Fourier inversion of Whittaker functions

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The Legacy of Joseph Fourier after 250 years
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Whittaker functions

Setting

- G connected real semisimple Lie group, finite center.
  example: $G = \mathrm{SL}(n, \mathbb{R})$.
- $N_0 < G$ nilpotent subgroup from Iwasawa deco.
  Example:

$$N_0 = \{x \in \mathrm{SL}(n, \mathbb{R}) \mid x = I + \text{upper triangular}\} .$$

- $\chi : N_0 \to \mathrm{U}(1)$ unitary character (regular: def^n postponed).
- $\mathcal{F}(G/N_0, \chi) := \{f : G \to \mathbb{C} \mid f(xn) = \chi(n)f(x) \quad (x \in G, n \in N_0)\}$.

$$L^2(G/N_0, \chi) := \{f \in \mathcal{F}(G/N_0, \chi) \mid |f| \in L^2(G/N_0)\}.$$

- L (left regular representation) $= \mathrm{Ind}_{N_0}^{G}(\chi^\vee)$, is unitary.

Whittaker Plancherel formula

Abstractly

- Since G is type I, $\mathrm{Ind}_{N_0}^{G}(\chi^\vee) = \int_{\widehat{G}}^{\oplus} m_\pi \pi d\mu(\pi)$.

Concrete realization

- Harish-Chandra, Announcement 1982.

  Details in Collected Papers 5 (posthumous), eds. R. Gangolli, V.S. Varadarajan, Springer 2018, 141-307.
  Final step "$\subset$" appears to be missing.

- N.R. Wallach, Independent treatment;

  Real reductive groups II, Acad. Press 1992, relies on erroneous estimate.
  Repair addressed in arXiv:1705.06787.

- Today: missing step by new inversion theorem.
  Bonus: Paley-Wiener theorem.

Regular character

- $G = KAN_0$ Iwasawa decomposition.
  Ex$^{\text{ple}}$: $G = \mathrm{SL}(n, \mathbb{R})$, $K = \mathrm{SO}(n)$, $A = \{a \in \mathrm{SL}(n, \mathbb{R}) \mid a \ \text{ diagonal}\}$.
- $\Sigma = \mathrm{Roots}(\mathfrak{g}, \mathfrak{a})$, $\Sigma^+ := \{\alpha \in \Sigma \mid \mathfrak{g}_\alpha \subset \mathfrak{n}_0\}$, $\Delta \subset \Sigma^+$ simple roots.

Definition $\chi : N_0 \to \mathrm{U}(1)$ regular means:

$$\forall \alpha \in \Delta : \quad d\chi(e)|_{\mathfrak{g}_\alpha} \neq 0.$$

- $P_0 := Z_K(A)AN_0$, minimal parabolic subgroup;
  $\mathcal{P}_{st}$: the (finite) set of all parabolic subgroups $P \supset P_0$.
- For $P \in \mathcal{P}_{st}$, Langlands decomposition: $P = M_P A_P N_P$.
  $\bar{P}N_0$ is open dense in G.

Theorem (Harish-Chandra's Thm 1)

Assume $u \in \mathcal{D}'(G)$ left $\bar{N}_P$-invariant, χ regular, and $R_n u = \chi(n)u$ for all $n \in N_0$. Then

$$u|_{\bar{P}N_0} = 0 \implies u = 0.$$

Ref for proof also: J.A.C. Kolk, V.S. Varadarajan, Indag. Math. 1996.

Discrete part of decomposition

Discrete part

$\pi \in \widehat{G}$ (unitary dual) is said to appear discretely in $L^2(G/N_0, \chi)$ if it can be realized as a closed subrepresentation.

Theorem (Harish-Chandra)

If $\pi \in \widehat{G}$ appears discretely in $L^2(G/N_0, \chi)$, then it appears discretely in $L^2(G)$, i.e., it belongs to the discrete series of G.

Proof by distributional asymptotics of matrix coefficients, combined with Thm 1.

Corollary

If $\pi \in \widehat{G}$ appears discretely in $L^2(G/N_0, \chi)$, then its infinitesimal character is real and regular.

