Plancherel formulas for reductive groups, symmetric spaces and Whittaker functions

I. Distribution vectors and Fourier transform

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Plancherel decomposition

Setting

- G real reductive group
- K maximal compact, $G = KAN_0$ Iwasawa decomposition
- $H < G$ a closed unimodular subgroup
- $\chi : H \to \mathrm{U}(1)$ unitary character

Function spaces

$$\mathcal{M}(G/H, \chi) := \{f : G \overset{\text{meas}}{\to} \mathbb{C} \mid f(xh) = \chi(h)^{-1} f(x) \quad (x \in G, h \in H)\}$$
$$L^2(G/H, \chi) := \{f \in \mathcal{M}(G/H, \chi) \mid |f| \in L^2(G/H)\}$$

- Left reg^r rep^n: $\quad L = \mathrm{Ind}_H^G(\chi)$ is unitary

Abstract Plancherel decomposition

- $\mathrm{Ind}_H^G(\chi) \simeq \int_{\widehat{G}}^{\oplus} 1_{\mathcal{V}_\pi} \otimes \pi \; d\mu(\pi) \qquad (\widehat{G} := \widehat{G}_u, \; \mathcal{V}_\pi : \text{multiplicity space}).$

Three cases

Symmetric space

- $(G^\sigma)_e < H < G^\sigma$ where $\sigma \in \mathrm{Aut}(G), \sigma^2 = I$.
  w.l.o.g.: $\sigma(K) = K$ (Riemannian special case: $\sigma = \theta$, $H = K$)
- $\chi = 1$

Subcase: group as symmetric space

- $\underline{G}$ real reductive group,
- $G := \underline{G} \times \underline{G} \curvearrowright \underline{G}$: $\qquad (x, y) \cdot g = xgy^{-1}$
  $\implies \underline{G} \simeq G/H$, where $H = \mathrm{diag}(\underline{G})$.
- $H = G^\sigma$, for $\sigma : (x, y) \mapsto (y, x)$.
- $\chi = 1$.

Case of Whittaker functions

- $H = N_0$ (maximal nilpotent),
- $\chi \in \widehat{N_0}$ regular, i.e.: $\forall \alpha \in \Sigma(\mathfrak{n}_0, \mathfrak{a})$ simple: $d\chi(e)|_{\mathfrak{g}_\alpha} \neq 0$.

Generalized vectors

Let $(\pi, \mathcal{H}_\pi) \in \widehat{G} :=$ unitary dual. Define

$$\begin{array}{rcl} \mathcal{H}_\pi^\infty & := & \{v \in \mathcal{H}_\pi \mid [g \mapsto \pi(g)v] \in C^\infty(G, \mathcal{H}_\pi)\} \\ \mathcal{H}_\pi^{-\infty} & := & (\overline{\mathcal{H}_\pi^\infty})'. \end{array}$$

Then $\mathcal{H}_\pi^\infty \subset \mathcal{H}_\pi \hookrightarrow \mathcal{H}_\pi^{-\infty}$.

Put $\mathcal{H}_\pi^{-\infty,\chi} := \{\eta \in H_\pi^{-\infty} \mid h \cdot \eta = \chi(h)\eta \quad (h \in H)\}$.

Frobenius rec^y: $\mathrm{Hom}_G(H_\pi, C^\infty(G/H, \chi)) \simeq \overline{\mathcal{H}_\pi^{-\infty,\chi}}$.

Corollary: π *in Plancherel formula* $\implies \mathcal{H}_\pi^{-\infty,\chi} \neq 0$.

Notation: $\widehat{G}_\chi := \{\pi \in \widehat{G} \mid \mathcal{H}_\pi^{-\infty,\chi} \neq 0\}$.

Group as symmetric space: $\underline{G} = G/H$, $\chi = 1$

$\widehat{G} = \{\pi \otimes \rho \mid \pi, \rho \in \widehat{\underline{G}}\}$.

$\widehat{G}_1 = \widehat{G}_H = \{\pi \otimes \pi^\vee \mid \pi \in \widehat{\underline{G}}\}$.

