# Uniform temperedness of Whittaker integrals

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Setting: $G$ real reductive 

$G = KAN_0$ Iwasawa deco

$\Sigma = R(\mathfrak{g}, \mathfrak{a})$, $\Sigma^+ \leftrightarrow N_0$, $\Delta = \{\text{simple roots}\}$

$\chi \in \hat{N}_0$ : unitary character

Regular: $\forall_{\alpha \in \Delta} \quad d\chi(e)|_{\mathfrak{g}_\alpha} \neq 0$

$C(G/N_0, \chi) := \{ f \in C^0(G) \mid f(gn_0) = \chi(n_0)^{-1} f(g) \}$

$C_c(G/N_0, \chi)$ : supp $f$ cpt mod $N_0$

$L^2(G/N_0, \chi)$ : $L^2$ completion, $L =: \mathrm{Ind}_{N_0}^G(\chi)$.

Whittaker Plancherel: HC '82 announcement

Harish-Chandra

History Whittaker Plancherel $\mathrm{Ind}_{N_0}^{G}(\chi)$

HC 1982: announcement

HC 2018: Collected Papers V, Posthumous

Discrete part OK

Remaining part not complete

W 1992: RRG2 discrete part OK

Remaining part, erroneous estimate

(Kuit & ~).

Wallach

# Whittaker vectors / coefficients

$(\pi, H_\pi)$ unitary rep$^n$, $H_\pi^{-\infty} := \overline{H_\pi^{\infty}}'$

$$H_\pi^\infty \subset H_\pi \subset H_\pi^{-\infty}$$

$\langle \cdot , \cdot \rangle$

Whittaker vectors

$$(H_\pi^{-\infty})_\chi := \{ j \in H_\pi^{-\infty} \mid \pi^{-\infty}(n) j = \chi(n) j \ (\forall n \in N_0) \}$$

**Lemma** (HC, W): $\pi$ irred $\Rightarrow \dim (H_\pi^{-\infty})_\chi < \infty$

$$Wh_j : H_\pi^\infty \longrightarrow C^\infty(G/N_0, \chi) \quad \text{equivariant}$$

$$Wh_j(v) : x \longmapsto \langle \pi(x)^{-1} v, j \rangle \qquad (v \in H_\pi^\infty)$$

W-coeff$^t$

## Schwartz space

$\mathcal{C}(G/N_0, \chi) := \{ f \in C^\infty(G/N_0, \chi) \mid \forall_{u \in U(\mathfrak{g})}, N \in \mathbb{N}:$

$$\sup_{k \in K,\, a \in A} (1 + |\log a|)^N a^\rho \, |L_u f(ka)| < \infty \}$$

discrete series

**Thm (HC, W):** For $\pi \in \widehat{G}_{ds}$, $j \in (H_\pi^{-\infty})_\chi$

$$wh_j : H_\pi^\infty \longrightarrow \mathcal{C}(G/N_0, \chi) \quad \text{continuous}$$

**Induced Reps:** $P = M_P A_P N_P$ standard

$\sigma \in \widehat{M}_{P,ds}$, $\nu \in \mathfrak{a}^*_{P\mathbb{C}}$

$\mathrm{Ind}_{\bar{P}}^{G}(\sigma \otimes \nu) := L$ in $L^2(G/\bar{P} : \sigma : \nu)$

$\mathrm{Ind}_{\bar{P}}^{G}(\sigma \otimes \nu) := L$ in $L^2(G/\bar{P} : \sigma : \nu) :=$

$$\{ f \in L^2_{loc}(G, \sigma) \mid f(g m a \bar{n}) = a^{-\nu + \rho_P} \sigma(m)^{-1} f(g) \}$$

$$\langle \cdot , \cdot \rangle : L^2(G/\bar{P} : \sigma : \nu) \times L^2(G/\bar{P} : \sigma : -\bar{\nu}) \longrightarrow \mathbb{C}$$

$$(f, g) \longmapsto \int_{K/K_P} \langle f(k), g(k) \rangle_\sigma \, d\dot{k}$$

sesquilinear
perfect
equivariant

$\nu \in i\mathfrak{a}_P^* \Longrightarrow \mathrm{Ind}_{\bar{P}}^{G}(\sigma \otimes \nu)$ unitary

compact picture: $L^2(G/\bar{P} : \sigma : \nu) \xrightarrow[\simeq]{\text{res}} L^2(K/K_P : \sigma_P)$