This result is crucial for the distinction of spectra in the Whittaker Plancherel decomposition.

Schwartz functions

Define $\rho \in \mathfrak{a}^*$ by $\rho(X) = \frac{1}{2}\mathrm{tr}(\mathrm{ad}(X)|_{N_0})$.

Def: Schwartz space (HC)

$\mathcal{C}(G/N_0, \chi)$: the space of $f \in C^\infty(G/N_0, \chi)$ such that

$$\sup_{k \in K, a \in A} (1 + |\log(a)|)^N a^\rho |L_u f(ka)| < \infty, \qquad (\forall u \in U(\mathfrak{g}), \forall N \geq 1).$$

For (τ, V_τ) a finite dimensional unitary representation of K, we define

$$\mathcal{C}(\tau, G/N_0, \chi) := \{f \in \mathcal{C}(G/N_0, \chi) \otimes V_\tau \mid f(ka) = \tau(k) f(x) \ \ (k \in K, x \in G)\}.$$

Finally, with $\mathfrak{Z} := \mathrm{center}\, U(\mathfrak{g})$,

$$\mathcal{A}(\tau, G/N_0, \chi) := \{f \in \mathcal{C}(\tau, G/N_0, \chi) \mid \dim \mathfrak{Z} f < \infty\}$$

Theorem (HC) $\mathcal{A}(\tau, G/N_0, \chi) = L^2_d(\tau, G/N_0, \chi)$.

The space is finite dimensional.

Whittaker integrals

Let $P = M_P A_P N_P \in \mathcal{P}_{st}$ and put $\mathcal{A}_{P,\tau} := \mathcal{A}(\tau, M_P/M_P \cap N, \chi)$.

For $\psi \in \mathcal{A}_{P,\tau}$ and $\lambda \in \mathfrak{a}^*_{P\mathbb{C}}$, define (for $k \in K$, $man \in M_P A_P N_P$):

$$\psi_{\bar{P},\lambda}(kman) := a^{-\lambda+\rho_P}\tau(k)\psi(m),$$

Definition (HC)

For $\psi \in \mathcal{A}_{P,\tau}$, $\lambda \in \mathfrak{a}^{*+}_{P\mathbb{C}}$, $x \in G$, the Whittaker integral is defined by

$$\mathrm{Wh}(P, \psi, \lambda, x) := \int_{N_P} \psi_{\bar{P},-\lambda}(xn)\chi(n)^{-1}\, dn.$$

It is essentially a finite sum of matrix coefficients of $\mathrm{Ind}_{\bar{P}}^G(\sigma \otimes -\lambda \otimes 1)$, *with* σ *appearing in* $L^2_d(\tau, M_P/M_P \cap N_0, \chi)$.

Remark: *For* $P = P_0$, *we have* $M_P \cap N_0 = \{e\}$ *and* $M_P \subset K$, *so* $\mathcal{A}(\tau, M_P/M_P \cap N, \chi) = L^2(\tau, M_P)$.

Holomorphy

$Wh(\psi, \lambda, \cdot)$ depends linearly on ψ and belongs to $C^\infty(\tau, G/N_0, \chi)$. It is convenient to write

$$\mathrm{Wh}(P, \lambda)(x)(\psi) := \mathrm{Wh}(P, \psi, \lambda, x);$$

Viewpoint: $\mathrm{Wh}(P, \lambda) \in C^\infty(G/N_0, \chi) \otimes \mathrm{Hom}(\mathcal{A}_{P,\tau}, V_\tau)$.

Theorem (Wallach)

The Whittaker integral $\mathrm{Wh}(P, \lambda)$, initially defined for $\lambda \in \mathfrak{a}^{*+}_{P\mathbb{C}}$, extends to an entire holomorphic function of $\lambda \in \mathfrak{a}^*_{P\mathbb{C}}$ with values in $C^\infty(G/N_0, \chi) \otimes \mathrm{Hom}(\mathcal{A}_{P,\tau}, V_\tau)$.