Discrete series

Definition $\pi \in \widehat{G}$ belongs to discrete series for $\mathrm{Ind}_H^G(\chi)$ iff $\mathrm{Hom}_G(\mathcal{H}_\pi, L^2(G/H,\chi)) \neq 0$. Notation: $\widehat{G}_{\chi,ds} = \{[\pi] \mid \text{such } \pi\}$.

Remark Group case: $G = \underline{G} \times \underline{G}$, $H = \mathrm{diag}(\underline{G}), \chi = 1$.

$$\widehat{G}_{H,ds} := \widehat{G}_{1,ds} = \{\pi \otimes \pi^\vee \mid \pi \in \widehat{\underline{G}}_{ds}\}.$$

Thm, group case (HC)

(a) *$\underline{G}$ has discrete series iff* $\mathrm{rk}\underline{G} = \mathrm{rk}\underline{K}$.

(b) $\pi \in \widehat{\underline{G}}_{ds} \implies$ *inf char of π is real and regular.*

Thm, symm space (Flensted-Jensen, Oshima–Matsuki)

(a) $\widehat{G}_{H,ds} \neq \emptyset$ *iff* $\mathrm{rk}\, G/H = \mathrm{rk}\, K/K \cap H$.

(b) *If π belongs to discrete series of $\underline{G}$ then its infinitesimal $\mathbb{D}(G/H)$-characters are real and regular.*

Thm, Whittaker case (Harish-Chandra, Wallach)

$$\widehat{G}_{\chi,ds} \subset \widehat{G}_{ds}$$

Induced representations

Setting

- $P = M_P A_P N_P$ psg with Langlands deco
- $\sigma \in \widehat{M}_P$ (unitary dual)
- $\nu \in \mathfrak{a}^*_{P\mathbb{C}} \rightsquigarrow$ character: $A_P \to \mathbb{C}^*$, $a \mapsto a^\nu = e^{\nu(\log a)}$]
- normalized induction $\mathrm{Ind}_P^G(\sigma \otimes \nu)$ (unitary for $\nu \in i\mathfrak{a}^*_P$)

Space of smooth vectors

$$C^\infty(G/P : \sigma : \nu) := \{f : G \overset{C^\infty}{\to} \mathcal{H}_\sigma \mid f(xman) = a^{-\nu-\rho_P}\sigma(m)^{-1}f(x)\}.$$

Equipped with $\pi_{P,\sigma,\nu}$:= restriction of left regular representation L.

Compact picture Set $K_P := K \cap M_P$, $\sigma_P = \sigma|_{K_P}$. Since $G = K \times_{K_P} P$, restriction from G to K induces topological linear iso:

$$C^\infty(G/P : \sigma : \nu) \simeq C^\infty(K/K_P : \sigma_P).$$

Define $\pi_{P,\sigma,\nu}$: ass^d transfer of $\mathrm{Ind}_P^G(\sigma \otimes \nu)$ to $C^\infty(K/K_P : \sigma_P)$.

Polar decomposition

$$G = K_{\mathrm{p}}AH, \qquad ({}_{\mathrm{p}}A \subset A).$$

Cases:

(a) Symmetric space: $\sigma(\mathfrak{a}) = \mathfrak{a}$, ${}_{\mathrm{p}}\mathfrak{a} := \mathfrak{a} \cap \mathfrak{h}^{\perp}$ maximal dim.

(b) Group: ${}_{\mathrm{p}}\mathfrak{a} = \{(X, -X) \mid X \in \underline{\mathfrak{a}}\}$.

(c) Whittaker: ${}_{\mathrm{p}}\mathfrak{a} =$ usual $\mathfrak{a}$ (so: Iwasawa decomposition).

Roots: $\Sigma(\mathfrak{g}, {}_{\mathrm{p}}\mathfrak{a})$ (non-red^{d}) root syst^{m}; w.l.o.g.: $\Sigma^{+}(\mathfrak{g}, {}_{\mathrm{p}}\mathfrak{a}) \subset \Sigma(\mathfrak{n}_0, {}_{\mathrm{p}}\mathfrak{a})$.

Standard psgp^{s}: Coxeter complex of $\Sigma^{+}(\mathfrak{g}, {}_{\mathrm{p}}\mathfrak{a}) \rightsquigarrow \mathcal{P}_{\mathrm{st}}$ (finite).

