5/6 – 2013

Analysis on real semisimple Lie groups, 3

1. G real semisimple, connected, $\# Z(G) < \infty$
K max cpt, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{q}$, $\mathfrak{a} \subset \mathfrak{q}$ max abelian, $\Sigma = \Sigma(\mathfrak{g}, \mathfrak{a})$,
Σ^+, $\mathfrak{n}$, $P = MAN$ minimal parabolic subgroup

$$\pi_\lambda^\infty \quad \text{in} \quad C^\infty(P:\lambda) = \{ f: G \to \mathbb{C} \mid f(xman) = a^{-\lambda-\rho} f(x) \}$$
$$(\pi_\lambda^\infty(x) f)(y) = f(x^{-1}y)$$

- $$\left. \begin{array}{l} C^\infty(P:\lambda) \times C^{\infty}(P:-\bar\lambda) \longrightarrow \mathbb{C} \\ (f,g) \longmapsto \int_{K/M} f(k)\overline{g(k)}\, dkM \end{array} \right\} \begin{array}{l}\text{sesquilinear pairing} \\ G\text{-equivariant}\end{array}$$

2. **Poisson transform** $\quad \mathcal{P}_\lambda : C^\infty(P:-\lambda) \to C^\infty(G/K)$
defined by
$$(\mathcal{P}_\lambda f)(x) = \int_{K/M} f(xk)\, dkM$$
G-equivariant transformation.

3. **Define** $\quad 1\!\!1_\lambda \in C^\infty(P:\lambda)$: $\begin{cases} K\text{-invariant} \\ 1\!\!1_\lambda(e) = 1 \end{cases}$

Then $1\!\!1_\lambda(kan) = a^{-\lambda-\rho}$, $\quad 1\!\!1_\lambda(x) = e^{(-\lambda-\rho)H(x)}$.

4. **Lemma** $\quad \mathcal{P}_\lambda f(x) = \langle f, \pi_{\bar\lambda}(x) 1\!\!1_{\bar\lambda} \rangle$
$$= \int_{K/M} f(k)\, e^{(-\lambda-\rho)H(x^{-1}k)}\, dkM$$

Proof: use G-equivariance:

$$\mathcal{P}_\lambda f(x) = \int_{K/M} f(xk)\, \overline{\mathbb{1}_{\bar\lambda}(k)}\, dkM$$

$$= \langle \pi_{-\lambda}(x^{-1})f,\ \mathbb{1}_{\bar\lambda} \rangle = \langle f,\ \pi_{\bar\lambda}(x)\mathbb{1}_{\bar\lambda} \rangle$$

$$= \int_{K/M} f(k)\, \overline{\mathbb{1}_{\bar\lambda}(x^{-1}k)}\, dkM = \text{RHS}. \quad \boxtimes$$

5. **Remark** Pairing $C^\infty(P:-\lambda) \times C^\infty(P:\lambda) \xrightarrow{(\cdot,\cdot)} \mathbb{C}$, $(f,g) \longmapsto \int_{K/M} f(k)g(k)\, dkM$ is bilinear G-invariant.

The representation π_λ^∞ of G on $C^\infty(P:\lambda)$ induces a representation of $\mathfrak{g}$ in $C^\infty(P:\lambda)$ by

$$Xf = \pi_\lambda(X)f = \frac{d}{dt}\Big|_{t=0} \pi_\lambda^\infty(\exp tX)f$$

By universal property of $U(\mathfrak{g})$, get rep'n of algebra $U(\mathfrak{g})$ in $C^\infty(P:\lambda)$.

6 **Lemma** $U(\mathfrak{g})^K \cdot \mathbb{1}_\lambda = \mathbb{C}\,\mathbb{1}_\lambda$.

