(E)}\right] = \left[\frac{\partial \varphi}{\partial t}(t, \cdot)\right]^{(E)}.$$

Proof. Let S be the space of C^1-maps $\mathbb{R} \times \Omega \to V$, equipped with the usual quasi-complete topology. Let S_0 be the subspace consisting of functions in S which are holomorphic in the second variable. Then S_0 is closed in S, hence quasi-complete. The identity (2.14) at (t, λ) can be viewed as an identity of continuous linear functionals on S_0. Hence, it suffices to check the identity on the dense subspace $C^1(\mathbb{R}) \otimes \mathcal{O}(\Omega) \otimes V$. This amounts to checking whether

$$\left(\frac{d}{dt} \otimes I \otimes I\right) \circ (I \otimes J^{(E)} \otimes I) = (I \otimes J^{(E)} \otimes I) \circ \left(\frac{d}{dt} \otimes I \otimes I\right).$$

The latter is obvious. □

2.4. Parabolic induction. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition associated with the maximal compact subgroup K and let θ be the associated involution of G. Let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal abelian subspace, and let $A = \exp \mathfrak{a}$. Let $\mathcal{P}(A)$ denote the collection of parabolic subgroups of G containing A. Let $\mathcal{L}(A)$ denote the collection of θ-stable Levi components of parabolic subgroups from $\mathcal{P}(A)$.

Let υ be a finite dimensional real linear space, and Ω an open subset of its complexification. For $L \in \mathcal{L}(A)$ we denote by HF_L the category defined as in Definition 2.32, with the group L in place of G.

The parabolic induction functor from HF_L^∞ to HF_G^∞ is defined as follows. Let $(\xi, V_\xi) \in \mathrm{HF}_L^\infty$ be a holomorphic family of smooth representations of L defined over Ω. Denote by $\bar{\pi}_{P,\xi_\lambda}$ the right regular representation of G on $C^\infty(G\colon P\colon \xi_\lambda)$, where

$$C^\infty(G\colon P\colon \xi_\lambda) := \left\{\psi \in C^\infty(G, V_\xi) \mid \psi(nmg) = \xi_\lambda(m)\psi(g),\ (g, m, n) \in G \times L \times N_P\right\}.$$

Let

(2.15)

$$C^\infty(K\colon \xi) := \left\{\psi \in C^\infty(K, V_\xi) \mid \psi(mk) = \xi(m)\psi(k),\ (k, m) \in K \times (K \cap L)\right\}.$$

Restriction of functions to K induces a continuous linear isomorphism between these spaces. Let π_{P,ξ_λ} denote the representation of G on $C^\infty(K\colon \xi)$, obtained from $\bar{\pi}_{P,\xi_\lambda}$ by transfer of structure, and set, for $g \in G$, $\lambda \in \upsilon_\mathbb{C}$,

$$\pi_{P,\xi,\lambda}(g) := \pi_{P,\xi_\lambda}(g).$$

Then it is readily seen that $(\pi_{P,\xi}, C^\infty(K\colon \xi))$ belongs to HF_G^∞. Here, the information that V_ξ is a Fréchet space is needed to conclude that $C^\infty(K\colon \xi)$ is Fréchet, in particular barrelled.

Let W_1, W_2 and W_3 be quasi-complete locally convex spaces, and let α be a continuous linear map from W_2 to W_3. Then the map

$$L_\alpha : \varphi \mapsto \alpha \circ \varphi, \quad \mathrm{Hom}(W_1, W_2) \to \mathrm{Hom}(W_1, W_3)$$

is continuous linear. It readily follows that the map

$$\mathrm{id}_{\mathcal{O}(\Omega)} \,\widehat{\otimes}\, L_\alpha : \mathcal{O}(\Omega, \mathrm{Hom}(W_1, W_2)) \to \mathcal{O}(\Omega, \mathrm{Hom}(W_1, W_3))$$

given by $f \mapsto L_\alpha \circ f$ is continuous linear. The notation for this map is explained by the fact that it may be viewed as the unique continuous linear extension of $\mathrm{id}_{\mathcal{O}(\Omega)} \otimes L_\alpha$.

Lemma 2.54. *Let L_α be as above and let E be a finite dimensional $\mathcal{O}_0$-module. Then we have the identity*

$$J^{(E)} \circ (\mathrm{id}_{\mathcal{O}(\Omega)} \,\widehat{\otimes}\, L_\alpha) = (\mathrm{id}_{\mathcal{O}(\Omega)} \,\widehat{\otimes}\, \mathrm{id}_{\mathrm{End}(E)} \,\widehat{\otimes}\, L_\alpha) \circ J^{(E)}$$

of maps

$$\mathcal{O}(\Omega, \mathrm{Hom}(W_1, W_2)) \to \mathcal{O}(\Omega, \mathrm{End}(E) \otimes \mathrm{Hom}(W_1, W_3)).$$

Proof. By continuity of the expressions on both sides of the identity it suffices to prove the identity on the dense subspace $\mathcal{O}(\Omega) \otimes \mathrm{Hom}(W_1, W_2)$ of $\mathcal{O}(\Omega, \mathrm{Hom}(W_1, W_2))$. But then the identity becomes obvious, in view of Remark A.7, last line. □

The following result may be phrased as 'derivation commutes with induction'.

Proposition 2.55. *Let E be a finite dimensional $\mathcal{O}_0$-module, P a parabolic subgroup with Levi component L, and $(\xi, V_\xi) \in \mathrm{HF}_L^\infty$. Then*

$$\pi_{P,\xi}^{(E)} = \pi_{P,\xi^{(E)}}.$$

Proof.   Let $g \in G$ and $k \in K$. Put $\alpha(k) := \mathrm{ev}_k : C^\infty(K\!:\!\xi) \to V_\xi$. Let

$$L_{\alpha(k)} : \mathrm{End}(C^\infty(K\!:\!\xi)) \to \mathrm{Hom}(C^\infty(K\!:\!\xi), V_\xi)$$

be defined as in the previous lemma. Accordingly,

$$(2.16) \qquad (\mathrm{id}_{\mathscr{O}(\Omega)} \,\widehat{\otimes}\, \mathrm{id}_E \,\widehat{\otimes}\, L_{\alpha(k)})(\pi_{P,\xi}^{(E)}(g)) = \big((\mathrm{id}_{\mathscr{O}(\Omega)} \,\widehat{\otimes}\, L_{\alpha(k)})(\pi_{P,\xi}(g))\big)^{(E)}.$$

Write $g = n p(g) \kappa(g)$ uniquely via the decomposition $G = N_P A_P (M_P \cap \exp \mathfrak{p}) K$, where $n \in N_P$, $p(g) \in A_P(M \cap \exp \mathfrak{p})$ and $\kappa(g) \in K$. We then have the following identity:

$$(\mathrm{id}_{\mathscr{O}(\Omega)} \,\widehat{\otimes}\, L_{\alpha(k)})(\pi_{P,\xi}(g)) = \xi(p(kg)) \circ L_{\alpha(\kappa(kg))}.$$

Hence, it follows from (2.16) that

$$\begin{aligned}
(\mathrm{id}_{\mathscr{O}(\Omega)} \,\widehat{\otimes}\, \mathrm{id}_E \,\widehat{\otimes}\, L_{\alpha(k)})(\pi_{P,\xi}^{(E)}(g)) &= \big(\xi(p(kg)) \circ L_{\alpha(\kappa(kg))}\big)^{(E)} \\
&= \big(\xi^{(E)}(p(kg)) \circ L_{\alpha(\kappa(kg))}\big) \\
&= (\mathrm{id}_{\mathscr{O}(\Omega)} \,\widehat{\otimes}\, \mathrm{id}_E \,\widehat{\otimes}\, L_{\alpha(k)})(\pi_{P,\xi^{(E)}}(g))
\end{aligned}$$

and the statement follows.   □

3. The Arthur–Campoli relations

Given a parabolic subgroup $P \in \mathscr{P}(A)$ we denote its Langlands decomposition by $P = M_P A_P N_P$.

Let (ξ, V_ξ) be a smooth, irreducible and admissible Fréchet representation of M_P. Given $\lambda \in \mathfrak{a}_{P\mathbb{C}}^*$ we denote by $\xi \otimes \lambda$ the smooth representation of $L_P := M_P A_P$ in V_ξ defined by

$$\xi \otimes \lambda(ma) = a^{\lambda + \rho_P} \xi(m).$$

As usual, here $\rho_P \in \mathfrak{a}_P^*$ is defined by $\rho_P = \frac{1}{2} \mathrm{tr}(\mathrm{ad}(\,\cdot\,)|_{\mathfrak{n}_P})$. The associated induced representation $\bar{\pi}_{P,\xi\otimes\lambda}$ of G in $C^\infty(G\!:\!P\!:\!\xi \otimes \lambda)$ is defined as in Section 2.4, with $v = \mathfrak{a}^*$, $\Omega = \mathfrak{a}_{\mathbb{C}}^*$ and $\xi_\lambda = \xi \otimes \lambda$. The family of these representations may be viewed as a holomorphic family $\pi_{P,\xi\otimes(\cdot)}$ of smooth representations of G over $\Omega = \mathfrak{a}_{\mathbb{C}}^*$ on the fixed space $C^\infty(K\!:\!\xi)$, defined in (2.15). As in the mentioned section, we agree to write $\pi_{P,\xi,\lambda} = \pi_{P,\xi\otimes\lambda}$. If P is a minimal parabolic subgroup, the representations $\pi_{P,\xi,\lambda}$ just defined are called representations of the smooth minimal principal series of G.

Let now $P \in \mathscr{P}(A)$ be arbitrary again. Then M_P is a group of the Harish-Chandra class, with maximal compact subgroup $K_P := M_P \cap K$. Two continuous admissible M_P-representations of finite length in a quasi-complete locally convex space are said to be infinitesimally equivalent if their Harish-Chandra modules are equivalent as $(\mathfrak{m}_P, K_P)$-modules. For each equivalence class ω of irreducible unitary representations of M_P, we fix a smooth admissible Fréchet representation $\xi = \xi_\omega$ which is infinitesimally equivalent to a representation of class ω, and which is topologically equivalent to a closed subrepresentation of a representation of the smooth minimal principal series of M_P. Indeed, this is possible by the subrepresentation theorem for the group M_P. The set of all these chosen representations ξ_ω is denoted by $M_P^\wedge$. Thus, $\omega \mapsto \xi_\omega$ defines a bijection from the set of equivalence classes of irreducible unitary representations of M_P onto $M_P^\wedge$.

Remark 3.1. In view of the theory of the Casselman–Wallach globalization functor, see [11, Section 11], the representation ξ_ω is a smooth Fréchet globalization of moderate growth of the Harish-Chandra module of any representative of ω. This characterization makes the choice of $M_P^\wedge$ more natural, but will not be needed in the present paper.

We denote by $M_{P,\mathrm{ds}}^\wedge$ the subset of $M_P^\wedge$ consisting of the representations ξ_ω, with ω a discrete series representation. In particular, if $P \in \mathcal{P}(A)$ is minimal, then M_P equals the centralizer M of $\mathfrak{a}$ in K, and $M_{\mathrm{ds}}^\wedge = M^\wedge$. For each $(\xi, V_\xi) \in M_{P,\mathrm{ds}}^\wedge$ we put

$$S(P,\xi) := \mathrm{End}(C^\infty(K:\xi))_{K \times K},$$

and we define the algebraic direct sum of linear spaces

$$(3.1) \qquad S(P) := \bigoplus_{\xi \in M_{P,\mathrm{ds}}^\wedge} S(P,\xi).$$

3.1. The Arthur–Campoli relations. Fix a minimal parabolic subgroup P_0 in $\mathcal{P}(A)$ and let $P_0 = MAN_0$ be its Langlands decomposition. In [1, Chapter III, §4], Arthur defines a Paley–Wiener space involving all minimal parabolic subgroups containing A. This definition is given in terms of Paley–Wiener growth conditions on the one hand, and the so-called Arthur–Campoli conditions on the other. In [3, Theorem 3.6] it is shown that the Arthur Paley–Wiener space is isomorphic to one defined in terms of the single minimal parabolic subgroup P_0. We shall now describe the Arthur–Campoli relations in the context of the operator valued Fourier transform $f \mapsto \hat{f}(P_0)$.

For $f \in C_c^\infty(G, K)$ and $\xi \in M^\wedge$, the Fourier transform $\hat{f}(P_0, \xi) \in \mathcal{O}(\mathfrak{a}_\mathbb{C}^*) \otimes S(P_0, \xi)$ is defined by

$$(3.2) \qquad \hat{f}(P_0, \xi, \lambda) := \int_G f(x) \pi_{P_0, \xi, \lambda}(x)\, dx, \quad \lambda \in \mathfrak{a}_\mathbb{C}^*.$$

Then $f \mapsto \hat{f}(P_0)$ maps $C_c^\infty(G, K)$ into $\mathcal{O}(\mathfrak{a}_\mathbb{C}^*) \otimes S(P_0)$.

We define Σ to be the set of 4-tuples (ξ, ψ, λ, u) with $\xi \in M^\wedge$, $\psi \in S(P_0, \xi)_{K \times K}^*$, $\lambda \in \mathfrak{a}_\mathbb{C}^*$ and $u \in S(\mathfrak{a}^*)$. An Arthur–Campoli sequence in Σ is defined to be a finite family $(\xi_i, \psi_i, \lambda_i, u_i)$ in Σ such that

$$\sum_i \langle \pi_{P_0, \xi_i, \lambda_i; u_i}(x), \psi_i \rangle = 0, \quad x \in G.$$

By integration over x it follows that this condition is equivalent to the condition that

$$(3.3) \qquad \sum_i \langle \hat{f}(P_0, \xi_i, \lambda_i; u_i), \psi_i \rangle = 0, \quad f \in C_c^\infty(G, K).$$

Definition 3.2. A function $\varphi \in \mathcal{O}(\mathfrak{a}_\mathbb{C}^*) \otimes S(P_0)$ is said to satisfy the Arthur–Campoli relations if

$$\sum_i \langle \varphi_{\xi_i, \lambda_i; u_i}, \psi_i \rangle = 0$$

for any Arthur–Campoli sequence $(\xi_i, \psi_i, \lambda_i, u_i)$ in Σ.