$K \cap M_P$ $\qquad$ $\sigma|_{K_P}$

$L \longleftrightarrow \pi_{\bar{P}, \sigma, \nu}$

## $C^\infty$ & $C^{-\infty}$ vectors

$$L^2(G/\bar{P}:\sigma:\nu)^\infty = C^\infty(G/\bar{P}:\sigma:\nu) \subseteq C^\infty(G, H_\sigma)$$

$$C^{-\infty}(G/\bar{P}:\sigma:\nu) := \overline{C^\infty(G/\bar{P}:\sigma:-\bar{\nu})}',$$

$$\begin{array}{rccccl} \langle\cdot,\cdot\rangle : & C^\infty(G/\bar{P}:\sigma:\nu) & \times & C^{-\infty}(G/\bar{P}:\sigma:-\bar{\nu}) & \to & \mathbb{C} \\ & & & \cup & & \\ & \text{—''—} & \times & L^2(G/\bar{P}:\sigma:-\bar{\nu}) & \xrightarrow{\langle\cdot,\cdot\rangle} & \end{array}$$

**Problem** Determine $C^{-\infty}(G/\bar{P}:\sigma:\nu)_\chi$

**Lemma** If $j \in$ then

(a) $j|_{N_P\bar{P}} \in C(N_P\bar{P}, H_\sigma^{-\infty})$

(b) $ev_e(j) = j(e) \in (H_\sigma^{-\infty})_{\chi_P}$

**Thm** (HC, W): $ev_e : C^{-\infty}(G/\bar{P} : \sigma : \nu)_\chi \hookrightarrow (H_\sigma^{-\infty})_{\chi_P}$

**Def** for $R \in \mathbb{R}$, $\mathfrak{a}^*_{P\mathbb{C}}(P, R) :=$

$$= \{ \nu \in \mathfrak{a}^*_{P\mathbb{C}} \mid \langle \mathrm{Re}\,\nu, \alpha \rangle > R \ (\alpha \in \Sigma(\mathfrak{n}_P, \mathfrak{a})) \}$$

**Thm** (HC, W): Let $\eta \in (H_\sigma^{-\infty})_\chi$, $\nu \in \mathfrak{a}^*_{P\mathbb{C}}(P, 0)$. Then

$$j(P, \sigma, \nu, \eta) : N_P \bar{P} \to H_\sigma^{-\infty}$$
$$(n m a \bar{n}) \longmapsto \chi(n)\, a^{-\nu + \rho_P} \xi(m)^{-1} \eta$$

belongs to $L^1(K, H_\sigma^{-\infty})$ and defines an element of $C^{-\infty}(G/\bar{P} : \sigma : \nu)_\chi$. Moreover,

$\nu \longmapsto j(P, \sigma, \nu, \eta)|_K$ is holomorphic $C^{-\infty}(K/K_P : \sigma_P)$-valued

**Goal:** extend $j(\bar{P}, \sigma, \nu, \eta)$ meromy with estimates

**Def**

$$C^{s}(K/K_P : \sigma_P) : \text{ Banach, } \|.\|_s \qquad (s \in \mathbb{N})$$

dual

$$C^{-s}(K/K_P : \sigma_P) := \overline{C^{s}(K/K_P : \sigma_P)}', \quad \|.\|_{-s}$$

Observe: $$C^{-\infty}(K/K_P : \sigma_P) = \bigcup_{s \in \mathbb{N}} C^{-s}(K/K_P : \sigma_P)$$

dir lim topology $\supset$ strong dual topology

Holomorphy w.r.t.

