Weyl, eigenfunction expansions and non-compact symmetric spaces

Eigenfunction expansions and symmetric spaces

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Outline

Influence of Weyl's work

'This is reminiscent of a result of Weyl [8(a), p. 266] on ordinary differential equations.'

This refers Weyl's work in Mathematatische Annalen, 1910.

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where dx is the Haar measure of G. Then the Plancherel formula asserts [2] the existence of a unique positive measure $d\mu$ on E (which is invariant under W) such that

$$\int |f(x)|^2 \, dx = \int |\bar{f}(\lambda)|^2 \, d\mu$$

for all such f. The problem is to determine this measure $d\mu$. Let $d\lambda$ denote the Euclidean measure on E. In this paper we shall give an asymptotic expansion for ϕ_λ on G. The leading terms of this expansion involve a certain coefficient $\mathbf{c}(\lambda)$, which, considered as a function of λ, is analytic on E except on certain hyperplanes (see Lemmas 37 and 52). In any case, the reciprocal $\mathbf{c}^{-1}$ is analytic on E and it will be shown in another paper that [3] $d\mu = |\mathbf{c}(\lambda)|^{-2} \, d\lambda$ (if dx and $d\lambda$ are suitably normalized). On the other hand the Fourier transform [2] of $\mathbf{c}$ is a distribution on E which is given by a simple formula in which the group structure of G enters in a very direct manner (see Theorem 5).

The above outline shows that our problem can be divided into three more or less distinct parts: (1) the asymptotic formula for ϕ_λ, (2) the investigation of the function $\mathbf{c}$ and (3) the proof of the relation $d\mu = |\mathbf{c}(\lambda)|^{-2} \, d\lambda$.

Harish-Chandra
Spherical functions on a semisimple Lie group I.
Amer. J. Math. **80** (1958), 241 - 310.

The Paper

Über gewöhnliche Differentialgleichungen mit Sigularitäten und die zugehörigen Entwicklungen willkürlicher Funktionen, Mathematische Annalen 68, 220–269 (1910).

8.

Über gewöhnliche Differentialgleichungen mit Singularitäten und die zugehörigen Entwicklungen willkürlicher Funktionen

Mathematische Annalen 68, 220—269 (1910)

Die vorliegende Arbeit verfolgt das Ziel, die Theorie der singulären Integralgleichungen, wie ich sie, auf die Untersuchungen von HILBERT und HELLINGER über die beschränkten quadratischen Formen von unendlichvielen Variablen[1]) gestützt, in einer kürzlich in den Mathematischen Annalen erschienenen Abhandlung[2]) entwickelt habe, für die Theorie der gewöhnlichen linearen Differentialgleichungen zweiter Ordnung nutzbar zu machen. Es handelt sich dabei um Differentialgleichungen, welche an dem einen Ende ihres reellen Integrationsintervalls eine Singularität von mehr oder minder kompliziertem Charakter aufweisen, und um die Aufstellung der aus solchen Differentialgleichungen entspringenden Entwicklungen willkürlicher Funktionen, wie sie in dem einfachsten Falle der Gleichung $d^2u/ds^2 = 0$ als Fouriersche Reihe und Fourier-

Sturm Liouville operator

On $]a,b[, \quad -\infty \leq a < b \leq +\infty$

$$L = -\frac{d}{dx} p \frac{d}{dx} + q \qquad \begin{cases} p \in C^1(]a,b[,\mathbb{R}) \\ q \in C^0(]a,b[,\mathbb{R}) \\ p > 0 \text{ on }]a,b[\end{cases}$$

Nature of operator

- singular: no restrictions on $p(x), q(x)$ as $x \downarrow a, x \uparrow b$.
- regular:
  - $-\infty < a, \ b < \infty$,
  - $q \in C^0([a,b]), p \in C^1([a,b])$
  - $p > 0$ on $[a,b]$.