3.2. Reformulation of the Arthur–Campoli relations. In the following, $\mathcal{O}_0$ denotes the ring of germs at 0 of holomorphic functions defined on a neighborhood of 0 in $\mathfrak{a}_{\mathbb{C}}^*$.

Let E be a finite dimensional $\mathcal{O}_0$-module and let $\Xi \subset M^{\wedge}$ and $\Lambda \subset \mathfrak{a}_{\mathbb{C}}^*$ be finite sets. We define the representation $\pi_{E,\Xi,\Lambda}$ of G by

$$(3.4) \qquad \pi_{E,\Xi,\Lambda} = \bigoplus_{(\xi,\lambda)\in\Xi\times\Lambda} \pi_{P_0,\xi,\lambda}^{(E)}.$$

Note that this representation is admissible and of finite length. Its underlying space is given by

$$V_{E,\Xi,\Lambda} = \bigoplus_{(\xi,\lambda)\in\Xi\times\Lambda} E \otimes C^{\infty}(K:\xi).$$

For each element $\varphi \in \mathcal{O}(\mathfrak{a}_{\mathbb{C}}^*) \otimes \mathcal{S}(P_0)$ we define the $K \times K$-finite endomorphism $\varphi_{E,\Xi,\Lambda}$ of $V_{E,\Xi,\Lambda}$ by taking the similar direct sum

$$(3.5) \qquad \varphi_{E,\Xi,\Lambda} := \bigoplus_{(\xi,\lambda)\in\Xi\times\Lambda} \varphi_{\xi}^{(E)}(\lambda).$$

We note that $\pi_{E,\Xi,\Lambda}(C_c^{\infty}(G,K))$ is a subset of the space $\mathrm{End}(V_{E,\Xi,\Lambda})$ and agree to write $\pi_{E,\Xi,\Lambda}(C_c^{\infty}(G,K))^{\perp}$ for its annihilator in the space

$$\mathrm{End}(V_{E,\Xi,\Lambda})_{K\times K}^*.$$

Proposition 3.3. *Let $\phi \in \mathcal{O}(\mathfrak{a}_{\mathbb{C}}^*) \otimes \mathcal{S}(P_0)$. Then the following conditions are equivalent:*

(a) *ϕ satisfies the Arthur–Campoli relations.*

(b) *For every finite dimensional $\mathcal{O}_0$-module E, every pair of finite sets $\Xi \subset M^{\wedge}$, $\Lambda \subset \mathfrak{a}_{\mathbb{C}}^*$ and all $\Psi \in \pi_{E,\Xi,\Lambda}(C_c^{\infty}(G,K))^{\perp}$,*

$$\langle \phi_{E,\Xi,\Lambda}, \Psi \rangle = 0.$$

We prove the result through a number of lemmas. In the following results a complication is caused by the circumstance that $\mathrm{End}(V_{E,\Xi,\Lambda})_{K\times K}$ is not the direct sum of the spaces $\mathrm{End}(V_{E,\xi,\lambda})_{K\times K}$, but rather that of the spaces $\mathrm{Hom}(V_{E,\xi_1,\lambda_1}, V_{E,\xi_2,\lambda_2})_{K\times K}$, for $\xi_1, \xi_2 \in \Xi$ and $\lambda_1, \lambda_2 \in \Lambda$. For $(\xi,\lambda) \in \Xi \times \Lambda$, let

$$\mathrm{i}_{\xi,\lambda} : \mathrm{End}(V_{E,\xi,\lambda})_{K\times K} \to \mathrm{End}(V_{E,\Xi,\Lambda})_{K\times K}$$

denote the associated embedding, and let

$$\mathrm{pr}_{\xi,\lambda} : \mathrm{End}(V_{E,\Xi,\Lambda})_{K\times K} \to \mathrm{End}(V_{E,\xi,\lambda})_{K\times K}$$

denote the associated projection map.

Lemma 3.4. *Let E, Ξ, Λ be as above. Then for each $\Psi \in \mathrm{End}(V_{E,\Xi,\Lambda})_{K\times K}^*$ there exists a finite sequence $(\xi_i, \psi_i, \lambda_i, u_i)$ in Σ such that for all $\varphi \in \mathcal{O}(\mathfrak{a}_{\mathbb{C}}^*) \otimes \mathcal{S}(P_0)$,*

$$(3.6) \qquad \langle \varphi_{E,\Xi,\Lambda}, \Psi \rangle = \sum_i \langle \varphi_{\xi_i,\lambda_i;u_i}, \psi_i \rangle.$$

Proof. Put

$$S(P_0, \Xi, \Lambda) := \bigoplus_{(\xi, \lambda) \in \Xi \times \Lambda} S(P_0, \xi).$$

We observe that $\mathrm{End}(V_{E,\Xi,\Lambda})^*_{K \times K}$ may be viewed as the direct sum of the $K \times K$-submodule $\mathrm{End}(E)^* \otimes S(P_0, \Xi, \Lambda)^*_{K \times K}$ and a unique $K \times K$-submodule $\mathrm{End}(E)^* \otimes \mathcal{T}$ (consisting of the 'cross terms'). Since every element of $\mathrm{End}(E)^* \otimes \mathcal{T}$ annihilates $\varphi_{E,\Xi,\Lambda}$, we may assume that $\Psi \in \mathrm{End}(E)^* \otimes S(P_0, \Xi, \Lambda)^*_{K \times K}$. By linearity, we may then reduce to the situation that $\Xi = \{\xi\}$ and $\Lambda = \{\lambda\}$. Again by linearity we may assume that Ψ is of the form $\eta \otimes \psi$, with $\eta \in \mathrm{End}(E)^*$ and $\psi \in S(P_0, \xi)^*_{K \times K}$. Let $u \in S(\mathfrak{a}^*)$ be associated with η as in Lemma 2.10. Then for all $\varphi \in \mathcal{O}(\mathfrak{a}^*_{\mathbb{C}}) \otimes S(P_0)$,

$$\langle \varphi_{E,\xi,\lambda}, \Psi \rangle = \langle \varphi^{(E)}_{\xi,\lambda}, \eta \otimes \psi \rangle = \langle \varphi_{\xi,\lambda;u}, \psi \rangle.$$

This finishes the proof. □

Lemma 3.5. *Let $(\xi_i, \psi_i, \lambda_i, u_i)$ be a finite sequence in Σ. Then there exists a finite dimensional $\mathcal{O}_0$-module E, finite sets $\Xi \subset M^\wedge$, $\Lambda \subset \mathfrak{a}^*_{\mathbb{C}}$ and an element Ψ of $\mathrm{End}(V_{E,\Xi,\Lambda})^*_{K \times K}$ such that for all $\varphi \in \mathcal{O}(\mathfrak{a}^*_{\mathbb{C}}) \otimes S(P_0)$,*

$$(3.7) \qquad \sum_i \langle \varphi_{\xi_i, \lambda_i; u_i}, \psi_i \rangle = \langle \varphi_{E,\Xi,\Lambda}, \Psi \rangle.$$

Proof. Let E and $\eta_i \in \mathrm{End}(E)^*$ be associated to the finite sequence u_i as in Lemma 2.11. Then $\Psi_i = \eta_i \otimes \psi_i$ belongs to $\mathrm{End}(E)^* \otimes S(P_0, \xi_i)^*_{K \times K} \simeq \mathrm{End}(V_{E,\xi_i,\lambda_i})^*_{K \times K}$. Let Ξ be the finite set of all ξ_i and let Λ be the finite set of all λ_i. Let $\mathrm{pr}_{\xi_i, \lambda_i}$ be defined as above. We define $\Psi \in \mathrm{End}(V_{E,\Xi,\Lambda})^*_{K \times K}$ by

$$\Psi = \sum_i \mathrm{pr}^*_{\xi_i, \lambda_i}(\eta_i \otimes \psi_i).$$

Then for all $\varphi \in \mathcal{O}(\mathfrak{a}^*_{\mathbb{C}}) \otimes S(P_0)$ we have

$$\sum_i \langle \varphi_{\xi_i, \lambda_i; u_i}, \psi_i \rangle = \sum_i \langle \varphi^{(E)}_{\xi_i, \lambda_i}, \eta_i \otimes \psi_i \rangle$$

$$= \sum_i \langle \mathrm{pr}_{\xi_i, \lambda_i} \varphi_{E,\Xi,\Lambda}, \eta_i \otimes \psi_i \rangle = \langle \varphi_{E,\Xi,\Lambda}, \Psi \rangle. \qquad \square$$

Proof of Proposition 3.3. Let ϕ be as stated and assume (a). Let E, Ξ, Λ and Ψ be as asserted in (b). In particular, $\Psi \in \mathrm{End}(V_{E,\Xi,\Lambda})^*_{K \times K}$. Let $(\xi_i, \psi_i, \lambda_i, u_i)$ be a sequence in Σ, associated to Ψ as in Lemma 3.4. Then using the relation (3.6) with $\varphi = \hat{f}(P_0)$ for $f \in C^\infty_c(G, K)$, we see that $(\xi_i, \psi_i, \lambda_i, u_i)$ is an Arthur–Campoli sequence (see (3.3)). Hence,

$$\langle \phi_{E,\Xi,\Lambda}, \Psi \rangle = \sum_i \langle \phi_{\xi_i, \lambda_i; u_i}, \psi_i \rangle = 0.$$

We have proved (b).

Conversely, assume (b) and let $(\xi_i, \psi_i, \lambda_i, u_i)$ be an Arthur–Campoli sequence in Σ. Let E, Ξ, Λ, Ψ be associated with this sequence as in Lemma 3.5. Then it follows from (3.7) with $\varphi = \hat{f}(P_0)$ for $f \in C_c^\infty(G, K)$ that Ψ belongs to $\pi_{E,\Xi,\Lambda}(C_c^\infty(G, K))^\perp$. This implies that

$$\sum_i \langle \phi_{\xi_i, \lambda_i; u_i}, \psi_i \rangle = \langle \phi_{E,\Xi,\Lambda}, \Psi \rangle = 0.$$

Hence (a). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

4. Delorme's intertwining conditions

4.1. Successive derivatives. Let υ be a finite dimensional vector space over $\mathbb{R}$, let $\Omega \subset \upsilon_{\mathbb{C}}^*$ be an open subset and let V be a quasi-complete locally convex space. Following Delorme [6], we define, for $\Phi \in \mathcal{O}(\Omega, \mathrm{End}(V))$ and $\eta \in \upsilon_{\mathbb{C}}^*$, the holomorphic function $\Phi^{(\eta)} : \Omega \to \mathrm{End}(V \oplus V)$ by

$$\Phi^{(\eta)}(\lambda)(v_1, v_2) := \left(\Phi(\lambda)v_1 + \frac{d}{dz}(\Phi(\lambda + z\eta)v_2)|_{z=0}, \Phi(\lambda)v_2 \right),$$

for all $\lambda \in \Omega$ and $v_1, v_2 \in V$. Still following [6], we agree to define, for any finite sequence $\eta = (\eta_1, \ldots, \eta_N)$ in $\upsilon_{\mathbb{C}}^*$, the iterated derivative

$$\Phi^{(\eta)} := (\cdots(\Phi^{(\eta_N)})^{(\eta_{N-1})} \cdots)^{(\eta_1)}$$

of Φ along η. Then $\Phi^{(\eta)}$ is a holomorphic function on Ω with values in $\mathrm{End}(V^{(\eta)})$, where $V^{(\eta)}$ denotes the direct sum of 2^N copies of V. Now assume that V is Fréchet (or more generally, barrelled). If π is a holomorphic family of continuous representations of G in V over the parameter set $\upsilon_{\mathbb{C}}^*$ then it follows by application of the methods of [6] that for each $\lambda \in \Omega$,

$$\pi_\lambda^{(\eta)}(x) := \pi(x)^{(\eta)}(\lambda), \quad x \in G,$$

defines a continuous representation of G in $V^{(\eta)}$.

4.2. The intertwining conditions. Let $\mathcal{D}$ be the set of 4-tuples $\delta = (P, \xi, \lambda, \eta)$, with $P \in \mathcal{P}(A)$, $\xi \in M_{P,\mathrm{ds}}^\wedge$, $\lambda \in \mathfrak{a}_{P\mathbb{C}}^*$ and η a finite sequence in $\mathfrak{a}_{P\mathbb{C}}^*$. Given $\delta \in \mathcal{D}$, we define the representation π_δ of G in $V_{\pi_\delta} := C^\infty(K:\xi)^{(\eta)}$ by

$$\pi_\delta := \pi_{P,\xi,\lambda}^{(\eta)}$$

For $P \in \mathcal{P}(A)$ we define the space $\mathcal{F}_P := \mathcal{O}(\mathfrak{a}_{P\mathbb{C}}^*) \otimes \mathcal{S}(P)$, with $\mathcal{S}(P)$ defined as in (3.1). Furthermore, we put

$$(4.1) \qquad\qquad\qquad \mathcal{F} := \bigoplus_{P \in \mathcal{P}(A)} \mathcal{F}_P.$$

Given $\varphi \in \mathcal{F}$ and $\delta = (P, \xi, \lambda, \eta) \in \mathcal{D}$, we define $\varphi_\delta \in \mathrm{End}(C^\infty(K:\xi)^{(\eta)})$ in a similar fashion as π_δ, by

$$\varphi_\delta := \varphi_{P,\xi}^{(\eta)}(\lambda).$$

Finally, given a sequence $\delta = (\delta_1, \ldots, \delta_N)$ of data from $\mathcal{D}$, we define

$$\pi_\delta := \pi_{\delta_1} \oplus \cdots \oplus \pi_{\delta_N}, \quad V_{\pi_\delta} := V_{\pi_{\delta_1}} \oplus \cdots \oplus V_{\pi_{\delta_N}}, \quad \varphi_\delta := \varphi_{\delta_1} \oplus \cdots \oplus \varphi_{\delta_N}.$$

Definition 4.1. We say that a function $\varphi \in \mathcal{F}$ satisfies Delorme's intertwining conditions (see [6, Definition 3 (4.4)]) if

(a) for every $N \in \mathbb{Z}_+$ and each $\delta \in \mathcal{D}^N$ the function φ_δ preserves all invariant subspaces of π_δ;

(b) for all $N_1, N_2 \in \mathbb{Z}_+$, all $\delta_1 \in \mathcal{D}^{N_1}$ and $\delta_2 \in \mathcal{D}^{N_2}$, and any two sequences of closed invariant subspaces $U_j \subset V_j$ for π_{δ_j}, the induced maps $\bar{\varphi}_{\delta_j} \in \mathrm{End}(V_j/U_j)$ are intertwined by all intertwining operators $T : V_1/U_1 \to V_2/U_2$.