- Harish-Chandra established existence of meromorphic extension, regular on $i\mathfrak{a}^*_P$.
- We found a new proof, using Thm 1.

Classical Whittaker functions

Example

- $G = \mathrm{SL}(2,\mathbb{R})$, $\tau \in \mathrm{SO}(2)^\wedge$, $M_0 = \{\pm I\}$, $\psi(-I) = \tau(-I) = \pm 1$.
- $Wh(P, \lambda, \psi)$ is essentially a classical Whittaker function on $\mathbb{R}$;
- satisfies ODE on $\mathbb{R}$ with regular singularity at ∞,
- but with irregular singularity at $-\infty$;

For $\alpha(\log a) \to -\infty$ have:

- $Wh(P, \lambda, \psi)(a) \sim e^{-a^{-\alpha}}$ (super fast decay);
- generic solution W of ODE:

$$W(a) \sim e^{a^{-\alpha}} \quad \text{(super exponential growth)}.$$

C-functions, Maass-Selberg relations

Asymptotic behavior (HC)

For $\psi \in \mathcal{A}_{P,\tau}$, $\lambda \in i\mathfrak{a}_P^*$, $m \in M_P$, $a \to \infty$ in A_P^+,

$$Wh(P,\lambda)(ma)\psi \sim \sum_{s \in W(\mathfrak{a}_P)} a^{s\lambda - \rho_P}[C_{P|P}(s,\lambda)\psi](m),$$

with $C_{P|P}(s,\lambda) \in \mathrm{End}(\mathcal{A}_{P,\tau})$ meromorphic in $\lambda \in \mathfrak{a}_{P\mathbb{C}}^*$.

For $a \to \infty$ in other chambers of A_P, $Wh(P,\lambda)(ma) = o(a^{-\rho_P})$.

Maass-Selberg relations (HC)

For all $s \in W(\mathfrak{a}_P)$, $\lambda \in i\mathfrak{a}_P^*$,

$$C_{P|P}^{\circ}(s,\lambda) := C_{P|P}(s,\lambda)C_{P|P}(1,\lambda)^{-1}$$

is unitary.

Fourier transform

Normalized Whittaker functions (HC)

$$\mathrm{Wh}^\circ(P, \lambda, x) := \mathrm{Wh}(P, \lambda, x) \circ C_{P|P}(s, \lambda)^{-1}.$$

Normalized Fourier transform

$${}^\circ\mathrm{Wh}^*(P, \lambda, x) := \mathrm{Wh}^\circ(P, -\bar{\lambda}, x)^* \in \mathrm{Hom}(V_\tau, \mathcal{A}_{P,\tau}).$$

For $f \in \mathcal{C}(\tau, G/N_0, \chi)$, $P \in \mathcal{P}_{st}$, $\lambda \in i\mathfrak{a}^*$,

$$\mathcal{F}_P^\circ f(\lambda) := \int_{G/N_0} {}^\circ\mathrm{Wh}^*(P, \lambda, x) f(x)\, dx \in \mathcal{A}_{P,\tau}.$$

Also: unnormalized versions all without $^\circ$.

Discrete part of Fourier transform

For $P = G$ one has $\mathfrak{a}_P^* = \{0\}$ and the normalized Fourier transform is given by the (finite rank) L^2-orthogonal projection

$$\mathcal{C}(\tau, G/N_0, \chi) \to L^2_d(\tau, G/N_0, \chi).$$

Plancherel formula

For $P, Q \in \mathcal{P}_{st}$, $P \sim Q$ means $\mathfrak{a}_P, \mathfrak{a}_Q$ conjugate under $W(\mathfrak{a})$.

Plancherel identity (HC)

For suitable normalization of the measures on $i\mathfrak{a}_P^*$,

$$\|f\|^2_{L^2(G/N_0,\chi)} := \sum_{P \in \mathcal{P}_{st}/\sim} \|\mathcal{F}_P^\circ f\|^2_{L^2(i\mathfrak{a}_P^*)}.$$

The issue of completeness Harish-Chandra proves this identity for f in a space spanned by wave packets, of which the density in $L^2(\tau, G/N_0, \chi)$ appears to remain unproven. In principle this allows a non-trival joint kernel of the Fourier transforms.
Speculation: perhaps Harish-Chandra intended to obtain completeness from the similar completeness related to his Plancherel decomposition of $L^2(G)$.