Regular case

- Boundary data: $\xi_a, \xi_b \in \mathbb{R}^2 \setminus \{0\}$
- Boundary conditions

$$C_\xi := \{u \in C^2([a,b]) \mid (u(c), p(c)u'(c)) \perp \xi_c, \;\; c = a, b\}$$

- Eigenspace

$$\begin{aligned} \mathcal{E}_\lambda &:= \ker(L - \lambda I), & \dim_{\mathbb{C}} = 2. \\ \mathcal{E}_{\lambda,\xi} &:= \mathcal{E}_\lambda \cap C_\xi, & \dim_{\mathbb{C}} = 1. \end{aligned}$$

Theorem (Expansion theorem)

- $\sigma(L) := \{\lambda \in \mathbb{C} \mid \mathcal{E}_{\lambda,\xi} \neq 0\}$ *is discrete*
- $L^2([a,b]) = \overline{\bigoplus}_{\lambda \in \sigma(L)} \mathcal{E}_{\lambda,\xi}$.

Regular case

Theorem (Expansion theorem)

- $\sigma(L) := \{\lambda \in \mathbb{C} \mid \mathcal{E}_{\lambda,\xi} \neq 0\}$ *is discrete*
- $L^2([a,b][) = \overline{\bigoplus}_{\lambda \in \sigma(L)} \mathcal{E}_{\lambda,\xi}.$

Proof.

- Use integral operator $\mathcal{G}_\lambda$ s.t. $(L - \lambda I)\mathcal{G}_\lambda = I$
- Integral kernel of $\mathcal{G}_\lambda$ symmetric and continuous
- $\Longrightarrow \mathcal{G}_\lambda$ compact self-adjoint.

□

Regular case

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$\square$

Boundary conditions

Why?

in regular case: to make L self-adjoint on C_ξ.

Problem

in singular case: when are boundary conditions necessary?

Recall

- $L = -\frac{d}{dx} p \frac{d}{dx} + q$
- $C_\xi := \{u \in C^2([a,b]) \mid (u(c), p(c)u'(c)) \perp \xi_c, \;\; c = a, b\}$

The singular case

Theorem (Weyl, 1910)

- *Only one of two alternatives possible at a (or b):*
  - ∘ $\forall \lambda \in \mathbb{C}: \quad \mathcal{E}_\lambda \subset L^2(]\,a, b\,[)$ (limit circle case),
  - · $\forall \lambda \in \mathbb{C}: \quad \mathcal{E}_\lambda \not\subset L^2(]\,a, b\,[)$ (limit point case).
-
  - ∘ *boundary condition needed at a;*
  - · *no boundary condition needed at a.*
- $\exists$ *eigenfunction expansion*

a	b	continuous or discrete expansion
∘	∘	discrete
∘	·	mixed
·	∘	mixed
·	·	mixed

Special singular case

Assumption

- L regular (hence $\circ$) at a,
- $\cdot$ at b;
- boundary datum ξ_a; select $\eta \in \xi_a^\perp \cap \mathbb{R}^2 \setminus \{0\}$;
- define $\varphi_\lambda \in \mathcal{E}_\lambda$ by

$$\varphi_\lambda(a) = \eta_1, \quad p(a)\varphi_\lambda'(a) = \eta_2.$$

Definition (associated integral transform)

For $f \in C_c(]\,a, b\,[)$ define

$$\mathcal{F}f(\lambda) := \int_a^b f(x)\,\varphi(x)\,dx, \qquad (\lambda \in \mathbb{R}).$$

Expansion in special singular case

Theorem (Weyl's expansion theorem)

$\exists$ *unique right continuous monotonically increasing function* $\rho : \mathbb{R} \to \mathbb{R}$ *with* $\rho(0) = 0$ *such that* $\mathcal{F}$ *extends to an isometry*

$$L^2(]a, b[, dx) \xrightarrow{\simeq} L^2(\mathbb{R}, d\rho).$$

In particular,

(1) $\int_a^b |f(x)|^2 \, dx = \int_{\mathbb{R}} |\mathcal{F}f(\lambda)|^2 \, d\rho(\lambda)$;

(2) $f(x) = \int_{\mathbb{R}} \mathcal{F}f(\lambda) \, d\rho(\lambda)$.

Problem

Determine $d\rho(\lambda)$.

Weyl's solution in special case

Theorem

Let $[a, b[= [0, \infty[;$, $p \sim 1, q \sim 0$ $(x \to \infty)$.