The space of functions $\varphi \in \mathcal{F}$ satisfying (a) and (b) is denoted by $\mathcal{F}(\mathcal{D})$.

4.3. A simplification of the intertwining conditions. In this section we will show that condition (b) of Definition 4.1 is in fact a consequence of condition (a) of the same definition.

Lemma 4.2. *Let $\varphi \in \mathcal{F}$. Then $\varphi \in \mathcal{F}(\mathcal{D})$ if and only if φ satisfies condition* (a) *of Definition* 4.1.

Proof. Assume that φ satisfies condition (a) of the mentioned definition. Then we must show that φ satisfies condition (b) as well. Let $\delta_j, \pi_{\delta_j}, U_j, V_j$ be as in (b), for $j = 1, 2$. Let $T : V_1/U_1 \to V_2/U_2$ be an intertwining operator. Let $p_j : V_j \to V_j/U_j$ denote the canonical projection, for $j = 1, 2$, and let $p := p_1 \oplus p_2$. Since T is equivariant, its graph W is an invariant subspace of $V_1/U_1 \oplus V_2/U_2$. Hence, $p^{-1}(W)$ is an invariant subspace of $V_1 \oplus V_2$. Since φ satisfies (a), it follows that $p^{-1}(W)$ is a $\varphi_{\delta_1} \oplus \varphi_{\delta_2}$-invariant subspace of $V_1 \oplus V_2$. This in turn is easily seen to imply that $T \circ \bar{\varphi}_{\delta_1} = \bar{\varphi}_{\delta_2} \circ T$. $\square$

4.4. Reduction to a single minimal parabolic subgroup. In this section, we will show that the space $\mathcal{F}(\mathcal{D})$ of functions satisfying Delorme's intertwining conditions is naturally isomorphic to a space of functions defined in terms of just the minimal parabolic subgroup P_0. We denote by $\mathcal{D}_{P_0}$ the set of 4-tuples (P, ξ, λ, η) in $\mathcal{D}$ for which $P = P_0$.

Definition 4.3. We say that a function $\varphi \in \mathcal{F}$ satisfies Delorme's intertwining conditions associated with P_0 if conditions (a) and (b) of Definition 4.1 are valid with everywhere $\mathcal{D}$ replaced by $\mathcal{D}_{P_0}$.

The space of functions $\varphi \in \mathcal{F}$ satisfying all such intertwining conditions is denoted by $\mathcal{F}(\mathcal{D}_{P_0})$.

The following analogue of Lemma 4.2 is now valid, with essentially the same proof.

Lemma 4.4. *Let $\varphi \in \mathcal{F}$. Then $\varphi \in \mathcal{F}(\mathcal{D}_{P_0})$ if and only if φ satisfies condition* (a) *of Definition* 4.1 *for every $N \in \mathbb{Z}_+$ and all $\delta \in (\mathcal{D}_{P_0})^N$.*

We consider the component

$$\mathcal{F}_{P_0} := \mathcal{O}(\mathfrak{a}_\mathbb{C}^*) \otimes \mathcal{S}(P_0)$$

of the direct sum $\mathcal{F}$ defined in (4.1), and denote the natural projection $\mathcal{F} \to \mathcal{F}_{P_0}$ by pr. Moreover, we define

$$\mathcal{F}_{P_0}(\mathcal{D}_{P_0}) := \mathcal{F}_{P_0} \cap \mathcal{F}(\mathcal{D}_{P_0}).$$

Proposition 4.5. *The natural projection* $\mathrm{pr} : \mathcal{F} \to \mathcal{F}_{P_0}$ *restricts to a linear isomorphism from* $\mathcal{F}(\mathcal{D})$ *onto* $\mathcal{F}_{P_0}(\mathcal{D}_{P_0})$.

Proof. If $\varphi \in \mathcal{F}$ and if δ is a finite sequence in $\mathcal{D}_{P_0}$, then $\varphi_\delta = (\varphi_{P_0})_\delta$. Hence $\varphi \in \mathcal{F}(\mathcal{D}_{P_0})$ if and only if $\mathrm{pr}(\varphi) \in \mathcal{F}(\mathcal{D}_{P_0})$.

From $\mathcal{D}_{P_0} \subset \mathcal{D}$ it follows that $\mathcal{F}(\mathcal{D}) \subset \mathcal{F}(\mathcal{D}_0)$. Therefore, pr maps $\mathcal{F}(\mathcal{D})$ into $\mathcal{F}_{P_0} \cap \mathcal{F}(\mathcal{D}_{P_0}) = \mathcal{F}_{P_0}(\mathcal{D}_{P_0})$. The proof will be completed by showing that the restricted projection $\mathrm{pr}_{\mathcal{D}} := \mathrm{pr}|_{\mathcal{F}(\mathcal{D})} : \mathcal{F}(\mathcal{D}) \to \mathcal{F}_{P_0}(\mathcal{D}_{P_0})$ is a linear isomorphism. We will do this by defining a map

$$\tau : \mathcal{F}_{P_0} \to \mathcal{F}$$

which will turn out to induce a two-sided inverse for $\mathrm{pr}_{\mathcal{D}}$.

Each parabolic subgroup $P \in \mathcal{P}(A)$ has a Langlands decomposition $P = M_P A_P N_P$. Here M_P is a real reductive group of the Harish-Chandra class, with Cartan decomposition $M_P = (M_P \cap K) \exp(\mathfrak{m}_P \cap \mathfrak{p})$. Moreover, $^*\mathfrak{a} := \mathfrak{m}_P \cap \mathfrak{a}$ is maximal abelian in $\mathfrak{m}_P \cap \mathfrak{p}$, and the centralizer of $^*\mathfrak{a}$ in $K_P := M_P \cap K$ equals $M = Z_K(\mathfrak{a})$. We select a minimal parabolic subgroup

$$^*Q = M \, ^*A \, ^*N$$

of M_P. In addition, we fix a minimal parabolic subgroup Q of G containing A such that $^*Q = Q \cap M_P$, and we fix an element $w \in N_K(\mathfrak{a})$ such that $P_0 = w^{-1} Q w$. In case $P = P_0$, we agree to make the special choice $Q = P_0$ and $w = e$.

Then by the subrepresentation theorem applied to the group M_P, for each $\xi \in M_{P,\mathrm{ds}}^\wedge$ we may fix a representation $\sigma \in M^\wedge$ and an element $\mu \in {}^*\mathfrak{a}_\mathbb{C}^*$ such that ξ is equivalent to a subrepresentation of the parabolically induced (smooth) representation $\pi_{^*Q,\sigma,\mu}^{M_P}$ of M_P. In addition, we fix an M_P-equivariant embedding

$$j_\xi : (\xi, V_\xi) \hookrightarrow \big(\pi_{^*Q,\sigma,\mu}^{M_P}, C^\infty(K \cap M_P : \sigma)\big).$$

Through induction by stages, this embedding induces an embedding

$$(4.2) \qquad\qquad j_{\xi P}^G : C^\infty(K : \xi) \to C^\infty(K : \sigma)$$

which intertwines $\pi_{P,\xi,\lambda}$ with $\pi_{Q,\sigma,\mu+\lambda}$, for all $\lambda \in \mathfrak{a}_{P\mathbb{C}}^*$. Here $^*\mathfrak{a}_\mathbb{C}^*$ and $\mathfrak{a}_{P\mathbb{C}}^*$ are viewed as subspaces of $\mathfrak{a}_\mathbb{C}^*$ via the direct sum decomposition $\mathfrak{a} = {}^*\mathfrak{a} \oplus \mathfrak{a}_P$. Thus, (4.2) is a morphism in the category $^\circ\mathrm{HF}_G$, see Definition 2.38. In the special case $P = P_0$, we agreed that $Q = P_0$ and $w = e$. In this case, we have $\mu = 0$, and as σ must be equivalent to ξ, it follows that $\sigma = w \cdot \sigma = \xi$. It is now readily verified that $j_{\xi P}^G$ is the identity map of $C^\infty(K : \xi)$.

Left translation by w induces a topological linear isomorphism

$$L(w) : C^\infty(K : \sigma) \to C^\infty(K : w\sigma)$$

which intertwines the induced representation $\pi_{Q,\sigma,\mu+\lambda}$ with $\pi_{P_0,w\sigma,w(\mu+\lambda)}$, for all $\lambda \in \mathfrak{a}_{P\mathbb{C}}^*$. Here $w\sigma$ denotes the representation of M in V_σ, given by $w\sigma(m) = \sigma(w^{-1}mw)$. We denote by $(w \cdot \sigma, V_{w\cdot\sigma})$ the unique element of $M^\wedge$ which is equivalent to $(w\sigma, V_\sigma)$. Fix an equivalence $t_w : V_\sigma \to V_{w\cdot\sigma}$ and denote the induced map $C^\infty(K : w\sigma) \to C^\infty(K : w \cdot \sigma)$ by T_w. Then writing $\mathcal{L}_w = T_w \circ L(w)$, we obtain a continuous linear injection

$$(4.3) \qquad\qquad \mathcal{L}_w \circ j_{\xi P}^G : C^\infty(K : \xi) \to C^\infty(K : w \cdot \sigma)$$

which intertwines the representation $\pi_{P,\xi,\lambda}$ with $\pi_{P_0,w\cdot\sigma,w(\mu+\lambda)}$, for all $\lambda \in \mathfrak{a}_{P\mathbb{C}}^*$. In the special case $P = P_0$, where $Q = P_0, w = e, \mu = 0$ and $\sigma = \xi$, we may take $t_w = \mathrm{id}_{V_\xi}$ and then $\mathscr{L}_w$ and hence (4.3) become the identity map of $C^\infty(K\!:\!\xi)$.

We are now ready to define the map τ. Let P, ξ, λ be as above, and let $\varphi \in \mathscr{F}_{P_0}(\mathscr{D}_0)$. Then the invariant subspace $\mathrm{im}(\mathscr{L}_w \circ j_{\xi P}^G)$ of the representation $\pi_{P_0,w\cdot\sigma,w(\mu+\lambda)}$ is preserved by $\varphi_{P_0,w\cdot\sigma,w(\mu+\lambda)}$ in view of condition (a) of Definition 4.1. Thus, we may define

$$\tau(\varphi)_{P,\xi,\lambda} : C^\infty(K\!:\!\xi) \to C^\infty(K\!:\!\xi)$$

to be the unique linear map such that the following diagram commutes:

(4.4)
$$
\begin{array}{ccc}
C^\infty(K\!:\!\xi) & \xrightarrow{\ \mathscr{L}_w \circ j_{\xi P}^G\ } & C^\infty(K\!:\!w \cdot \sigma) \\
{\scriptstyle \tau(\varphi)_{P,\xi,\lambda}}\Big\downarrow & & \Big\downarrow{\scriptstyle \varphi_{P_0,w\cdot\sigma,w(\mu+\lambda)}} \\
C^\infty(K\!:\!\xi) & \xrightarrow[\ \mathscr{L}_w \circ j_{\xi P}^G\ ]{} & C^\infty(K\!:\!w \cdot \sigma).
\end{array}
$$

Since $\varphi_{P_0,w\cdot\sigma} \in \mathcal{O}(\mathfrak{a}_\mathbb{C}^*) \otimes \mathrm{End}(C^\infty(K\!:\!w\!:\!\sigma))_{K\times K}$, it is readily seen that $\tau(\varphi)_{P,\xi}$ defines a holomorphic function on $\mathfrak{a}_{P\mathbb{C}}^*$, with values in $\mathcal{S}(P,\xi)$. Accordingly, τ defines a linear map $\mathscr{F}_{P_0} \to \mathscr{F}$.

We will now finish the proof by showing that

(i) τ maps $\mathscr{F}_{P_0}(\mathscr{D}_{P_0})$ into $\mathscr{F}(\mathscr{D})$,

(ii) τ restricts to a linear isomorphism $\tau_\mathscr{D} : \mathscr{F}_{P_0}(\mathscr{D}_{P_0}) \to \mathscr{F}(\mathscr{D})$ which is a two-sided inverse for $\mathrm{pr}_\mathscr{D}$.

We will first establish (i). Let $\varphi \in \mathscr{F}_{P_0}(\mathscr{D}_0)$, and let $N \in \mathbb{Z}_+$ and $\delta \in \mathscr{D}^N$. We claim that it suffices to show that there exists a $\delta' \in \mathscr{D}_{P_0}^N$ and a linear embedding $j : V_\delta \to V_{\delta'}$ intertwining π_δ with $\pi_{\delta'}$ such that

(4.5)
$$\varphi_{\delta'} \circ j = j \circ \tau(\varphi)_\delta.$$

Indeed assume the claim to hold, and let $W \subset V_\delta$ be an invariant subspace. Then $j(W)$ is an invariant subspace of $V_{\delta'}$. Moreover, since $\varphi \in \mathscr{F}_{P_0}(\mathscr{D}_0)$, it follows that $\varphi_{\delta'}$ leaves $j(W)$ invariant. As j is injective, it now follows from (4.5) that $\tau(\varphi)_\delta$ leaves W invariant. Thus, the validity of the claim would imply that the above condition (i) holds.