Different idea for obtaining completeness Use Paley-Wiener shift argument such as developed in collaboration with Henrik Schlichtkrull for proof of the Plancherel formula for a semisimple symmetric space G/H.

Series expansion

Let $P = P_0$ be minimal. Then $\mathcal{A}_{P,\tau} \simeq V_\tau^M$ and,

$$\mathrm{Wh}(P, \lambda) \in C^\infty(\tau, G, \chi) \otimes \mathcal{A}^*_{P,\tau}$$

holomorphic in $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$. The function is $\mathfrak{Z}$-finite, hence satisfies a cofinite system of differential equations, which has regular singularities at infinity in the direction of A^+.

Expansion at infinity

$$\mathrm{Wh}(P, \lambda) = \sum_{s \in W} \mathrm{Wh}_+(P, s\lambda) C_{P|P}(s, \lambda)$$

where $\mathrm{Wh}_+(P, \lambda) \in C^\infty(\tau, G, \chi) \otimes \mathcal{A}^*_{P,\tau}$ depends meromorphically on $\lambda \in \mathfrak{a}^*_{\mathrm{q}\mathbb{C}}$, and admits a converging series expansion

$$\mathrm{Wh}_+(P, \lambda)(a) = a^{\lambda - \rho} \sum_{\mu \in \mathbb{N}\Delta} a^{-\mu} \Gamma_\mu(\lambda) \qquad (a \in A),$$

with $\Gamma_\mu(\lambda) \in \mathrm{End}(\mathcal{A}_{P,\tau})$ meromorpic, $\Gamma_0(\lambda) = \mathrm{id}_{\mathcal{A}_{P,\tau}}$.

Fourier inversion

$$C_c^\infty(\tau, G/N_0, \chi) := \{f \in C^\infty(\tau, G/N_0, \chi) \mid \mathrm{supp}\, f \text{ cpt mod } N_0\}.$$

Key theorem: Fourier inversion

$$f(x) = |W(\mathfrak{a})| \int_{i\mathfrak{a}^* + \eta} \mathrm{Wh}_+(P, \lambda, x)\, \mathcal{F}^0_{P_0} f(\lambda)\, d\lambda,$$

for all $f \in C_c^\infty(\tau, G/N_0, \chi)$ *and all* $x \in G$, *provided* $\langle \eta, \alpha \rangle << 0$ $(\forall \alpha \in \Sigma^+)$.

NB: For $\lambda \in \mathfrak{a}^*_{\mathrm{q}\mathbb{C}}$, the function $\mathrm{Wh}_+(P, \lambda)$ is globally defined on X, but may exhibit super exponential growth in directions different from A^+.

Notation: denote expression on right by $\mathcal{T}_\eta f(x)$. Then

$$\mathcal{T}_\eta : C_c^\infty(\tau, G/N_0, \chi) \to C^\infty(\tau, G/N_0, \chi).$$

Proof of Fourier inversion

Sketch of proof:

- There exists a non-trivial symmetric differential operator $D = L_Z$, $Z \in \mathfrak{Z}$ which cancels singularities if η moves to 0. By Cauchy's thm:

$$\begin{aligned} D(\mathcal{T}_\eta f)(x) &= D(\mathcal{T}_0 f)(x) \\ &= \int_{i\mathfrak{a}^*} D\mathrm{Wh}^0(P, \lambda, x)\mathcal{F}^0 f(\lambda)\, d\lambda. \end{aligned}$$

- By Paley-Wiener shift for $\langle\, \eta, \alpha \,\rangle \to -\infty$ $(\forall \alpha \in \Delta)$, one sees

$$\mathrm{supp}\, f \subset KaCN_0 \implies \mathrm{supp}(\mathcal{T}_\eta f) \subset KaCN_0.$$

Here $C = \exp \underline{C}$, $\underline{C} := -\mathfrak{a}^{++}$, the cone negative dual to $\mathfrak{a}^+$.