Write $\mu = \sqrt{\lambda}$, *for* $\lambda > 0$. *Then*

- $\exists$ *unique functions* $c(\mu)$ *on* $\mu > 0$ *such that:*

$$\varphi_\lambda(x) \sim c(\mu)\, e^{i\mu x} + \overline{c(\mu)}\, e^{-i\mu x}, \qquad (x \to \infty);$$

- *on* $]-\infty, 0]$: $d\rho$ *discretely supported;*
- *on* $]0, \infty[$: $d\rho(\lambda) = \frac{2}{\pi |c(\mu)|^2}\, d\mu.$

$\uparrow$

Weyl's principle

General solution

Calculation of $d\rho(\lambda)$ in general:

- Titschmarsh, 1941: complex contour methods;
- Kodaira, 1947: extended Weyl's result to the general singular case.

A result of Kodaira

Setting

- $L = -\frac{d^2}{dx^2} + q \quad \text{on} \quad]0, \infty[, \qquad (\text{so } p = 1);$
- $q(x) \sim 0, \quad (x \to \infty);$
- $q(x) \sim \nu(\nu + 1)x^{-2}, \quad (x \downarrow 0), \qquad \nu \geq -\frac{1}{2}.$

Eigenfunction

- indicial exponents: $\nu + 1, -\nu$;
- take $\varphi_\lambda \in \mathcal{E}_\lambda$, $\varphi_\lambda(x) \sim x^{\nu+1} (x \downarrow 0)$.

A result of Kodaira

Theorem (Kodaira, 1947)

Weyl's formula is still valid:

$$d\rho(\lambda) = \frac{2}{\pi |c(\mu)|^2}\, d\mu, \qquad (\mu = \sqrt{\lambda}).$$

Moreover,

- $c(\mu)$ *extends*
  - *analytically to* $\operatorname{Im}\mu > 0$,
  - *continuously to* $\operatorname{Im}\mu \geq 0$,
- $\operatorname{supp}(d\rho|_{]-\infty,0]}) \subset \{\mu^2 \mid \ \mu \in i\,\mathbb{R}_{\geq 0},\ c(\mu) = 0\}$.

Weyl's opinion

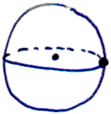
Gibbs Lecture, 1948

years for axiomatized Hilbert space. ... which Hellinger had founded Hilbert's general theory [9], had been directly applied to the special differential problem by E. Hilb [10]; but he did not carry it so far as to obtain the explicit construction of the differential $d\rho$. Recently E. C. Titchmarsh in several papers and in his book on *Eigenfunction expansions* [11] resumed this direct approach. The basic equation (12) is due to him. Yet his construction of $w(\lambda)$ and of $d\rho$ is not as direct as I should wish them. Also a number of contributions made by A. Wintner and P. Hartman during the last two years ought to be mentioned [12]. The formula (12) was rediscovered by Kunihiko Kodaira (who of course had been cut off from our Western mathematical literature since the end of 1941); his construction of ρ and his proofs for (12) and the expansion formula (9), still unpublished, seem to clinch the issue. It is remarkable that forty years had to pass before such a thoroughly satisfactory direct treatment emerged; the fact is a reflection on the degree to which mathematicians during this period got absorbed in abstract generalizations and lost sight of their task of finishing up some of the more concrete problems of undeniable importance.

4. **Inequalities and asymptotic laws for eigenvalues.** But let us drop this matter now and turn to another subject, that of the asymptotic distribution of the eigen-frequencies for the two- or more-

Symmetric spaces, basic examples

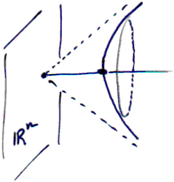
(1) $S^n \subset \mathbb{R}^{n+1}$: $x_1^2 + \cdots + x_{n+1}^2 = 1.$



$S^n = \mathrm{SO}(n+1)e_1 =$
$\mathrm{SO}(n+1)/\mathrm{SO}(n)$

compact Riemannian

(2) $H_n \subset \mathbb{R}^{n+1}$: $x_1^2 - (x_2^2 + \cdots + x_{n+1}^2) = 1.$

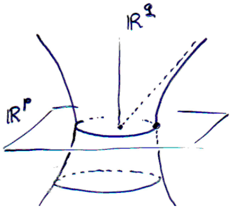


$H_n = \mathrm{SO}(1, n+1)e_1 =$
$\mathrm{SO}(1, n+1)/\mathrm{SO}(n)$

non-compact Riemannian

Symmetric spaces, basic examples II

(3) $H_{p,q} \subset \mathbb{R}^{n+1}: \quad (x_1^2 + \cdots + x_p^2) - (x_{p+1}^2 + \cdots + x_{n+1}^2) = 1.$



$p + q = n + 1, \quad p > 1$
$H_{p,q} = \mathrm{SO}(p, q)e_1 =$
$\mathrm{SO}(p, q)/\mathrm{SO}(p - 1, q)$

pseudo-Riemannian

Symmetric spaces

General setting: $X = G/H$.