We turn to the proof of the claim. It clearly suffices to prove the claim in case $N = 1$, so that $\delta \in \mathscr{D}$. Thus, δ is a 4-tuple of the form $(P, \xi, \lambda_0, \eta)$ with notation as in the beginning of Section 4.2. In particular, η is a finite sequence in $\mathfrak{a}_{P\mathbb{C}}^*$. Let Q, σ, μ, w be associated with the data P, ξ as in the first part of this proof, where the definition of τ was given. Then the injective linear map (4.3) intertwines $\pi_{P,\xi,\lambda}$ with $\pi_{P_0,w\cdot\sigma,w(\mu+\lambda)}$ for all $\lambda \in \mathfrak{a}_{P\mathbb{C}}^*$.

It follows that the map

$$j : C^\infty(K\!:\!\xi)^{(\eta)} \to C^\infty(K\!:\!w \cdot \sigma)^{(\eta)}$$

induced by (4.3) intertwines π_δ with $\pi_{\delta'}$, where $\delta' = (P_0, w \cdot \sigma, w(\lambda_0 + \mu), w\eta)$. Moreover, the commutativity of the diagram (4.4) implies that

$$j \circ \tau(\varphi)_{P,\xi,\lambda}^{(\eta)} = \varphi_{P_0,w\cdot\sigma,w(\mu+\lambda)}^{(w\eta)} \circ j,$$

or, abbreviated, $j \circ \tau(\varphi)_\delta = \varphi_{\delta'} \circ j$. This establishes the claim.

We now consider the induced map $\tau_{\mathcal{D}} : \mathcal{F}_{P_0}(\mathcal{D}_0) \to \mathcal{F}(\mathcal{D})$, obtained by restriction of τ, and will finish the proof by establishing (ii).

Let $\varphi \in \mathcal{F}_{P_0}(\mathcal{D}_{P_0})$. Moreover, let Q, σ, μ, w be associated to $P = P_0$ as above. By the special choices we made for this particular parabolic subgroup, the diagram (4.4) becomes

$$
\begin{array}{ccc}
C^\infty(K{:}\xi) & \xrightarrow{\ \text{id}\ } & C^\infty(K{:}\xi) \\
{\scriptstyle \tau(\varphi)_{P_0,\xi,\lambda}}\Big\downarrow & & \Big\downarrow{\scriptstyle \varphi_{P_0,\xi,\lambda}} \\
C^\infty(K{:}\xi) & \xrightarrow[\ \text{id}\ ]{} & C^\infty(K{:}\xi).
\end{array}
$$

It follows that $\mathrm{pr} \circ \tau(\varphi) = \varphi$, and we see that $\tau_{\mathcal{D}}$ is a right inverse to $\mathrm{pr}_{\mathcal{D}}$.

We will finish the proof by showing that $\tau_{\mathcal{D}}$ is also a left inverse. Let $\psi \in \mathcal{F}(\mathcal{D})$, and let $P \in \mathcal{P}(A)$ and $\xi \in M^{\wedge}_{P,\mathrm{ds}}$. Let Q, σ, μ, w be as above. As ψ satisfies condition (b) of Definition 4.1, it follows that the following diagram commutes, for all $\lambda \in \mathfrak{a}^*_{P\mathbb{C}}$,

$$
\begin{array}{ccc}
C^\infty(K{:}\xi) & \xrightarrow{\ \mathscr{L}_w \circ j^G_{\xi P}\ } & C^\infty(K{:}w \cdot \sigma) \\
{\scriptstyle \psi_{P,\xi,\lambda}}\Big\downarrow & & \Big\downarrow{\scriptstyle \psi_{P_0,w\cdot\sigma,w(\mu+\lambda)}} \\
C^\infty(K{:}\xi) & \xrightarrow[\ \mathscr{L}_w \circ j^G_{\xi P}\ ]{} & C^\infty(K{:}w \cdot \sigma).
\end{array}
$$

By comparison with (4.4) we see that $\psi_{P,\xi} = \tau(\psi_{P_0})_{P,\xi}$. Hence, $\tau \circ \mathrm{pr}(\psi) = \psi$, for $\psi \in \mathcal{F}(\mathcal{D})$. Therefore, $\tau_{\mathcal{D}}$ is a left inverse to $\mathrm{pr}_{\mathcal{D}}$. $\qquad\square$

4.5. Reformulation in our setting. We shall now compare Delorme's derivation process with the process defined in the present paper. In the following we shall identify $V \oplus V$ with $\mathbb{C}^2 \otimes V$ via the map $(v_1, v_2) \mapsto (1,0) \otimes v_1 + (0,1) \otimes v_2$. This identification induces an identification $\mathrm{End}(V \oplus V) \simeq \mathrm{End}(\mathbb{C}^2) \otimes \mathrm{End}(V)$. Let now $\eta \in \mathfrak{a}^*_{\mathbb{C}}$ and let Ω be an open subset of $\mathfrak{a}^*_{\mathbb{C}}$. For $\varphi \in \mathcal{O}(\Omega)$ we define $D^{(\eta)}\varphi \in \mathcal{O}(\Omega) \otimes \mathrm{End}(\mathbb{C}^2)$ by

$$
D^{(\eta)}\varphi(\lambda) = \begin{pmatrix} \varphi(\lambda) & \varphi(\lambda;\eta) \\ 0 & \varphi(\lambda) \end{pmatrix}.
$$

Let V be a quasi-complete locally convex space. Then for $\Phi \in \mathcal{O}(\Omega) \otimes \mathrm{End}(V)$, Delorme's derivative $\Phi^{(\eta)}$ is given by

$$
\Phi^{(\eta)} = (D^{(\eta)} \otimes \mathrm{id}_{\mathrm{End}(V)})\Phi.
$$

Here we note that

$$(4.6) \qquad\qquad V^{(\eta)} = \mathbb{C}^2 \otimes V,$$

so that $\mathrm{End}(V^{(\eta)}) = \mathrm{End}(\mathbb{C}^2) \otimes \mathrm{End}(V)$. It follows that Delorme's differentiation map $(\,\cdot\,)^{(\eta)}$ from $\mathcal{O}(\Omega, \mathrm{End}(V))$ to $\mathcal{O}(\Omega, \mathrm{End}(V^{(\eta)}))$ is the unique continuous linear extension of the map $D^{(\eta)} \otimes \mathrm{id}_{\mathrm{End}(V)}$.

Now assume that $\eta \neq 0$ and let $\mathcal{I}_\eta$ be the cofinite ideal of $\mathcal{O}_0 = \mathcal{O}_0(\mathfrak{a}^*_{\mathbb{C}})$ defined in (2.11). Then $\mathcal{I}_\eta$ has codimension 2 by Lemma 2.27. More precisely, let $X \in \mathfrak{a}_{\mathbb{C}}$ be such that $\eta(X) = 1$. Then the map $(z_1, z_2) \mapsto z_1 + z_2 X + \mathcal{I}_\eta$ defines a linear isomorphism κ from $\mathbb{C}^2$ onto the $\mathcal{O}_0$-module

$$
E_\eta := \mathcal{O}_0/\mathcal{I}_\eta.
$$

Now assume that V is barrelled. Then in view of (4.6) the isomorphism κ induces an isomorphism from $\mathcal{O}(\Omega, \operatorname{End}(V^{(\eta)}))$ onto $\mathcal{O}(\Omega, \operatorname{End}(E_\eta \otimes V))$, denoted κ_*.

Lemma 4.6. *With notation as above, the following diagram commutes:*

$$
\begin{array}{ccc}
\mathcal{O}(\Omega, \operatorname{End}(V)) & & \\
{\scriptstyle (\cdot)^{(\eta)}}\Big\downarrow & \xrightarrow{\makebox[2cm]{$J^{(E_\eta)}$}} & \\
\mathcal{O}(\Omega, \operatorname{End}(V^{(\eta)})) & \xrightarrow[\kappa_*]{} & \mathcal{O}(\Omega, \operatorname{End}(E_\eta \otimes V)).
\end{array}
$$

Proof. By the same calculation as in the proof of Lemma 2.28, it follows that the multiplication action $m(\varphi)$ of an element $\varphi \in \mathcal{O}_0$ on E_η with respect to the basis $\{\bar{1}, \bar{X}\}$ is given by the matrix

$$
M(\varphi) := \begin{pmatrix} \varphi(0) & \varphi(0; \eta) \\ 0 & \varphi(0) \end{pmatrix}.
$$

Let $\tilde{\kappa}$ be the isomorphism $\operatorname{End}(\mathbb{C}^2) \to \operatorname{End}(E_\eta)$ induced by κ. Then it follows that for all $\varphi \in \mathcal{O}(\Omega)$ and $\lambda \in \Omega$,

$$
J^{(E_\eta)}\varphi(\lambda) = m(\gamma_0(T_\lambda \varphi)) = \tilde{\kappa}(M(\gamma_0(T_\lambda \varphi))) = \tilde{\kappa}(D^{(\eta)}\varphi(\lambda)).
$$

This immediately implies the commutativity of the diagram in case $V = \mathbb{C}$. From this we see that the diagram commutes with the spaces $\mathcal{O}(\Omega, \operatorname{End}(W))$ replaced by the subspaces $\mathcal{O}(\Omega) \otimes \operatorname{End}(W)$, for W equal to $V, V^{(\eta)}$ or $E_\eta \otimes V$. By density of the mentioned spaces and continuity of the maps involved, the result follows. $\qquad\square$

Given a finite sequence $\eta = (\eta_1, \ldots, \eta_N)$ of elements in $\mathfrak{a}_\mathbb{C}^*$, we define the $\mathcal{O}_0$-module $E_\eta := E_{\eta_1} \otimes \cdots \otimes E_{\eta_N}$.

Corollary 4.7. *For every finite sequence η as above, and all $\xi \in M^\wedge$ and $\lambda \in \mathfrak{a}_\mathbb{C}^*$,*

$$
\pi_{P_0, \xi, \lambda}^{(\eta)} \simeq \pi_{P_0, \xi, \lambda}^{(E_\eta)}.
$$

Proof. For η of length one this follows from the above lemma. For arbitrary sequences it follows from the recurrent nature of the definition of $\pi^{(\eta)}$ and Proposition 2.26. $\qquad\square$

Definition 4.8. Let V be a Harish-Chandra module. We denote by $\operatorname{End}(V)^\#$ the space of endomorphisms $\varphi \in \operatorname{End}(V)_{K \times K}$ such that for every $n \in \mathbb{N}$ the product map $\varphi^{\times n} \in \operatorname{End}(V^{\times n})$ leaves all $(\mathfrak{g}, K)$-invariant subspaces of $V^{\times n}$ invariant. If (π, V_π) is an admissible representation of G of finite length, we define $\operatorname{End}(\pi)^\# := \operatorname{End}((V_\pi)_K)^\#$.

For E a finite dimensional $\mathcal{O}_0$-module (where $\mathcal{O}_0 = \mathcal{O}_0(\mathfrak{a}_\mathbb{C}^*)$), and for $\Xi \subset M^\wedge$ and $\Lambda \subset \mathfrak{a}_\mathbb{C}^*$ finite subsets, we recall the definition of the representation $\pi_{E, \Xi, \Lambda}$ by (3.4). Moreover, given $\varphi \in \mathcal{F}_{P_0} = \mathcal{O}(\mathfrak{a}_\mathbb{C}^*) \otimes \mathcal{S}(P_0)$, we recall the definition of $\varphi_{E, \Xi, \Lambda}$ in (3.5).

Lemma 4.9. *Let E, E' be finite dimensional $\mathcal{O}_0$-modules such that E is a subquotient of E'. Let $\Xi \subset \Xi' \subset M^\wedge$ and $\Lambda \subset \Lambda' \subset \mathfrak{a}_\mathbb{C}^*$ be finite subsets. Then for all $\varphi \in \mathcal{F}_{P_0}$,*

$$
\varphi_{E', \Xi', \Lambda'} \in \operatorname{End}(\pi_{E', \Xi', \Lambda'})^\# \implies \varphi_{E, \Xi, \Lambda} \in \operatorname{End}(\pi_{E, \Xi, \Lambda})^\#.
$$

Proof. The result follows from the crucial observation that the representation $\pi_{E,\Xi,\Lambda}$ is a subquotient of $\pi_{E',\Xi',\Lambda'}$ and the map $\varphi_{E',\Xi',\Lambda'}$ induces the map $\varphi_{E,\Xi,\Lambda}$. Obviously, it suffices to prove this observation for $\Xi = \Xi'$ and $\Lambda = \Lambda'$. In this case, we put, for $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$,

$$\pi_\lambda := \bigoplus_{\xi \in \Xi} \pi_{P_0,\xi,\lambda}.$$

Then π is a holomorphic family of admissible smooth representations of G over $\mathfrak{a}_{\mathbb{C}}^*$, so that the functor $X_\pi : E \mapsto \pi^{(E)}$ has the exactness properties of Lemma 2.44. Hence, $\pi_\lambda^{(E)}$ is a subquotient of $\pi_\lambda^{(E')}$, and $\varphi_{E,\Xi,\lambda}$ is induced by $\varphi_{E',\Xi,\lambda}$, for all $\lambda \in \Lambda$. The result now follows by taking the direct sum over $\lambda \in \Lambda$. $\qquad\square$

Proposition 4.10. *Let $\varphi \in \mathscr{F}_{P_0}$. Then the following conditions are equivalent:*

(a) $\varphi \in \mathscr{F}_{P_0}(\mathscr{D}_{P_0})$.