Cases:

(a) Symmetric space: $\mathcal{P}_{\mathrm{st}} = \{P \mid P \ \text{psgp} \ \supset MAN_0, \sigma(P) = \bar{P}\}$.

(b) Group: $G = \underline{G} \times \underline{G}$, $\mathcal{P}_{\mathrm{st}} = \{P \times \bar{P} \mid P \in \mathcal{P}_{\mathrm{st}}(\underline{G})\}$.

(c) Whittaker: $\mathcal{P}_{\mathrm{st}} =$ usual collection of $P \supset MAN_0$

H-orbits on G/P, multiplicity spaces

Lemma *Let $P \in \mathcal{P}_{\mathrm{st}}$. Then there exists a finite subset ${}_P\mathcal{W} \subset N_K(\mathfrak{a}_{\mathfrak{p}})$ such that*

$$(H \backslash G/P)_{\mathrm{open}} = \cup_{v \in {}_P\mathcal{W}} \; Hv^{-1}P \text{ (disjoint)}.$$

Cases

(a) Symmetric spaces: ${}_P\mathcal{W} \overset{1-1}{\longleftrightarrow} W_P(\mathfrak{a}_{\mathfrak{p}}) \backslash W(\mathfrak{a}_{\mathfrak{p}}) / W_{K \cap H}(\mathfrak{a}_{\mathfrak{p}})$.

(b) Group: ${}_P\mathcal{W} = \{e\}$.

(c) Whittaker: ${}_P\mathcal{W} = \{e\}$.

Multiplicity spaces

For $P \in \mathcal{P}_{\mathrm{st}}$, $v \in {}_P\mathcal{W}$ put $(v\chi)_P := [\chi \circ \mathrm{Ad}(v)^{-1}]|_{M_P \cap vHv^{-1}}$.

For $\sigma \in (M_P)^{\wedge}_{\chi, ds}$ define $\mathcal{V}^{\chi}_{\sigma} := \oplus_{v \in {}_P\mathcal{W}} \; \mathcal{H}_{\sigma}^{-\infty, (v\chi)_P}$.

Cases:

(a) Symmetric spaces: $\mathcal{V}_{\sigma} = \oplus_{v \in {}_P\mathcal{W}} \; \mathcal{H}_{\sigma}^{-\infty, M_P \cap vHv^{-1}}$.

(b) Group: $\mathcal{V}^{\chi}_{\sigma} = \mathcal{V}_{\sigma} = \mathcal{H}_{\sigma}^{-\infty, H}$.

(c) Whittaker: $\mathcal{V}^{\chi}_{\sigma} = \mathcal{H}_{\sigma}^{-\infty, \chi|_{M_P \cap N_0}}$.

Generalized vectors

G-equivariant sesquilinear pairing

$$C^\infty(G/P:\sigma:\nu)\times C^\infty(G/P:\sigma:-\bar\nu)\to\mathbb{C},$$
$$(f,g)\mapsto \int_K\langle\, f(k),g(k)\,\rangle_\sigma\,dk.$$

Define:

$$C^{-\infty}(G/P:\sigma:\nu):=\overline{C^\infty(G/P:\sigma:\nu)'}$$

Pairing induces continuous linear injection

$$C^\infty(G/P:\sigma:\nu)\hookrightarrow C^{-\infty}(G/P:\sigma:\nu).$$

Lemma For $P\in\mathcal{P}$, $\sigma\in(M_P)^\wedge_{\chi,ds}$ and $\nu\in{}_{\mathrm{P}}\mathfrak{a}^*_{\mathrm{P}\mathbb{C}}\simeq(\mathfrak{a}_P/\mathfrak{a}_P\cap\mathfrak{h})^*_{\mathbb{C}}$ the evaluation map $j\mapsto(j(v)\mid v\in{}_P\mathcal{W})$ is a well-defined linear map

$$\mathrm{ev}:C^{-\infty}(G/\bar P:\sigma:\nu)^\chi\to\mathcal{V}^\chi_\sigma.$$

Parametrization of generalized vectors

Thm: For $\nu \in {}_{\mathrm{p}}\mathfrak{a}^*_{\mathrm{P}\mathbb{C}} := (\mathfrak{a}_P/\mathfrak{a}_P \cap \mathfrak{h})^*_{\mathbb{C}}$ generic, the map ev is injective.