$D\mathcal{T}_\eta$

- is symmetric, hence support preserving;
- is essentially a differential operator on A.

Support preservation

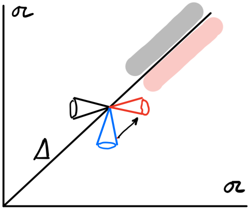
$$S = D \circ \mathcal{T}_\eta = D \circ \mathcal{T}_0 \implies S \text{ symmetric}$$

$$\begin{array}{ccc} C_c^\infty(\tau, G/N_0, \chi) & \xrightarrow{S} & C^\infty(\tau, G/N_0, \chi) \\ \downarrow & & \downarrow \\ \varphi \in \quad C_c^\infty(\mathfrak{a}) \otimes V_\tau^M & \xrightarrow{\exists ! \underline{S}} & C^\infty(\mathfrak{a}) \otimes V_\tau^M \end{array} \qquad \forall \varphi : \operatorname{supp} \underline{S}\varphi \subset \operatorname{supp} \varphi + \underline{C}.$$

Hence distribution kernel K of $\underline{S}$ satisfies:

1. $\operatorname{supp} K \subset \Delta + \underline{C} \times \{0\}$. By symmetry, get
2. $\operatorname{supp} K \subset \Delta + \{0\} \times \underline{C} = \Delta + (-\underline{C}) \times \{0\}$.

1 & 2 $\implies \operatorname{supp} K \subset \Delta \implies \underline{S}$ support preserving.



Proof of Fourier inversion, continued

$D\mathcal{T}_\eta$

- is essentially a differential operator on A;
- commutes with $\mathfrak{Z}$-action;
- satisfies cofinite system of DE's with regular singularities at ∞ in A^+;
- is determined by highest order asymptotic part.

$D\mathcal{T}_\eta = D$ by highest order term asymptotic analysis.

Application of Holmgren's uniquess theorem

- $\mathrm{rad}\,(D)$ analytic with highest order part in $\mathbb{D}(A) \otimes I$.

$$\left.\begin{array}{l} \mathrm{supp}(\mathcal{T}_\eta f - f) \subset K(\mathrm{supp} f \cap A)CN \\ D(\mathcal{T}_\eta f - f) = 0 \end{array}\right\} \implies \mathcal{T}_\eta f - f = 0.$$

Residual kernels

By Fourier inversion:

$$f(x) = |W(\mathfrak{a})| \int_{i\mathfrak{a}^*+\eta} \mathrm{Wh}_+(P, \lambda, x)\, \mathcal{F}^0_{P_0} f(\lambda)\, d\lambda.$$

Shifting η towards zero and organizing residues, one gets

$$f(x) = |W| \sum_{F \subset \Delta} t(F)\, T^t_F f(x),$$

where

$$T^t_F f(x) = \int_{i\mathfrak{a}^*_F+\varepsilon_F} \int_{G/N_0} K^t_F(\lambda, x, y) f(y)\, dy\, d\mu_F(\lambda).$$

Here $P_F = M_F A_F N_F$ is the standard parabolic subgroup associated with $F \subset \Delta$, with $t : 2^\Delta \to [0, 1]$ a weight function describing a certain organisation of residue shifts, and with $\varepsilon_F \in \mathfrak{a}^{*+}_F$ sufficiently close to 0.

Completeness

Theorem

$$K_F^t(\lambda, x, y) = \mathrm{Wh}^\circ(P_F, \lambda)(x) \circ {}^\circ\mathrm{Wh}^*(P_F, \lambda)(y)$$

This identification follows from a vanishing theorem for families. By the Maass-Selberg relations the functions $K_F^t(\lambda, x, y)$ are seen to be regular on $i\mathfrak{a}^*$, hence we may let $\varepsilon_F \to 0$ and then:

Fourier inversion

$$f(x) = |W| \sum_{F \subset \Delta} t(F) \int_{i\mathfrak{a}_F^*} \mathrm{Wh}^\circ(P_F, \lambda, x) \mathcal{F}_{P_F}^\circ f(\lambda) d\mu_F(\lambda).$$

This result implies the completeness of the given collection of Fourier transforms.