(b) *For every finite dimensional $\mathcal{O}_0$-module E and every pair of finite sets $\Xi \subset M^\wedge$ and $\Lambda \subset \mathfrak{a}_{\mathbb{C}}^*$, the endomorphism $\varphi_{E,\Xi,\Lambda}$ belongs to $\mathrm{End}(\pi_{E,\Xi,\Lambda})^\#$.*

Proof. First assume (b). Let $\delta = (\delta_1, \ldots, \delta_N)$ be a sequence of data in $\mathscr{D}_{P_0}$. Then for (a) it suffices to show that

(4.7)
$$\varphi_\delta \in \mathrm{End}(V_{\pi_\delta})^\#,$$

in view of Lemma 4.4. We note that, with $\delta_j = (P_0, \xi_j, \lambda_j, \eta_j)$,

$$\pi_\delta = \bigoplus_{j=1}^{n} \pi_{P_0,\xi_j,\lambda_j}^{(E_{\eta_j})}.$$

Let E be the direct sum of the modules E_{η_j}. Put $\Xi := \{\xi_1, \ldots, \xi_N\}$ and $\Lambda := \{\lambda_1, \ldots, \lambda_N\}$. For each $1 \le j \le N$ and $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$, let $(\pi_j)_\lambda := \pi_{P_0,\xi_j,\lambda}$. Then π_j is a holomorphic family of admissible smooth representations of G over $\mathfrak{a}_{\mathbb{C}}^*$, so that Lemma 2.44 applies to the functors $X_{\pi_j} : E \mapsto \pi_j^{(E)}$. It follows that

$$\pi_{\delta_j} \preceq \pi_{j\lambda_j}^{(E)}, \quad \text{and} \quad \varphi_{\delta_j} = \varphi_{P_0,\xi_j,\lambda_j}^{(E)}\big|_{V_{\delta_j}}$$

in a natural fashion (here $\rho \preceq \pi$ indicates that ρ is a subrepresentation of π). This in turn implies that

$$\pi_\delta \preceq \pi_{E,\Xi,\Lambda}, \quad \text{and} \quad \varphi_\delta = \varphi_{E,\Xi,\Lambda}\big|_{V_\delta}.$$

Hence (4.7) follows.

Conversely, assume (a). Let E, Ξ, Λ be as stated in (b). First we consider the case that E is of the form E_η, with η a finite sequence in $\mathfrak{a}_{\mathbb{C}}^*$. Then

$$\pi_{E_\eta,\Xi,\Lambda} = \bigoplus_{\xi \in \Xi,\, \lambda \in \Lambda} \pi_{P_0,\xi,\lambda}^{(\eta)}$$

and it follows from (a) that $\varphi_{E_\eta,\Xi,\Lambda}$ belongs to $\mathrm{End}(\pi_{E_\eta,\Xi,\Lambda})^\#$. In view of Lemma 2.44, this implies that $\varphi_{E,\Xi,\Lambda}$ belongs to $\mathrm{End}(\pi_{E,\Xi,\Lambda})^\#$ for E a direct sum of copies of E_η. Finally, let E be arbitrary. Then by Proposition 2.31 the module E is a subquotient of E_η^N for a suitable finite sequence η and a suitable $N \in \mathbb{N}$. This implies, again by Lemma 2.44, that $\pi_{E,\Xi,\Lambda}$ is a subquotient of $\pi_{E_\eta^N,\Xi,\Lambda}$ and that $\varphi_{E,\Xi,\Lambda}$ is induced by $\varphi_{E_\eta^N,\Xi,\Lambda}$. From this and Lemma 4.9 it follows that $\varphi_{E,\Xi,\Lambda}$ belongs to $\mathrm{End}(\pi_{E,\Xi,\Lambda})^\#$. Hence (b). $\qquad\square$

5. Conditions in terms of the Hecke algebra

Let k be a field and A a k-algebra with an approximate identity $(\alpha_j)_{j \in J}$. This means that

(a) J is a partially ordered set;

(b) for all j_1, j_2 in J with $j_1 \leq j_2$ we have $\alpha_{j_1} \alpha_{j_2} = \alpha_{j_2} \alpha_{j_1} = \alpha_{j_1}$;

(c) for every $a \in A$ there exists a $j \in J$ such that $\alpha_j a = a \alpha_j = a$.

The Hecke algebra $\mathbb{H}(G, K)$ is an example of such an algebra (see [9, Chapter I, §6]). Indeed, let $(\vartheta_j)_{j \in \mathbb{N}}$ be an increasing sequence of finite subsets of $K^\wedge$, whose union is $K^\wedge$. Then

$$\alpha_j := \sum_{\delta \in \vartheta_j} \dim(\delta) \chi_{\delta^\vee}, \quad j \in \mathbb{N},$$

defines an approximate identity in $\mathbb{H}(G, K)$.

We denote the opposite algebra of A by A^{opp}. It is readily seen that the elements $\alpha_j \otimes \alpha_j$ form an approximate identity for the algebra $A \otimes A^{\mathrm{opp}}$, so that this algebra is approximately unital.

For $j \in J$, let A_j be the set of $a \in A$ with $\alpha_j a = a \alpha_j = a$. Then it is readily seen that A_j is a subalgebra of A. Moreover, (b) implies that $A_{j_1} \subset A_{j_2}$ whenever $j_1 \leq j_2$, and (c) implies that A is the union of the subalgebras A_j.

5.1. Some general facts on approximately unital A-modules.

Let V be a left (or right) A-module (in particular, V is a k-linear space). We say that V is approximately unital if, for every $v \in V$, there exists a $j \in J$ such that $\alpha_j \cdot v = v$.

For $j \in J$, let

$$V_j := \{\alpha_j \cdot v \mid v \in V\}.$$

Then $V_{j_1} \subset V_{j_2}$ whenever $j_1 \leq j_2$. We note that V is approximately unital if and only if $\cup_{j \in J} V_j = V$.

Let $\mathrm{End}(V)$ denote the algebra of k-linear endomorphisms of V and let

$$\pi : A \to \mathrm{End}(V)$$

denote the canonical algebra homomorphism. Then V_j is the image of $\pi(\alpha_j)$. We note that V_j is invariant under the action of A_j, turning this space into a left A_j-module. Let V^j denote the kernel of $\pi(\alpha_j)$. Then since α_j is an idempotent, it follows that

(5.1) $$V = V^j \oplus V_j.$$

For any A-module V, the submodule $V_{\mathrm{au}} := A \cdot V$ is approximately unital, and it is in fact the maximal submodule with this property. Note that

$$V_{\mathrm{au}} = \bigcup_{j \in J} V_j.$$

We now assume that V is an approximately unital A-module. The k-linear space $\mathrm{Hom}_k(V, k)$ has a natural A^{opp}-module structure. Accordingly, we define

$$V^\vee := (\mathrm{Hom}_k(V, k))_{\mathrm{au}}.$$

We denote by $\mathrm{End}(V)_0$ the image of the natural linear map $V \otimes V^\vee \to \mathrm{End}(V)$ induced by $v \otimes v' \mapsto (u \mapsto v'(u)v)$.

We will say that an A-module V is admissible if V_j is finite dimensional for every $j \in J$.

Lemma 5.1. *Let V be an admissible approximately unital A-module. Then, as $A \otimes A^{\mathrm{opp}}$-modules,*

$$\mathrm{End}(V)_0 = \mathrm{End}(V)_{\mathrm{au}}.$$

Proof. It is readily seen that $\mathrm{End}(V)_0$ is an $A \otimes A^{\mathrm{opp}}$-submodule of $\mathrm{End}(V)$ which is approximately unital. This implies that $\mathrm{End}(V)_0 \subset \mathrm{End}(V)_{\mathrm{au}}$.

Conversely, let $j \in J$. We consider the inclusion map $\iota_j : V_j \to V$ and the epimorphism $p_j : V \to V_j$ determined by $\iota_j \circ p_j = \pi(\alpha_j)$. These maps induce linear maps $S_j : \mathrm{End}(V) \to \mathrm{End}(V_j)$ and $T_j : \mathrm{End}(V_j) \to \mathrm{End}(V)$ given by

$$S_j : f \mapsto p_j \circ f \circ \iota_j \quad \text{and} \quad T_j : g \mapsto \iota_j \circ g \circ p_j.$$

It is readily checked that

$$S_j \circ T_j = \mathrm{id}_{\mathrm{End}(V_j)}, \quad T_j \circ S_j(f) = \pi(\alpha_j) \circ f \circ \pi(\alpha_j) = \pi_{\mathrm{End}(V)}(\alpha_j)f.$$

Hence, T_j is an injective linear map $\mathrm{End}(V_j) \to \mathrm{End}(V)$ with image $\mathrm{End}(V)_j$. One now readily checks that the following diagram commutes:

$$
\begin{array}{ccc}
\mathrm{End}(V)_j & \xrightarrow{\ \mathrm{i}_j\ } & \mathrm{End}(V) \\
\uparrow{\scriptstyle t_j} & & \uparrow{\scriptstyle \mathrm{i}_j \otimes p_j^*} \\
\mathrm{End}(V_j) & \longleftarrow & V_j \otimes (V_j)^*.
\end{array}
$$

Here i_j is the inclusion map, and t_j is the map uniquely determined by $\mathrm{i}_j \circ t_j = T_j$. The map at the bottom is the canonical inclusion.

We now observe that the map at the bottom is a linear isomorphism by admissibility of V. Moreover, p_j^* maps $(V_j)^*$ into $(V^*)_j \subset V^\vee$, and we infer that $\mathrm{End}(V)_j \subset \mathrm{End}(V)_0$. Finally, by taking the union over all j, we conclude that $\mathrm{End}(V)_{\mathrm{au}} \subset \mathrm{End}(V)_0$. □

5.2. A double commutant theorem.

Definition 5.2. Let (π, V) be an approximately unital A-module. We define $\mathrm{End}(\pi)^\#$ to be the space of $\varphi \in \mathrm{End}(V)_0$ such that for every $n \in \mathbb{N} \setminus \{0\}$ and every A-submodule $W \subset V^{\times n}$ we have $\varphi^{\times n}(W) \subset W$.

Lemma 5.3. *Let (π, V) be an admissible approximately unital A-module. Then*

$$\pi(A) = \mathrm{End}(\pi)^\#.$$

Proof. Since $\pi(A)$ is an approximately unital $A \otimes A^{\mathrm{opp}}$-submodule of $\mathrm{End}(V)$, it is contained in $\mathrm{End}(V)_{\mathrm{au}} = \mathrm{End}(V)_0$. Moreover, if $a \in A$, then for every $n \in \mathbb{N} \setminus \{0\}$ and every invariant subspace W of $V^{\times n}$ we have $\pi(a)^{\times n}(W) \subset W$. Hence, $\pi(A) \subset \mathrm{End}(\pi)^\#$.

We now turn to the converse inclusion. By definition $\mathcal{E} := \mathrm{End}(\pi)^\#$ is a subspace of $\mathrm{End}(V)_0 = \mathrm{End}(V)_{\mathrm{au}}$. Moreover $\mathcal{E}$ is $A \otimes A^{\mathrm{opp}}$-invariant, hence an approximately unital $A \otimes A^{\mathrm{opp}}$-module. Fix $\varphi \in \mathcal{E}$. Then it follows that there exists a $j \in J$ such that

$(\alpha_j \otimes \alpha_j) \cdot \varphi = \varphi$. By admissibility, the space V_j has a finite basis $u_1, \ldots, u_n$ over k. Let W be the A-submodule of $V^{\times n}$ generated by $(u_1, \ldots, u_n)$. Then by definition of $\operatorname{End}(\pi)^{\#}$, the space W is $\varphi^{\times n}$-invariant. This implies the existence of an element $a \in A$ such that $\varphi(u_k) = \pi(a)u_k$ for all $1 \leq k \leq n$. Hence $\varphi = \pi(a)$ on V_j. It follows that $\varphi = \pi(a\alpha_j)$ on V_j. On the other hand, both $\varphi = (\alpha_j \otimes \alpha_j) \cdot \varphi = \pi(\alpha_j) \circ \varphi \circ \pi(\alpha_j)$ and $\pi(a\alpha_j)$ vanish on V^j. In view of (5.1) this implies that $\varphi = \pi(a\alpha_j)$ on V. Hence $\varphi = \pi(a\alpha_j) \in \pi(A)$ and the proof is complete. □

5.3. Application to Harish-Chandra modules. Every admissible $(\mathfrak{g}, K)$-module V is a module for the Hecke algebra $\mathbb{H}(G, K)$ in a natural way, and as such it is admissible and approximately unital. This assignment of an $\mathbb{H}(G, K)$-module to an admissible $(\mathfrak{g}, K)$-module defines a functor which establishes an isomorphism of categories, from the category of admissible $(\mathfrak{g}, K)$-modules onto the category of admissible approximately unital $\mathbb{H}(G, K)$-modules (see [9, Chapter I, §6, Theorem 1.117]).

Accordingly, if V is an admissible $(\mathfrak{g}, K)$-module, then the associated algebra $\operatorname{End}(\pi)^{\#}$ consists of all $K \times K$-finite endomorphisms φ of V with the property that, for every positive integer n, the map $\varphi^{\times n}$ preserves all $(\mathfrak{g}, K)$-invariant subspaces of $V^{\times n}$. In particular, the present notation is compatible with the notation introduced earlier in Definition 4.8.

Corollary 5.4. *Let (π, V) be an admissible $(\mathfrak{g}, K)$-module. Then*

$$\pi(\mathbb{H}(G, K)) = \operatorname{End}(\pi)^{\#}.$$

Proof. This follows from Lemma 5.3. □

5.4. Proof of the main theorem. For $P \in \mathcal{P}(A)$ we define $\mathrm{PW}_P^{\mathrm{pre}}(G, K)$ to be the subspace of $\mathcal{F}_P$ (see (4.1)), consisting of all functions $\varphi \in \mathcal{F}_P$ such that there exist an $R > 0$ and for all $n \in \mathbb{N}$ a constant $C_n > 0$ such that the estimate (0.2) holds for all $\xi \in M_{P,\mathrm{ds}}^{\wedge}$ and all $\lambda \in \mathfrak{a}_{P\mathbb{C}}^*$. Furthermore, we define

$$\mathrm{PW}^{\mathrm{pre}}(G, K) = \bigoplus_{P \in \mathcal{P}(A)} \mathrm{PW}_P^{\mathrm{pre}}(G, K).$$

Then $\mathrm{PW}^{\mathrm{pre}}(G, K)$ is a subspace of $\mathcal{F}$. We recall that $P_0 \in \mathcal{P}(A)$ is a fixed minimal parabolic subgroup of G.

Definition 5.5. (a) The Delorme Paley–Wiener space $\mathrm{PW}^D(G, K)$ is defined to be the intersection $\mathrm{PW}^{\mathrm{pre}}(G, K) \cap \mathcal{F}(\mathcal{D})$.