Thm: The space $\mathcal{V}^{\chi}_{\sigma}$ is finite dimensional. There exists a unique map

$$j(\bar{P}, \sigma, \nu) : \mathcal{V}^{\chi}_{\sigma} \to C^{-\infty}(G/\bar{P} : \sigma : \nu)^{\chi}$$

which is meromorphic in $\nu \in {}_{\mathrm{p}}\mathfrak{a}^*_{\mathrm{P}\mathbb{C}}$ in the compact picture, i.e., as a function with values in $(\mathcal{V}^{\chi}_{\sigma})^* \otimes C^{-\infty}(K/K_P : \sigma_P)$.

Standard intertwining operators

Setting: $P \in \mathcal{P}_{\rm st}$, $\sigma \in (M_P)^\wedge_{\chi,ds}$, $\nu \in {}_{\rm p}\mathfrak{a}^*_{\rm P\mathbb{C}}$.

Theorem: For ${}_{\rm p}\mathfrak{a}^*_{\rm P\mathbb{C}}$ sufficiently $\Sigma(P, {}_{\rm p}\mathfrak{a})$-dominant the following is valid. For each $f \in C^\infty(G/P : \sigma : \nu)$ and all $x \in G$ the integral

$$A(\bar{P}, P, \sigma, \nu) f(x) = \int_{\bar{N}_P} f(x\bar{n})\, d\bar{n}$$

is abs^y converg^t and defines $A(\bar{P}, P, \sigma, \nu) f \in C^\infty(G/\bar{P} : \sigma : \nu)$. The operator

$$A(\bar{P}, P, \sigma, \nu) : C^\infty(G/P : \sigma : \nu) \to C^\infty(G/H : \sigma : \nu)$$

is continuous linear and G-equivariant.

Theorem (Knapp–Stein, Vogan–Wallach): The assignment $\nu \mapsto A(\bar{P}, P, \sigma, \nu)$ extends to ${}_{\rm p}\mathfrak{a}^*_{\rm P\mathbb{C}}$ as a meromorphic function with values in $\mathrm{End}(C^\infty(K/K_P : \sigma_P))$. Each regular value $A(\bar{P}, P, \sigma, \nu)$ intertwines the (induced) representations $\pi_{P,\sigma,\nu}$ and $\pi_{\bar{P},\sigma,\nu}$.

Intertwiners on generalized vectors

Theorem For regular $\nu \in {}_{\mathrm{p}}\mathfrak{a}^*_{\mathrm{P}\mathbb{C}}$ the operator $A(\bar{P}, P, \sigma, \nu)$ uniquely extends to a continuous linear endomorphism of $C^{-\infty}(K/K_P : \sigma_P)$. This extension depends meromorphically on ν and intertwines the representations $\pi^{-\infty}_{P,\sigma,\nu}$ and $\pi^{-\infty}_{\bar{P},\sigma,\nu}$.
There exists a non-trivial meromorphic scalar function $\nu \mapsto \eta(P, \sigma, \nu)$ such that

$$A(\bar{P}, P, \sigma, \nu)A(P, \bar{P}, \sigma, \nu) = \eta(P, \sigma, \nu)\,\mathrm{id}$$

Definition: $j^\circ(P, \sigma, \nu) := A(\bar{P}, P, \sigma, \nu)^{-1} j(\bar{P}, \sigma, \nu)$.

Easy consequence: For generic $\nu \in {}_{\mathrm{p}}\mathfrak{a}^*_{\mathrm{P}\mathbb{C}}$, the map $j^\circ(P, \sigma, \nu)$ is bijective

$$\mathcal{V}^\chi_\sigma \xrightarrow{1-1} C^{-\infty}(G/P : \sigma : \nu)^\chi.$$

Multiplicity space

Setting: $P \in \mathcal{P}_{\mathrm{st}}, \sigma \in (M_P)^{\wedge}_{\chi, ds}$.