G Lie group, $H = G^\sigma$, $\sigma \in \mathrm{Aut}(G)$ involution.

(1) compact Riemannian	G compact
(2) non-compact Riemannian	G semisimple H compact
(3) pseudo-Riemannian	G semisimple H non-compact

Eigenfunctions

$$\begin{aligned} \mathbb{D}(G/H): &= \{G - \text{invariant differential operators on } G/H\} \\ &\simeq \mathbb{C}[D_1, \dots, D_r] \quad \text{polynomial algebra, } r = \mathrm{rk}(G/H). \end{aligned}$$

Example: Case 1, Riemannian compact

$$\begin{aligned} L^2(G) &\simeq \bigoplus_{\delta \in \widehat{G}} V_\delta \otimes V_\delta^*, & \text{(Peter-Weyl);} \\ L^2(G/H) &\simeq \bigoplus_{\delta \in \widehat{G}} V_\delta \otimes (V_\delta^*)^H, & \dim(V_\delta^*)^H \in \{0, 1\}. \end{aligned}$$

Matrix coefficient

$V_\delta \otimes (V_\delta^*)^H \ni v \otimes \eta \mapsto m_{v,\eta} \in C^\infty(G/H)$,

$$m_{v,\eta}(x) := \langle v, \pi^\vee(x)\eta \rangle, \qquad (x \in G).$$

Joint eigenfunction because $\mathbb{D}(G/H) \curvearrowright (V_\delta^*)^H$.

Eigenfunctions

Example: Case 2, Riemannian non-compact

$X = G/K$, $\mathfrak{a} \subset \mathfrak{k}^{\perp}$, maximal abelian, $\dim \mathfrak{a} = r = \mathrm{rk}(X)$.
$R = R(\mathfrak{g}, \mathfrak{a})$ is a root system; $W :=$ Weyl group.

$$\begin{array}{ccc} \mathfrak{a} & \overset{\exp}{\hookrightarrow} & G \\ \downarrow & & \downarrow \\ \mathfrak{a}/W & \overset{1-1}{\longrightarrow} & K\backslash G/K \\ \downarrow \simeq & & \\ \overline{\mathfrak{a}^+} & & \end{array}$$

Polar decomposition: for $f \in C_c(G/K)$,

$$\int_{G/K} f(x)\, dx = \int_K \int_{\overline{\mathfrak{a}^+}} f(k \exp X\, K)\, J(X)\, dX\, dk.$$

$\uparrow$ Jacobian

Spherical functions

Spherical principal series

Unitary representation π_μ in $L^2(K/M)$, for $\mu \in \mathfrak{a}^*$; here $M = Z_K(\mathfrak{a})$.

Elementary spherical function

$$\varphi_\mu(x) := \langle 1_{K/M}, \pi_\mu^\vee(x) 1_{K/M} \rangle.$$

Properties:

- $\varphi_\mu \in C^\infty(K\backslash G/K)$;
- $\varphi_{w\mu} = \varphi_\mu$;
- $\mathbb{D}(G/H)\varphi_\mu \subset \mathbb{C}\varphi_\mu$;

Harish-Chandra's Plancherel theorem

Asymptotics

$$\varphi_\mu(\exp X) \sim J(X)^{-1/2} \sum_{w\in W} c(w\mu)\, e^{i\, w\mu(X)} \qquad (X \xrightarrow{\mathfrak{a}^+} \infty).$$

Fourier transform

For $f \in C_c(K\backslash G/K)$, define

$$\mathcal{F}f(\mu) := \int_{G/K} f(x)\varphi_{-\mu}(x)\, dx.$$

Theorem (Harish-Chandra, 1958)