(b) The restricted Delorme Paley–Wiener space $\mathrm{PW}_{P_0}^D(G, K)$ is defined to be the intersection $\mathrm{PW}_{P_0}^{\mathrm{pre}}(G, K) \cap \mathcal{F}_{P_0}(\mathcal{D}_{P_0})$.

Theorem 5.6. *The natural projection* $\mathrm{pr} : \mathcal{F} \to \mathcal{F}_{P_0}$ *restricts to an isomorphism of* $\mathrm{PW}^D(G, K)$ *onto* $\mathrm{PW}_{P_0}^D(G, K)$.

Proof. It is clear that pr maps $\mathrm{PW}^{\mathrm{pre}}(G, K)$ into $\mathrm{PW}_{P_0}^{\mathrm{pre}}(G, K)$. Combining this with Proposition 4.5, we see that $\mathrm{pr}_{\mathcal{D}}$ maps $\mathrm{PW}^D(G, K)$ injectively into $\mathrm{PW}_{P_0}^D(G, K)$.

Let $\tau : \mathcal{F}_{P_0} \to \mathcal{F}$ be defined as in the proof of Proposition 4.5. Then it is readily checked that τ maps $\mathrm{PW}_{P_0}^{\mathrm{pre}}(G, K)$ into $\mathrm{PW}^{\mathrm{pre}}(G, K)$. This implies that $\tau_{\mathcal{D}}$ maps $\mathrm{PW}_{P_0}^D(G, K)$

into $\mathrm{PW}^D(G, K)$. As $\mathrm{pr}_{\mathcal{D}} \circ \tau_{\mathcal{D}} = \mathrm{id}$ on $\mathrm{PW}^D_{P_0}(G, K)$, it follows that $\mathrm{pr}_{\mathcal{D}}$ maps the space $\mathrm{PW}^D(G, K)$ surjectively onto $\mathrm{PW}^D_{P_0}(G, K)$. □

Definition 5.7. We define $\mathrm{PW}^H(G, K)$ to be the space of functions $\varphi \in \mathrm{PW}^{\mathrm{pre}}_{P_0}(G, K)$, such that for every finite dimensional $\mathcal{O}_0$-module E, and every pair of finite sets $\Xi \subset M^\wedge$, $\Lambda \subset \mathfrak{a}^*_{\mathbb{C}}$, we have

$$\varphi_{E,\Xi,\Lambda} \in \pi_{E,\Xi,\Lambda}(\mathbb{H}(G, K)).$$

Theorem 5.8. *The space* $\mathrm{PW}^D_{P_0}(G, K)$ *equals* $\mathrm{PW}^H(G, K)$.

Proof. Both spaces are subspaces of $\mathrm{PW}^{\mathrm{pre}}_{P_0}(G, K)$. Let φ be an element of the latter space. Then $\varphi \in \mathrm{PW}^D_{P_0}(G, K)$ is equivalent to $\varphi \in \mathcal{F}_{P_0}(\mathcal{D}_{P_0})$, which in turn is equivalent to condition (b) of Proposition 4.10. In view of Corollary 5.4, the latter condition is equivalent to $\varphi \in \mathrm{PW}^H(G, K)$. □

Definition 5.9. We define the Arthur Paley–Wiener space $\mathrm{PW}^A(G, K)$ to be the space of functions $\varphi \in \mathrm{PW}^{\mathrm{pre}}_{P_0}(G, K)$ such that φ satisfies the Arthur–Campoli relations (see Definition 3.2).

Theorem 5.10. *The space* $\mathrm{PW}^H(G, K)$ *equals* $\mathrm{PW}^A(G, K)$.

Proof. Both are subspaces of $\mathrm{PW}^{\mathrm{pre}}_{P_0}(G, K)$. Let φ be a function in the latter space. Then by Proposition 3.3 the assertion $\varphi \in \mathrm{PW}^A(G, K)$ is equivalent to the assertion that for every finite dimensional $\mathcal{O}_0$-module, and every pair of finite subsets $\Xi \subset M^\wedge$, $\Lambda \subset \mathfrak{a}^*_{\mathbb{C}}$, we have

(5.2) $$\varphi_{E,\Xi,\Lambda} \in \left(\pi_{E,\Xi,\Lambda}(C^\infty_c(G, K))^\perp\right)^\perp.$$

As every function from $\mathrm{PW}^{\mathrm{pre}}_{P_0}(G, K)$ has values in

$$\bigoplus_{\xi \in M^\wedge} \mathrm{End}(C^\infty(K{:}\xi))_{\vartheta\vartheta}$$

for some finite subset $\vartheta \subset K^\wedge$, it follows by admissibility that (5.2) is equivalent to

$$\varphi_{E,\Xi,\Lambda} \in \pi_{E,\Xi,\Lambda}(C^\infty_c(G, K)).$$

By Proposition 1.2 this in turn is equivalent to

$$\varphi_{E,\Xi,\Lambda} \in \pi_{E,\Xi,\Lambda}(\mathbb{H}(G, K)).$$

If follows that $\varphi \in \mathrm{PW}^A(G, K) \iff \varphi \in \mathrm{PW}^H(G, K)$. □

We now come to the main result of our paper.

Theorem 5.11. *The map* $\mathrm{pr} : \varphi \mapsto \varphi_{P_0}$ *defines a linear isomorphism from* $\mathrm{PW}^D(G, K)$ *onto* $\mathrm{PW}^A(G, K)$.

Proof. This follows from combining Theorems 5.6, 5.8 and 5.10. □

5.5. Another useful characterization of the Paley–Wiener space. In this subsection we will obtain another useful characterization of the Paley–Wiener space $\mathrm{PW}^A(G, K)$. This will allow us to derive the Paley–Wiener theorem due to Helgason [8] and Gangolli [7] from Arthur's Paley–Wiener theorem.

We define the Fourier transform $\mathfrak{F} : C_c^\infty(G, K)_{K \times K} \to \mathcal{O}(\mathfrak{a}_\mathbb{C}^*) \otimes \mathcal{S}_{P_0}$ by

$$\mathfrak{F}(f)_\xi(\lambda) = \hat{f}(P_0, \xi, \lambda),$$

see (3.2). From this definition it is immediate that $\mathfrak{F}$ is a $K \times K$-equivariant linear map.

In terms of this Fourier transform, Arthur's Paley–Wiener theorem may be stated as follows (see [1, 3]).

Theorem 5.12 (Arthur's Paley–Wiener theorem). *The map $\mathfrak{F}$ is a $K \times K$-equivariant linear isomorphism from $C_c^\infty(G, K)_{K \times K}$ onto $\mathrm{PW}^A(G, K)$.*

By Theorem 5.10 the Paley–Wiener space $\mathrm{PW}^A(G, K)$ equals the space $\mathrm{PW}^H(G, K)$ introduced in Definition 5.7. We shall now give another characterization of that space.

We use the notation $\mathrm{PW}(\mathfrak{a})$ for the (Euclidean) Paley–Wiener space associated with $\mathfrak{a}$, i.e., $\mathrm{PW}(\mathfrak{a})$ is the image of the classical Fourier transform $C_c^\infty(\mathfrak{a}) \to \mathcal{O}(\mathfrak{a}_\mathbb{C}^*)$. Then the space $\mathrm{PW}_{P_0}^{\mathrm{pre}}(G, K)$, introduced in the beginning of Section 5.4, equals $\mathrm{PW}(\mathfrak{a}) \otimes \mathcal{S}(P_0)$.

Proposition 5.13. *The space $\mathrm{PW}^H(G, K)$ is equal to the space of all functions $\varphi \in \mathrm{PW}(\mathfrak{a}) \otimes \mathcal{S}(P_0)$ with the following property.*

For each finite number of triples $(u_j, \xi_j, \lambda_j) \in S(\mathfrak{a}^*) \times M^\wedge \times \mathfrak{a}_\mathbb{C}^*$, $1 \le j \le n$, there exists an element $h \in \mathbb{H}(G, K)$ such that

$$\varphi(\xi_j, \lambda_j; u_j) = \pi_{P_0, \xi_j, \lambda_j; u_j}(h) \quad \text{for all } 1 \le j \le n.$$

Proof. We first assume that $\varphi \in \mathrm{PW}^H(G, K)$. Then $\varphi \in \mathrm{PW}(A) \otimes \mathcal{S}(P_0)$. Let a finite number of triples (u_j, ξ_j, λ_j) be given. For each u_j there exists a finite dimensional $\mathcal{O}_0$-module E_j, and a linear functional $\eta_j \in \mathrm{End}(E_j)^*$ such that $\eta_j \circ f^{(E_j)} = f(\lambda_j; u_j)$, for all $f \in \mathcal{O}(\mathfrak{a}_\mathbb{C}^*)$; see Lemma 2.11. Put $E = \oplus E_j$, and let $\mathrm{pr}_j : E \to E_j$ denote the associated projection maps. There exists an element $h \in \mathbb{H}(G, K)$ such that $\varphi_{E, \Xi, \Lambda} = \pi_{E, \Xi, \Lambda}(h)$. This implies that $\varphi_{E, \xi_j, \lambda_j} = \pi_{E, \xi_j, \lambda_j}(h)$. for each j. By application of $\eta_j \circ \mathrm{pr}_j \otimes I$ to the latter expression it follows that $\varphi(\xi_j, \lambda_j; u_j) = \pi_{P_0, \xi_j, \lambda_j; u_j}(h)$, for all j. This proves that $\mathrm{PW}^H(G, K)$ is included in the space described.

To obtain the other inclusion, let $\varphi \in \mathrm{PW}(\mathfrak{a}) \otimes \mathcal{S}(P_0)$ satisfy the conditions of the space described. Let E be a finite dimensional $\mathcal{O}_0$-module, and let $\Xi \subset M^\wedge$ and $\Lambda \subset \mathfrak{a}_\mathbb{C}^*$ be finite sets. Fix a basis $\{\eta^k\}$ of $\mathrm{End}(E)^*$. We may number the elements of Λ by λ_j. Then for each j, k there exists a $u_j^k \in S(\mathfrak{a}^*)$ such that $\eta^k f^{(E)}(\lambda_j) = f(\lambda_j; u_j^k)$ for all $f \in \mathcal{O}(\mathfrak{a}_\mathbb{C}^*)$, see Lemma 2.10. By the assumption on φ, there exists an element $h \in \mathbb{H}(G, K)$ such that $\varphi(\xi, \lambda_j; u_j^k) = \pi_{P_0, \xi, \lambda_j; u_j^k}(h)$ for all $\xi \in \Xi$ and all j, k. It follows that

$$(\eta^k \otimes 1) \circ \varphi_{E, \xi, \lambda_j} = (\eta^k \otimes 1) \circ \pi_{E, \xi, \lambda_j}(h),$$

for all k, j and $\xi \in \Xi$. This implies that $\varphi_{E, \xi, \lambda_j} = \pi_{E, \xi, \lambda_j}(h)$ for all j and all $\xi \in \Xi$. Hence, $\varphi_{E, \Xi, \Lambda} = \pi_{E, \Xi, \Lambda}(h)$, and we conclude that $\varphi \in \mathrm{PW}^H(G, K)$. $\square$

Corollary 5.14. *The space* $PW^H(G, K)_{11}$ *of* $K \times K$*-fixed elements in* $PW^H(G, K)$ *consists of all functions* $\varphi \in PW(\mathfrak{a}) \otimes S(P_0, 1)_{11}$ *such that for each finite number of pairs* $(u_i, \lambda_i) \in S(\mathfrak{a}^*) \times \mathfrak{a}_{\mathbb{C}}^*$, $1 \leq j \leq n$, *there exists an element* $h \in \mathbb{H}(G, K)_{11}$ *such that*

$$\varphi(\lambda_j; u_j) = \pi_{P_0, 1, \lambda_j; u_j}(h) \quad \text{for all } 1 \leq j \leq n.$$

Proof. Since $S(P_0)_{11} = S(P_0, 1)_{11}$, this follows from the previous result by projection onto the $K \times K$-type $(1, 1)$. $\qquad\square$

Let $P_1 \in \text{End}(C^\infty(K:1))$ denote the K-equivariant projection onto the one-dimensional subspace of $C^\infty(M\backslash K)$ consisting of the constant functions. Then we observe that

$$S(P_0, 1)_{11} \simeq \text{End}(C^\infty(K:1))_{11} = \mathbb{C} P_1.$$

Accordingly, we may view $PW^H(G, K)_{11}$ as a subspace of $PW(\mathfrak{a}) \otimes \mathbb{C} P_1$.

The inclusion map $\iota : K \to G$ induces a continuous linear map $\iota^* : C^\infty(G) \to C^\infty(K)$ by pull-back. The transposed of ι^* is an injective continuous linear map $\iota_* : \mathcal{E}'(K) \to \mathcal{E}'(G)$. Accordingly, we shall use this map to view $\mathcal{E}'(K)$ as a subspace of $\mathcal{E}'(G)$. In particular, the normalized Haar measure dk will be viewed as the element of $\mathbb{H}(G, K)$ given by $dk(\varphi) = \int_K \varphi(k) \, dk$, for $\varphi \in C^\infty(G)$. Clearly, $dk \in \mathbb{H}(G, K)_{11}$.

In the following we will use the notation R (resp. L) for the right (resp. left) regular action of G on $\mathcal{E}'(G)$. The associated representations of $U(\mathfrak{g})$ are denoted by the same symbols.

Lemma 5.15. *The map* $u \mapsto R_u dk$ *induces a linear isomorphism from the space* $U(\mathfrak{g})^K / U(\mathfrak{g})^K \cap U(\mathfrak{g})\mathfrak{k}$ *onto* $\mathbb{H}(G, K)_{11}$.