Recall: $\mathcal{V}^{\chi}_{\sigma} := \oplus_{v \in {}_P\mathcal{W}} \mathcal{V}_{\sigma,v}$, where

$$\begin{array}{ccc} \mathcal{V}^{\chi}_{\sigma,v} := \mathcal{H}_{\sigma}^{-\infty,(v\chi)_P} & \simeq & \mathrm{Hom}_G(\overline{\mathcal{H}}_{\sigma}^{\infty}, C^{\infty}(M_P/M_P \cap vHv^{-1}, (v\chi)_P)) \\ \eta & \mapsto & i_{\eta} \end{array}$$

Define: $\mathcal{V}^{\chi}_{\sigma,ds,v}$ the space of $\eta \in \mathcal{V}^{\chi}_{\sigma,v}$ such that i_η extends to a continuous linear map $\overline{\mathcal{H}}_{\sigma} \to L^2(M_P, M_P \cap vHv^{-1}, (v\chi)_P)$. Equip with inner product such that

$$\mathcal{V}^{\chi}_{\sigma,ds,v} \otimes \overline{\mathcal{H}}_{\sigma} \to L^2(M_P, M_P \cap vHv^{-1}(v\chi)_P) \quad \text{isometry}$$

Define: $\mathcal{V}^{\chi}_{\sigma,ds} := \oplus_{v \in {}_P\mathcal{W}} \; \mathcal{V}^{\chi}_{\sigma,ds,v} \qquad \text{(orthogonal)}.$

Remark

- Group case: $\mathcal{V}^{\chi}_{\sigma,ds} = \mathcal{V}^{\chi}_{\sigma} \simeq \mathbb{C} \rightsquigarrow$ formal degree.
- Whittaker case: $\mathcal{V}^{\chi}_{\sigma,ds} = \mathcal{V}^{\chi}_{\sigma}$ (Harish-Chandra, Wallach).

Plancherel identity

Definition Fourier transform

For $f \in C_c^\infty(G/H, \chi)$, the Fourier transform $\hat{f}(P, \sigma, \nu)$ is the element of $(\mathcal{V}^\chi_{\sigma,ds})^* \otimes L^2(K/K_P : \sigma_P)$, defined by

$$\hat{f}(P, \sigma, \nu)(\eta) := \int_{G/H} f(x)\, \pi^{-\infty}_{P,\sigma,\nu}(x)\, j^\circ(P, \sigma, \nu)(\eta)\; dx$$

Theorem (Plancherel)

$$\|f\|^2_{L^2} = \sum_{P \in \mathcal{P}_{\rm st}/\sim}\ \sum_{\sigma \in (M_P)^\wedge_\chi} \int_{i\,{}_{\rm p}\mathfrak{a}^*_{\rm P}} \|\hat{f}(P, \sigma, \nu)\|^2\; d\lambda_P(\nu)$$

- $P \sim Q$ iff ${}_{\rm p}\mathfrak{a}_P$ and ${}_{\rm p}\mathfrak{a}_Q$ are $W({}_{\rm p}\mathfrak{a})$-conjugate.
- $d\lambda_P$ is Lebesgue measure on $i\,{}_{\rm p}\mathfrak{a}^*_{\rm P}$, suitably normalized.

Examples

The Riemannian case: $H = K$, $\chi = 1$. In this case, $\mathcal{V}_\sigma^\chi = \mathbb{C}$,

$j(\bar{P}_0, 1, \nu) = 1_{\bar{P}_0,\nu} : k \mapsto 1.$

$j^\circ(P_0, 1, \nu) = c(\nu)^{-1} 1_{P_0,\nu}$

$\|\hat{f}(P_0, 1, \nu)\|^2 = |c(\nu)|^{-2} \|\pi_{P_0,1,\nu}(f) 1_{P_0,\nu}\|^2.$

Case of the group: $G = \underline{G} \times \underline{G}$, $H = \mathrm{diag}(\underline{G})$, $\chi = 1$. In this case:

$\mathcal{V}_\sigma^\chi = \mathbb{C}$.

inner product is formal degree d_σ times standard inner product of $\mathbb{C}$