$\mathcal{F}$ extends to an isometry

$$L^2(K\backslash G/K) \xrightarrow{\simeq} L^2(\mathfrak{a}^{*+}, \frac{d\mu}{|c(\mu)|^2}).$$

Rank one case

Setting

$\dim \mathfrak{a} = 1, \quad \mathfrak{a} \simeq \mathbb{R}, \ W = \{\pm I\}.$

$$\tilde{\varphi}_\mu := J(X)^{1/2}\,\varphi_\mu \ \sim \ c(\mu)e^{i\,\mu X} + c(-\mu)e^{-i\,\mu X}$$

Differential equation

$L\tilde{\varphi}_\mu = \mu^2\tilde{\varphi}_\mu, \qquad$ where $\qquad L = -\mathrm{rad}\,(J^{1/2}\circ\Delta\circ J^{-1/2}) - \rho^2$

Lemma

$$L \quad = \quad -\tfrac{d^2}{dX^2} + \underbrace{J^{-1/2}\frac{d^2}{dX^2}(J^{1/2}) - \rho^2}_{q}$$

satisfies Kodaira's conditions !

Plancherel theorem for a semisimple group

Definitions

- G semisimple,
- $\widehat{G}_{\mathrm{ds}} = \{\text{discrete series representations}\}$,
- $H_1, \ldots, H_l$ representatives of G-conjugacy classes of Cartan subgroups,
- $H_j = T_j A_j$ (compact, vectorial), $M_j := Z_G(A_j)/A_j$.

Theorem (Plancherel, Harish-Chandra early 70's)

- $L^2(G) \simeq \bigoplus_{j=1}^{l} \bigoplus_{\xi \in \widehat{M}_{j,\mathrm{ds}}} \int_{\mathfrak{a}_j^*}^{\oplus} \deg(\xi) \; \pi_{\xi,\mu} \otimes \pi_{\xi,\mu}^* \; dm_\xi(\mu)$.
- $dm_\xi(\mu) = \|c_{\xi,\delta}(\mu)\|_{\mathrm{HS}}^{-2} d\mu, \qquad \forall\, K - \text{type} \;\; \delta \prec \pi_{\xi,\mu}$.

Semisimple symmetric spaces

Motivation

- they generalize Riemannian symmetric spaces;
- G is a semisimple symmetric space:

  $G \simeq G \times G/\mathrm{diag}, \quad \sigma : (x, y) \mapsto (y, x).$

  Plancherel decomposition becomes multiplicity free.

History

- early 1980's: classification of discrete series of G/H

  (Flensted-Jensen, Oshima & Matsuki)
- 1990's: Plancherel theorem

  (Delorme, vdB & Schlichtkrull)

Semisimple symmetric spaces

New phenomenon

Occurrence of finite multiplicities in Plancherel deco.

Setting

- $X = G/H, \quad \exists K \subset G \text{ max cpt}: \quad \sigma(K) = K;$
- Polar decomposition: $G = K \exp \overline{\mathfrak{a}_\mathfrak{q}^+} H$, Jacobian J;

Theorem

Part of Plancherel decomposition

$$L^2(G/H)^K_{\mathrm{mc}} = \int^{\oplus}_{\mathfrak{a}_\mathfrak{q}^+} \mathbb{C}^{N*} \otimes \pi_\mu \, dm(\mu).$$

where

- π_μ *principal series, in* $\mathcal{H}_\mu \simeq L^2(K/M)$;
- $(\mathcal{H}_\mu^{-\infty})^H \simeq \mathbb{C}^N$, *for generic* μ.

Weyl's principle still holds

Spherical functions

- $\mathbb{C}^N \ni \eta \mapsto \varphi_\mu(x)(\eta) = \langle 1_K, \pi_\mu^\vee(x)\eta \rangle$;
- $\varphi_\mu \in C^\infty(K\backslash G/H) \otimes (\mathbb{C}^N)^*$.

Asymptotics

$$\varphi_\mu \sim J^{-1/2} \sum_{w\in W} e^{i\,w\mu} \underbrace{C(w\mu)}_{\in (\mathbb{C}^N)^*}$$

Theorem (Plancherel measure)

$$dm(\mu) = \frac{d\mu}{\|C(\mu)\|^2}.$$