Proof. We define the linear map $\alpha : U(\mathfrak{g}) \otimes \mathcal{E}'(K) \to \mathcal{E}'(G)$ by

$$\alpha(u \otimes T) = R_u T.$$

Then α factors to a linear isomorphism

$$\bar{\alpha} : U(\mathfrak{g}) \otimes_{U(\mathfrak{k})} \mathcal{E}'(K) \to \mathbb{H}(G, K),$$

see [9, Chapter I, §6]. This map intertwines the $K \times K$-actions $(\text{Ad} \otimes R) \times (1 \otimes L)$ and $R \times L$, hence restricts to an isomorphism

$$(U(\mathfrak{g}) \otimes_{U(\mathfrak{k})} \mathcal{E}'(K))_{11} \xrightarrow{\simeq} \mathbb{H}(G, K)_{11}.$$

The space of left K-invariants in $\mathcal{E}'(K)$ equals $\mathbb{C} dk$. As dk is also right K-invariant, we see that

$$(U(\mathfrak{g}) \otimes_{U(\mathfrak{k})} \mathcal{E}'(K))_{11} \simeq U(\mathfrak{g})^K \otimes_{U(\mathfrak{k})} \mathbb{C} dk \simeq U(\mathfrak{g})^K / U(\mathfrak{g})^K \cap U(\mathfrak{g})\mathfrak{k}.$$

The result now follows. $\qquad\square$

In the following lemma, W denotes the Weyl group of $\mathfrak{a}$ in $\mathfrak{g}$.

Lemma 5.16. *The image of* $\mathbb{H}(G, K)_{11}$ *under Fourier transform* $h \mapsto \hat{h}$ *equals the subspace* $P(\mathfrak{a}_{\mathbb{C}}^*)^W \otimes \mathbb{C} P_1$ *of* $\mathcal{O}(\mathfrak{a}_{\mathbb{C}}^*) \otimes S(P_0)$.

Proof. Denote the image by S. Then S is contained in

$$\mathcal{O}(\mathfrak{a}_\mathbb{C}^*) \otimes \mathcal{S}(P_0)_{11} \simeq \mathcal{O}(\mathfrak{a}_\mathbb{C}^*) \otimes \mathbb{C}P_1.$$

Let $h \in \mathbb{H}(G, K)_{11}$. Then it follows that the Fourier transform of h is of the form $\psi \otimes P_1$, with $\psi \in \mathcal{O}(\mathfrak{a}_\mathbb{C}^*)$. In view of the previous result, $h = R_u(dk)$, with $u \in U(\mathfrak{g})^K$. Hence, for all $\lambda \in \mathfrak{a}_\mathbb{C}^*$,

$$\psi(\lambda)1_{M \backslash K} = (\psi(\lambda) \otimes P_1)1_{M \backslash K} = \pi_{P_0,1,\lambda}(h)1_{M \backslash K}$$
$$= \pi_{P_0,1,\lambda}(u)1_{M \backslash K} = \gamma(u, \lambda)1_{M \backslash K},$$

where γ denotes the Harish-Chandra algebra homomorphism $U(\mathfrak{g})^K \to S(\mathfrak{a})$ which has image $S(\mathfrak{a})^W = P(\mathfrak{a}_\mathbb{C}^*)^W$. The result now follows by application of Lemma 5.15. $\square$

Corollary 5.17. *The space* $\mathrm{PW}^H(G, K)_{11}$ *equals* $\mathrm{PW}(\mathfrak{a})^W \otimes \mathbb{C}P_1$.

Proof. Let $\varphi \in \mathrm{PW}^H(G, K)_{11}$. Let $\lambda \in \mathfrak{a}_\mathbb{C}^*$ and $w \in W$. Then by Corollary 5.14 there exists an element $h \in \mathbb{H}(G, K)_{11}$ such that $\varphi(\lambda) = \hat{h}(P_0, 1, \lambda)$ and $\varphi(w\lambda) = \hat{h}(P_0, 1, w\lambda)$. As $\hat{h}(P_0, 1, \lambda) = \hat{h}(P_0, 1, w\lambda)$ by Lemma 5.16, we see that φ is W-invariant. Hence, it follows that $\varphi \in \mathrm{PW}(\mathfrak{a})^W \otimes \mathbb{C}P_1$.

Conversely, assume that $\varphi \in \mathrm{PW}(\mathfrak{a})^W \otimes \mathbb{C}P_1$. Write $\varphi = \psi \otimes P_1$, then ψ is a W-invariant holomorphic function. Let $(u_j, \lambda_j) \in S(\mathfrak{a}^*) \times \mathfrak{a}_\mathbb{C}^*$, $1 \le j \le n$. Then there exists an element $p \in P(\mathfrak{a}^*)^W$ such that

$$\psi(\lambda_j; u_j) = p(\lambda_j; u_j) \quad \text{for all } 1 \le j \le n.$$

In view of Lemma 5.16 there exists an element $h \in \mathbb{H}(G, K)_{11}$ such that $\hat{h}(P_0, 1, \lambda) = p \otimes P_1$. Hence,

$$\varphi(\lambda_j; u_j) = \hat{h}(P_0, 1, \lambda_j; u_j) = \pi_{P_0,1,\lambda_j;u_j}(h) \quad \text{for all } 1 \le j \le n.$$

In view of Corollary 5.14 it follows that $\varphi \in \mathrm{PW}^H(G, K)_{11}$. $\square$

We now note that the spherical Fourier transform

$$\mathcal{F}_{11} : C_c^\infty(G, K)_{11} \to \mathcal{O}(\mathfrak{a}_\mathbb{C}^*)$$

is given by the formula

$$\mathcal{F}_{11}f(\lambda)1_{K/M} = \pi_{P_0,1,\lambda}(f)1_{K/M}.$$

This implies that for all $f \in C_c^\infty(G, K)_{11}$ we have

$$\mathcal{F}_{11}f(\lambda) \otimes P_1 = \mathfrak{F}f(\lambda).$$

We can now finally deduce the Paley–Wiener theorem of Helgason [8] and Gangolli [7].

Corollary 5.18. $\mathcal{F}_{11}(C_c^\infty(G, K)_{11}) = \mathrm{PW}(\mathfrak{a})^W$.

Proof. By $K \times K$-equivariance, it follows from Arthur's Paley–Wiener theorem that

$$\mathcal{F}_{11}(C_c^\infty(G, K)_{11}) \otimes \mathbb{C}P_1 = \mathfrak{F}(C_c^\infty(G, K)_{11}) = \mathrm{PW}^A(G, K)_{11}.$$

The latter space equals $\mathrm{PW}^H(G, K)_{11}$, by Theorem 5.10. Now apply Corollary 5.17. $\square$

A. Some topological results

In this appendix, all locally convex spaces will be assumed to be complex and Hausdorff. If X is a locally compact Hausdorff space, and V a quasi-complete locally convex space, then $C(X, V)$, equipped with the topology of uniform convergence on compact subsets, is quasi-complete as well. Let Ω be an open subset of a finite dimensional complex linear space $v_{\mathbb{C}}$. Then $\mathcal{O}(\Omega, V)$, the space of holomorphic functions $\Omega \to V$, is a closed subspace of $C(\Omega, V)$. Moreover, the map $(f, v) \mapsto (z \mapsto f(z)v)$ induces an embedding of the algebraic tensor product $\mathcal{O}(\Omega) \otimes V$ onto a subspace of $\mathcal{O}(\Omega, V)$. Accordingly, we shall view this tensor product as a subspace. Let $P(v_{\mathbb{C}})$ denote the space of polynomial functions $v_{\mathbb{C}} \to \mathbb{C}$.

Lemma A.1. *The algebraic tensor product $\mathcal{P}(v_{\mathbb{C}}) \otimes V$ is dense in $\mathcal{O}(\Omega, V)$.*

In particular, $\mathcal{O}(\Omega) \otimes V$ is a dense subspace of $\mathcal{O}(\Omega, V)$.

Proof. By using partitions of unity, one readily sees that $C(\Omega) \otimes V$ is dense in $C(\Omega, V)$. On the other hand, by application of the Stone–Weierstrass theorem, it follows that $P(v_{\mathbb{C}})$ is dense in $C(\Omega)$. The lemma now readily follows. $\square$

Let V_1, V_2 and V_3 be quasi-complete locally convex spaces and let $\beta : V_1 \times V_2 \to V_3$ be a bilinear map. If X is a set, then β induces a $\mathbb{C}^X$-bilinear map

$$\beta_* : V_1^X \times V_2^X \to V_3^X, \quad (f_1, f_2) \mapsto \big(x \mapsto \beta(f_1(x), f_2(x))\big).$$

If X is a locally compact Hausdorff space, we will say that β preserves continuity on X if β_* maps $C(X, V_1) \times C(X, V_2)$ into $C(X, V_3)$. We note that $\beta_* : C(X, V_1) \times C(X, V_2) \to C(X, V_3)$ is $C(X)$-bilinear. Similarly, if Ω is an open subset of $v_{\mathbb{C}}$, we will say that β preserves holomorphy on Ω if β_* maps $\mathcal{O}(\Omega, V_1) \times \mathcal{O}(\Omega, V_2)$ into $\mathcal{O}(\Omega, V_3)$. Note that the map β_* is $\mathcal{O}(\Omega)$-bilinear. If X, Y are topological spaces, and $A \subset X$, then a map $f : X \to Y$ is said to be continuous relative to A if $f|_A : A \to Y$ is continuous (with respect to the restriction topology on A).

Lemma A.2. *Assume that β is separately continuous and in addition continuous relative to every compact subset of $V_1 \times V_2$. Let X be a locally compact Hausdorff space. Then β preserves continuity on X. Moreover,*

(a) *if V_1 is barrelled, then $\beta_* : C(X, V_1) \times C(X, V_2) \to C(X, V_3)$ is continuous in the first variable;*

(b) *if V_2 is barrelled, then $\beta_* : C(X, V_1) \times C(X, V_2) \to C(X, V_3)$ is continuous in the second variable.*

Proof. Let $K \subset X$ be compact, and $f_1 \in C(X, V_1)$, $f_2 \in C(X, V_2)$. Put $C_1 := f_1(K)$, $C_2 := f_2(K)$. Since β is continuous relative to $C_1 \times C_2$ and $x \mapsto (f_1(x), f_2(x))$ has continuous restriction to K, with values in $C_1 \times C_2$, it follows that $\beta_*(f_1, f_2)$ is continuous relative to K. Since X is locally compact, we see that β_* has values in $C(X, V_3)$. This establishes the first assertion.

We now turn to the remaining assertions. By symmetry, it suffices to establish (a). Thus, assume that V_1 is barrelled and let $f_2 \in C(X, V_2)$ be fixed. From the first part of the proof

we know that $f_1 \mapsto \beta_*(f_1, f_2)$ is a linear map from $C(X, V_1)$ to $C(X, V_3)$. To establish its continuity, fix a seminorm q_3 of V_3, and a compact subset K of X. Put $C_2 = f_2(K)$. Then the family of continuous linear maps $\beta(\cdot, v_2) \in \text{Hom}(V_1, V_3)$ is pointwise bounded, for $v_2 \in C_2$. By barrelledness of V_1 it follows that the family is equicontinuous. Hence, there exists a continuous seminorm q_1 on V_1 such that

$$q_3(\beta(v_1, v_2)) \le q_1(v_1), \quad v_1 \in V_1, v_2 \in C_2.$$

Let $f_1 \in C(X, V_1)$. Then substituting $f_1(x)$ for v_1 and $f_2(x)$ for v_2 ($x \in K$) in the above estimate, we find that

$$\sup_{x \in K} q_3(\beta_*(f_1, f_2)(x)) \le \sup_{x \in K} q_1(f_1(x)).$$

This establishes the continuity. □

Lemma A.3. *Assume that V_1, V_2 are barrelled, and let β be as in Lemma A.2. If $\Omega \subset \mathfrak{v}_\mathbb{C}$ is open, then β preserves holomorphy on Ω. Moreover, the map*

$$\beta_* : \mathcal{O}(\Omega, V_1) \times \mathcal{O}(\Omega, V_2) \to \mathcal{O}(\Omega, V_3)$$

is a separately continuous $\mathcal{O}(\Omega)$-bilinear map.

Proof. It is readily seen that β_* maps the subspace $(\mathcal{O}(\Omega) \otimes V_1) \times (\mathcal{O}(\Omega) \otimes V_2)$ into the closed subspace $\mathcal{O}(\Omega, V_3)$ of the quasi-complete space $C(\Omega, V_3)$. By continuity of β_* in the first variable (see Lemma A.2), and by density and closedness, it follows that β_* maps $\mathcal{O}(\Omega, V_1) \times \mathcal{O}(\Omega) \otimes V_2$ into $\mathcal{O}(\Omega, V_3)$. By the same kind of argument applied to the second variable, the result follows. □

Examples of a different nature are provided by the composition of maps. If V, W are two locally convex spaces then by $\text{Hom}(V, W)$ we denote the space of continuous complex linear maps $V \to W$. Unless otherwise specified this space is equipped with the strong operator topology. If W is quasi-complete, and V barrelled, then by application of the principle of uniform boundedness, it follows that $\text{Hom}(V, W)$ is quasi-complete (see, e.g., [4, Chapitre III, 27, §4, no. 2, Corollaire 4]). Assume now that V_1, V_2 and V_3 are arbitrary locally convex spaces, not necessarily quasi-complete.

Lemma A.4. *The map*

(A.1) $\qquad \beta : \text{Hom}(V_1, V_2) \times \text{Hom}(V_2, V_3) \to \text{Hom}(V_1, V_3), \quad (A, B) \mapsto B \circ A$

is bilinear and separately continuous. If V_2 is barrelled, then β is continuous relative to subsets of the form $\text{Hom}(V_1, V_2) \times C$, with $C \subset \text{Hom}(V_2, V_3)$ compact.