Thm ($\sim$) (Functional equation) There exists

a diff op $D(\nu) : C^{-s} \to C^{-s-t}(K/K_P : \sigma_P)$

of the form

dual hw vector

f.d. hw $\mu = 4\rho_P$

$$D(\nu) = m \circ (I \otimes e^{N_0}) \circ \left[ \pi^{-\infty}_{\bar{P},\sigma,\nu} \otimes \pi_\mu(\underline{z}(\nu)) \right] \circ (\,\cdot \otimes e_K)$$

$\in P(\mathfrak{a}^*_{PC}, \mathfrak{Z})$

K-fixed

s.t.

$$\boxed{j(\bar{P},\sigma,\nu) = D(\nu) \circ j(\bar{P},\sigma,\nu+\mu) \circ R(\nu)}$$

rational, $\in \mathrm{End}(H^{-\infty}_{\sigma,\chi_P})$

**Cor (~):** $\forall_{R \in \mathbb{R}} \; \exists p_R \in P(\mathfrak{a}^*_{P\mathbb{C}}) \; \exists_{s>0,\, N>0}$ s.t.

$p_R(\cdot)\, j(P, \sigma, \cdot)$ extends to holomorphic function

$$\mathfrak{a}^*_{P\mathbb{C}}(0, R) \longrightarrow C^{-s}(K/K_P : \sigma_P) \otimes (H^{-\infty}_{\sigma, \chi_P})'$$

&

$$\forall \nu \in \quad \| p_R(\nu)\, j(P, \sigma, \cdot) \|_{-s} \leq C(1 + |\nu|)^N$$

**Thm (~)** **Holomorphy:** Above valid with $p_R = 1$.

**Pf** involves injectivity of $ev_e$ & Hartog's lemma

**Rem** weak holomorphy due to W.

(11)

Fix $\eta \in H^{-\infty}_{\sigma,\chi_P}$ and put

$I^\infty_{P,\sigma} :=$

$$\mathrm{wh}_\nu := \mathrm{wh}_{j(\bar P,\sigma,\nu,\eta)} : C^\infty(K/K_P : \sigma) \to C^\infty(G/N_0, \chi) .$$

<u>Def</u> $\mathfrak{a}^*_{P\mathbb{C}}(\varepsilon) = \{ \nu \in \mathfrak{a}^*_{P\mathbb{C}} \mid |\mathrm{Re}\,\nu| < \varepsilon \}$

<u>Notation</u> $\mathrm{csn}(I^\infty_{P,\sigma}) := \{$ cont$^s$ seminorms on $I^\infty_{P,\sigma} \}$

$$|(\nu, a)| := (1 + |\nu|)(1 + |\log a|)$$

<u>Cor</u> (iv) (uniformly moderate estimate) $\exists\, \xi \in \mathfrak{a}^* :$

$\exists\, \varepsilon > 0$ $\exists\, s, N > 0$ $\exists\, n \in \mathrm{csn}(I^\infty_{P,\sigma})$ : $\forall f \in I^\infty_{P,\sigma}$

$$|\mathrm{wh}_\nu(f)(a)| \leq |(\nu, a)|^N a^\xi e^{s|\mathrm{Re}\,\nu||\log a|} n(f)$$

$$(\nu \in \mathfrak{a}^*_{P\mathbb{C}}(\varepsilon),\ a \in A).$$

**Cor** ($\nu$) (uniformly moderate estimate) $\exists\, \xi \in \mathfrak{a}^*$

$$\boxed{\begin{array}{l} \exists_{\varepsilon>0}\ \exists_{s,N>0}\ \exists_{n \in \mathrm{csn}(I^\infty_{P,\sigma})} : \forall f \in I^\infty_{P,\sigma} \\ |Wh_\nu(f)(a)| \le |(\nu,a)|^N a^{\xi} e^{s|\mathrm{Re}\,\nu|\,\|\log a\|} n(f) \\ \qquad (\nu \in \mathfrak{a}^*_{P\mathbb{C}}(\varepsilon),\ a \in A) \end{array}}$$

"$\xi$ dominates $(Wh_\nu)$" : $\Longleftrightarrow$ $\boxed{\phantom{xxxxxxxx}}$