Vanishing Theorem

Let $P \in \mathcal{P}_{st}$. The vanishing thm is about certain meromorphic $\mathfrak{a}^*_{P\mathbb{C}} \ni \lambda \mapsto f_\lambda \in C^\infty(\tau, G/N_0, \lambda)$ such that

- f_λ behaves finitely under $\mathfrak{Z}$ in a specific λ-dependent way,
- f_λ satisfies certain mild restrictions on leading exponents along $\mathfrak{a}_P^+$,
- certain asymptotic coefficients along codimension one walls are of moderate growth in the transversal Levi variable.

Vanishing theorem

Let f_λ be a family as above. If the asymptotic coefficient of $a^{\lambda-\rho}$ in the expansion of f_λ along A_P^+ vanishes identically as a function of $(\lambda, m) \in \mathfrak{a}^*_{P\mathbb{C}} \times M_P$ then $f_\lambda = 0$ for all $\lambda \in \mathfrak{a}^*_{P\mathbb{C}}$.

Importance

This result allows identification of families by looking at top order asymptotic behavior.

Paley-Wiener theorem

Definition

Recall: $C = \exp(-\mathfrak{a}^{++})$. A function $f \in \mathcal{C}(\tau, G/N_0, \chi)$ is said to be *cone supported* (notation $\mathcal{C}_{cs}$) if $\exists a_0 \in A$ s.t.

$$\mathrm{supp} f \subset K a_0 C N_0.$$

Lemma

For every $f \in \mathcal{C}_{cs}(\tau, G/N_0, \chi)$, all $u \in U(\mathfrak{g})$ and all $m > 0$,

$$\sup_{k \in K, a \in A} e^{m|\log a|} \|L_u f(ka)\| < \infty.$$

Paley-Wiener theorem

Let $P = P_0$ (minimal). Then $\mathcal{F}_P$ is injective on $\mathcal{C}_{cs}(\tau, G/N_0, \chi)$. The image of this space under $\mathcal{F}_P$ equals the space $\mathrm{PW}(\chi, \tau)$ of holomorphic functions $\varphi : \mathfrak{a}_{\mathbb{C}}^* \to \mathcal{A}_{P,\tau}$ satisfying

- *estimates of Paley–Wiener type;*
- *relations of Arthur–Campoli type.*

Arthur–Campoli type relations

More precisely, the definition of this space is as follows.

Definition Paley–Wiener space

$\mathrm{PW}(\chi, \tau)$ is the space of holomorphic functions $\varphi : \mathfrak{a}_{\mathbb{C}}^* \to \mathcal{A}_{P,\tau}$ satisfying

- $\exists R > 0$ and $\forall \lambda_0 \in \mathfrak{a}_{\mathbb{C}}^*$ $\forall N \in \mathbb{N}$ $\exists C > 0$ s.t.

$$|\varphi(\lambda)| \leq C(1 + \|\lambda\|)^{-N} e^{R\|Re\lambda\|} \qquad (\lambda \in \lambda_0 - \mathfrak{a}_{\mathbb{C}}^{*+}).$$

- For all finite collections $\lambda_i \in \mathfrak{a}_{\mathbb{C}}^*, u_i \in S(\mathfrak{a}^*),\ \xi_i \in \mathrm{Hom}(V_\tau, \mathcal{A}_{P,\tau})^*$, $1 \leq i \leq N$,

$$\sum_{i=1}^{N} \langle\, \xi_i, \partial_{u_i} \mathrm{Wh}^*(P, \cdot)(\lambda_i)\,\rangle = 0 \quad \Longrightarrow \quad \sum_{i=1}^{N} \langle\, \xi_i, \partial_{u_i} \varphi(\lambda_i)\,\rangle = 0.$$