Proof. As V_2 is barrelled, every compact subset of $\text{Hom}(V_2, V_3)$ is equicontinuous. The result now follows from [4, Chapitre III, 33, § 5, no. 5, Proposition 9]. □

Corollary A.5. *Let V_1, V_2, V_3 be quasi-complete, and assume V_2 is barrelled. Let β be as in* (A.1). *Then β preserves continuity on X, for any locally compact Hausdorff space X.*

Proof. This follows from Lemma A.4 combined with Lemma A.2. □

Lemma A.6.   *Let V_1, V_2, V_3 be quasi-complete, and assume both V_1 and V_2 to be bar-relled. Let β be the composition map given by* (A.1) *and let $\Omega \subset \upsilon_{\mathbb{C}}$ be open. Then β preserves holomorphy on Ω.*

Proof.   Let $A \in \mathcal{O}(\Omega, \mathrm{Hom}(V_1, V_2))$ and $B \in \mathcal{O}(\Omega, \mathrm{Hom}(V_2, V_3))$. Then by the previous result it follows that $\beta_*(A, B) : \mu \mapsto B(\mu)A(\mu)$ is continuous. By barrelledness of V_1, the function $f : (\mu, v) \mapsto A(\mu)v$, $\Omega \times V_1 \to V_2$ is continuous and holomorphic in μ. Similarly, the function $g : (v, v_2) \mapsto B(v)v_2$, $\Omega \times V_2 \to V_3$ is continuous and holomorphic in v. It follows that $h : (\mu, v, v_1) \mapsto g(v, f(\mu, v_1))$ is continuous, and separately holomorphic in μ, v. In particular, for fixed v_1, the map $(\mu, v) \mapsto h(\mu, v, v_1)$ is holomorphic in each of the two variables. If $\xi \in V_3'$ then $(\mu, v) \mapsto \xi h(\mu, v, v_1)$ is continuous and separately holomorphic, hence holomorphic. It follows that $(\mu, v) \mapsto h(\mu, v, v_1)$ is weakly holomorphic, hence holomorphic. We conclude that $\mu \mapsto h(\mu, \mu, v_1)$ is holomorphic. This implies that the function $(\mu, v_1) \mapsto \beta_*(A, B)(\mu)v_1$ is continuous on $\Omega \times V_1$, holomorphic in μ and linear in v. This in turn implies that $\beta_*(A, B)$ is holomorphic as a function with values in $\mathrm{Hom}(V_1, V_3)$.   □

Given an element u of the symmetric algebra $S(v)$ of $\upsilon_{\mathbb{C}}$, we will write ∂_u for the naturally associated constant coefficient (holomorphic) differential operator on $\upsilon_{\mathbb{C}}$, see Section 2.1. This operator acts on the space of holomorphic functions $\mathcal{O}(\Omega)$, for any open subset $\Omega \subset \upsilon_{\mathbb{C}}$. We agree to write $f(\cdot\,; u) = \partial_u f$, for $f \in \mathcal{O}(\Omega)$, see (2.2).

If V is a quasi-complete locally convex space, then $\partial_u \otimes \mathrm{id}_V$ has a unique extension to a continuous linear endomorphism of $\mathcal{O}(\Omega, V)$. Indeed, uniqueness is obvious from density of $\mathcal{O}(\Omega) \otimes V$ in $\mathcal{O}(\Omega, V)$. Existence follows for instance by application of the Cauchy integral formula. Let A be a finite dimensional associative algebra. Suppose

$$M : \mathcal{O}(\Omega) \to \mathcal{O}(\Omega, A)$$

is a continuous algebra homomorphism which can be represented by an element from $S(v) \otimes A$, i.e., there exist $u_i \in S(v)$ and $a_i \in A$ such that

$$(\text{A.2}) \qquad\qquad Mf = \sum_i \partial_{u_i} f \cdot a_i.$$

Then it follows from the above that the map $M \otimes \mathrm{id}_V$ has a unique extension to a continuous linear map $\mathcal{O}(\Omega, V) \to \mathcal{O}(\Omega, A \otimes V)$. The extension is denoted by $M \,\widehat{\otimes}\, \mathrm{id}_V$, or more briefly by M again.

Remark A.7.   In the present paper, the following examples are of particular importance. Let V be a quasi-complete locally convex space and let $\mathcal{I}$ be a cofinite ideal in $\mathcal{O}_0$. Then $M = J_{\mathcal{I}}$ is of the form (A.2), see (2.4), and defines an algebra homomorphism $\mathcal{O}(\Omega) \to \mathcal{O}(\Omega, \mathcal{O}_0/\mathcal{I})$. It follows that $J_{\mathcal{I}} \otimes \mathrm{id}_V : \mathcal{O}(\Omega) \otimes V \to \mathcal{O}(\Omega, \mathcal{O}_0/\mathcal{I}) \otimes V$ has a unique extension to a continuous linear map

$$J_{\mathcal{I}} \,\widehat{\otimes}\, \mathrm{id}_V : \mathcal{O}(\Omega, V) \to \mathcal{O}(\Omega, \mathcal{O}_0/\mathcal{I} \otimes V),$$

which will often briefly be denoted by $J_{\mathcal{I}}$ again. Similarly, let E be a finite dimensional $\mathcal{O}_0$-module. Then $J^{(E)} : \mathcal{O}(\Omega) \to \mathcal{O}(\Omega, \mathrm{End}(E))$ is an algebra homomorphism of the form (A.2), see Definition 2.7 and Corollary 2.12. Hence, $J^{(E)} \otimes \mathrm{id}_V$ has a unique extension to a

continuous linear map

$$J^{(E)} \widehat{\otimes} \, \mathrm{id}_V : \mathcal{O}(\Omega, V) \to \mathcal{O}(\Omega, \mathrm{End}(E) \otimes V),$$

which will often briefly be denoted by $J^{(E)} : f \mapsto f^{(E)}$.

Lemma A.8. *Let V_1, V_2 and V_3 be quasi-complete locally convex Hausdorff spaces, and let $\beta : V_1 \times V_2 \to V_3$ be a bilinear form which preserves holomorphy on the open subset Ω of $v_{\mathbb{C}}$. Let $\beta' : E_1 \times E_2 \to E_3$ be a bilinear form of finite dimensional spaces. Then the bilinear form $\beta' \otimes \beta : E_1 \otimes V_1 \times E_2 \otimes V_2 \to E_3 \otimes V_3$ preserves holomorphy on Ω as well.*

Proof. This is an easy consequence of the finite dimensionality of the spaces E_j. ☐

Let m^A denote the (continuous) bilinear product map

$$m^A : (a, b) \mapsto ab, \quad A \times A \to A.$$

Proposition A.9. *Let V_1, V_2, V_3 be quasi-complete locally convex Hausdorff spaces, and let $\beta : V_1 \times V_2 \to V_3$ be a bilinear form which preserves holomorphy on Ω. Let A be a finite dimensional associative algebra, and let $M : \mathcal{O}(\Omega) \to \mathcal{O}(\Omega, A)$ be as in (A.2). Then, for all $f_j \in \mathcal{O}(\Omega, V_j)$, $j = 1, 2$,*

$$M\beta_*(f_1, f_2) = (m^A \otimes \beta)_*(Mf_1, Mf_2).$$

The following lemma prepares for the proof. Given $\mu \in \Omega$, let $\mathcal{M}_\mu$ be the maximal ideal of the algebra $\mathcal{O}(\Omega)$ consisting of the functions vanishing at μ.

Lemma A.10. *Let M be as in (A.2). There exists a number $d \in \mathbb{N}$ with the following property. Let V be a quasi-complete locally convex Hausdorff space. Then for every $k \geq d$, the operator $M \widehat{\otimes} \, \mathrm{id}_V$ maps $\mathcal{M}_\mu^k \mathcal{O}(\Omega, V)$ into $\mathcal{M}_\mu^{k-d} \mathcal{O}(\Omega, A \otimes V)$.*

Proof. It suffices to show that, for any element $u \in S(v)$ of order d, the differentiation ∂_u maps $\mathcal{M}_\mu^k \mathcal{O}(\Omega, V)$ into $\mathcal{M}_\mu^{k-d} \mathcal{O}(\Omega, V)$. Clearly, it suffices to do this for $u = X \in v$ and $d = 1$. In this case the result follows from the Leibniz rule

$$(\partial_u \widehat{\otimes} \, \mathrm{id}_V)(\varphi F) = (\partial_u \varphi) F + \varphi (\partial_u \widehat{\otimes} \, \mathrm{id}_V)(F),$$

for all $\varphi \in \mathcal{O}(\Omega)$ and $F \in \mathcal{O}(\Omega, V)$. ☐

Proof of Proposition A.9. We will first prove the identity for $f_j \in \mathcal{O}(\Omega) \otimes V_j$, $j = 1, 2$. Let $m : \mathcal{O}(\Omega) \times \mathcal{O}(\Omega) \to \mathcal{O}(\Omega)$ denote the multiplication map. Then

$$\beta_*(f_1, f_2) = (m \otimes \beta)(f_1, f_2) \in \mathcal{O}(\Omega) \otimes V_3.$$

Moreover, since $M : \mathcal{O}(\Omega) \to \mathcal{O}(\Omega, A)$ is an algebra homomorphism, it follows that

$$\begin{aligned}
M\beta_*(f_1, f_2) &= (M \otimes \mathrm{id}_{V_3}) \circ (m \otimes \beta)(f_1, f_2) \\
&= (m^A \otimes \beta)_*([M \otimes \mathrm{id}_{V_1}] f_1, [M \otimes \mathrm{id}_{V_2}] f_2) \\
&= (m^A \otimes \beta)_*(Mf_1, Mf_2).
\end{aligned}$$

We will now establish the identity for general $f_j \in \mathcal{O}(\Omega, V_j)$. Clearly, it suffices to prove the identity of holomorphic functions at a fixed point μ of Ω. Moreover, since the operations $(f_1, f_2) \mapsto \beta_*(f_1, f_2)$ and M commute with restriction, we may assume that Ω is a polydisk centered at μ. Let d be as in Lemma A.10. Fix $k > d$. From the power series expansions of f_1, f_2 at the point μ it follows that there exist polynomials $p_j \in P(\mathfrak{v}) \otimes V_j$ and holomorphic functions $r_j \in \mathcal{M}_\mu^k \mathcal{O}(\Omega, V_j)$ such that

$$f_j = p_j + r_j, \quad j = 1, 2.$$

From the $\mathcal{O}(\Omega)$-bilinearity of β_* it readily follows that

(A.3) $\beta_*(f_1, f_2) = \beta_*(p_1, p_2) + R,$

with $R \in \mathcal{M}_\mu^k \mathcal{O}(\Omega, V_3)$. From the $\mathcal{O}(\Omega)$-bilinearity of β_* combined with Lemma A.10 it follows that

(A.4) $(m^A \otimes \beta)_*(Mf_1, Mf_2) = (m^A \otimes \beta)_*(Mp_1, Mp_2) + \rho,$

with $\rho \in \mathcal{M}_\mu^{k-d} \mathcal{O}(\Omega, V_3 \otimes A)$. Evaluation at the point μ gives $\mathrm{ev}_\mu(R) = R(\mu) = 0$ and $\mathrm{ev}_\mu(\rho) = \rho(\mu) = 0$. Combining this with (A.3), (A.4) and the first part of the proof applied to (p_1, p_2) we find

$$\begin{aligned}
\mathrm{ev}_\mu \, M\beta_*(f_1, f_2) &= \mathrm{ev}_\mu \, M\beta_*(p_1, p_2) \\
&= \mathrm{ev}_\mu (m^A \otimes \beta)_*(Mp_1, Mp_2) \\
&= \mathrm{ev}_\mu (m^A \otimes \beta)_*(Mf_1, Mf_2).
\end{aligned}$$

This establishes the desired identity at the point μ. $\square$

Corollary A.11.   *Let V_1, V_2, V_3 and β, Ω be as in Proposition A.9.*

(a) *For any cofinite ideal $\mathcal{I}$ of $\mathcal{O}_0$,*

$$J_{\mathcal{I}} \circ \beta_* = (m^{\mathcal{O}_0/\mathcal{I}} \otimes \beta)_* \circ (J_{\mathcal{I}}, J_{\mathcal{I}})$$

   on $\mathcal{O}(\Omega, V_1) \times \mathcal{O}(\Omega, V_2)$.

(b) *For any finite dimensional $\mathcal{O}_0$-module E,*

$$J^{(E)} \circ \beta_* = (m^{\mathrm{End}(E)} \otimes \beta)_* \circ (J^{(E)}, J^{(E)})$$

   on $\mathcal{O}(\Omega, V_1) \times \mathcal{O}(\Omega, V_2)$.

Proof.   In view of Remark A.7, this follows from Proposition A.9. $\square$

Corollary A.12.   *Let V_1, V_2, V_3 be quasi-complete locally convex Hausdorff spaces, and assume that V_1, V_2 are barrelled. Let $S \in \mathcal{O}(\Omega, \mathrm{Hom}(V_1, V_2))$ and $T \in \mathcal{O}(\Omega, \mathrm{Hom}(V_2, V_3))$. Then the map $TS : \mu \mapsto T(\mu)S(\mu)$ belongs to $\mathcal{O}(\Omega, \mathrm{Hom}(V_1, V_3))$. Moreover, the following holds.*

(a) *For any cofinite ideal $\mathcal{I}$ of $\mathcal{O}_0$,*

$$J_{\mathcal{I}}(TS) = J_{\mathcal{I}}(T)J_{\mathcal{I}}(S).$$

(b) *For any finite dimensional $\mathcal{O}_0$-module E,*

$$(TS)^{(E)} = T^{(E)}S^{(E)}.$$

The expression on the right-hand side of (a) should be read as the natural pointwise product of the functions

$$J_{\mathcal{I}}S \in \mathcal{O}(\Omega, \mathcal{O}_0/\mathcal{I} \otimes \mathrm{Hom}(V_1, V_2)) \quad \text{and} \quad J_{\mathcal{I}}T \in \mathcal{O}(\Omega, \mathcal{O}_0/\mathcal{I} \otimes \mathrm{Hom}(V_2, V_3)).$$

For (b) we note that $S^{(E)}$ is holomorphic on Ω with values in

$$\mathrm{End}(E) \otimes \mathrm{Hom}(V_1, V_2) \simeq \mathrm{Hom}(E \otimes V_1, E \otimes V_2).$$

Likewise, $T^{(E)}$ is a $\mathrm{Hom}(E \otimes V_2, E \otimes V_3)$-valued holomorphic function on Ω. Accordingly, the expression on the right-hand side of (b) should be read as the pointwise composition.

Proof. This follows from the previous corollary combined with Lemma A.6.   □

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