**Goal**: reduce to $\xi = -\rho$, (w.l.o.g. $Z(G)$ cpt)

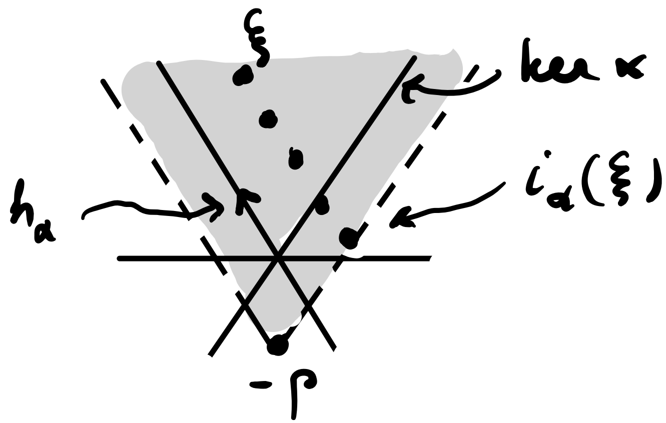
**Def**
- $\preceq$ on $\mathfrak{a}^*$ : $\xi_1 \preceq \xi_2 :\Longleftrightarrow \xi_2 - \xi_1 \geq 0$ on $\mathfrak{a}^+$
- $(h_\alpha)_{\alpha\in\Delta}$ basis of $\mathfrak{a}^*$ dual to $\Delta$
- for $\xi \in \mathfrak{a}^*$, $\alpha \in \Delta$ $\quad i_\alpha(\xi) := \begin{cases} -\rho & \text{on } h_\alpha \\ \xi & \text{on } \mathrm{span}\{h_\beta \mid \beta \in \Delta\setminus\{\alpha\}\}. \end{cases}$

**Lemma** (improvement step) Let $\alpha \in \Delta$, $\xi \in \mathfrak{a}^*$.

Suppose $\xi$ dominates $(wh_\nu)$ and $\xi \succeq -\rho$.

(a) if $\xi - \alpha \succeq -\rho$ then $\forall_{0 \leq c < 1}$ : $\xi - c\alpha$ dominates $(wh_\nu)$ and $\succeq -\rho$.

(b) if not, then $i_\alpha(\xi)$ dominates $(wh_\nu)$ and $i_\alpha(\xi) \succeq -\rho$.




**Proof** By using the differential equations

$$L_Z \, wh_\nu = \gamma(\Lambda_\sigma - \nu, Z) \, wh_\nu \quad (Z \in \mathfrak{Z}),$$

asymptotic analysis along walls $\mathbb{R} h_\alpha$, $\alpha \in \Delta$,
pin down leading exponents with parameter dependence

**Thm:** $-\rho$ dominates $(\mathrm{wh}_\nu)$.

**Meaning**

$$\exists_{\varepsilon>0}\ \exists_{\substack{N>0\\ s>0}}\ \exists_{n\in \mathrm{CSN}(I^\infty_{P,\sigma})}\ \forall_{f\in I^\infty_{P,\sigma}}$$

$$|\mathrm{wh}_\nu(f)(a)| \leq \|(\nu,a)\|^N a^{-\rho} e^{s\,|\mathrm{Re}\,\nu|\,|\log a|}\, n(f)$$

$$(\nu\in \mathfrak{a}^*_{P\mathbb{C}}(\varepsilon),\ a\in A).$$

**Cor** (Uniformly tempered estimate)

Let $f\in I^\infty_{P,\sigma,K}$. Then $\forall_{u\in U(\mathfrak{g})}\ \forall_{D\in S(\mathfrak{a}^*_P)}\ \exists\ N, C>0$:

$$|L_u(\mathrm{wh}_{\nu;D})(f)(a)| \leq C\,\|(\nu,a)\|^N a^{-\rho}$$

$$(\nu\in i\mathfrak{a}^*_P,\ a\in A).$$

**Cor** If $v \in I^{\infty}_{P,\sigma,K}$ then

$$\mathcal{F}_{v,\eta}\varphi(\nu) = \int_{G/N_0} \varphi(x)\, \overline{wh_{\nu}(v)(x)}\, d\dot{x}$$

defines $\mathcal{F}_{v,\eta} : \mathcal{C}(G/N_0, \chi) \to \mathcal{S}(i\mathfrak{a}_P^*)$.

& is continuous linear

Euclidean Schwartz space

**Final Remark:**

Striking analogy with $G/H$ (symmetric space)