Proof let $u \in U(\mathfrak{g})^K$. Then $\pi_\lambda(k)\pi_\lambda(u)\mathbb{1}_\lambda$

$= \pi_\lambda(\mathrm{Ad}(k)u)\,\pi_\lambda(k)^{-1}\mathbb{1}_\lambda = \pi_\lambda(u)\mathbb{1}_\lambda$. Hence

$$\pi_\lambda(u)\,\mathbb{1}_\lambda = \chi_\lambda(u)\,\mathbb{1}_\lambda, \quad \chi_\lambda(u) = \underbrace{(u\,\mathbb{1}_\lambda)(e)}_{\in\,\mathbb{C}} \qquad \boxtimes$$

<u>Remark:</u> $\chi_\lambda : U(\mathfrak{g})^K \to \mathbb{C}$ is an algebra homomorphism with kernel containing $U(\mathfrak{g})^K \cap U(\mathfrak{g})\mathfrak{k}$ & factors through $\chi_\lambda : U(\mathfrak{g})^K / U(\mathfrak{g})^K \cap U(\mathfrak{g})\mathfrak{k} \to \mathbb{C}$.

7 <u>Lemma</u> Let $u \in U(\mathfrak{g})^K$. Then for all $f \in C^\infty(P : -\lambda)$,

$$R_u(\mathcal{P}_\lambda f) = \chi_\lambda(u)\,\mathcal{P}_\lambda f \qquad (7.1)$$

<u>Proof:</u> Let $g \in C^\infty(P : \lambda)$ and define $T_g : C^\infty(P : -\lambda) \to C^\infty(G/K)$ by

$$T_g f(x) = \int_{K/M} f(xk)\, g(k)\, dk.$$

Then by equivariance of the pairing described in Remark 5, we see that for $X \in \mathfrak{g}$

$$\begin{aligned} R_X T_g f(x) &= \frac{d}{dt}\Big|_{t=0} \int_{K/M} f(x \exp tX\, k)\, g(k)\, dkM \\ &= \frac{d}{dt}\Big|_{t=0} \int_{K/M} f(xk)\, g(\exp(-tX)k)\, dkM \\ &= \int_{K/M} f(xk)\,(\pi_\lambda(X) g)(k)\, dkM \\ &= T_{\pi_\lambda(X) g}(f). \end{aligned}$$

By repeated application one sees that

$$R_u T_g f(x) = T_{\pi_\lambda(u)g}(f)$$

Substituting $g = 1_X$, we find (7.1).

8. From (7.1) we see that that the Poisson transform $\mathcal{P}_\lambda f$ satisfies a system of differential equations depending on λ. We will describe this system in more detail.

First we note that for $X \in \mathfrak{g}$ the map

$$R_X : C^\infty(G) \to C^\infty(G)$$
$$R_X f(x) = \frac{d}{dt}\Big|_{t=0} f(x \exp tX)$$

is a left G-invariant derivation. Let $\mathbb{D}(G)$ denote the algebra of left G-invariant linear partial differential operators with smooth coefficients on G. Then $X \mapsto R_X$ defines a linear map $\mathfrak{g} \to \mathbb{D}(G)$ with

$$R_{[X,Y]} = R_X \circ R_Y - R_Y \circ R_X.$$

By the universal property of the universal enveloping algebra. The map $X \mapsto R_X$ extends to an algebra homomorphism

$$u \mapsto R_u, \quad U(\mathfrak{g}) \to \mathbb{D}(G) \qquad (8.1)$$

By using the natural filtrations by order, and invoking the PBW thm, we find that (8.1) is an isomorphism of algebras, so

$$\mathbb{D}(G) \simeq U(\mathfrak{g}).$$

9. Let $\mathbb{D}(G/K)$ denote the algebra of left G-invariant smooth linear partial differential operators on G/K. In terms of the universal enveloping algebra, $\mathbb{D}(G/K)$ may be characterized as follows.

If $u \in U(\mathfrak{g})^K$ then $R_u \in \mathbb{D}(G)$ maps $C^\infty(G/K)$ to itself and acts like an operator $\overline{R}_u$ from $\mathbb{D}(G/K)$. It can be shown that the algebra homomorphism $u \mapsto \overline{R}_u$, $U(\mathfrak{g})^K \to \mathbb{D}(G/K)$ is surjective with kernel

$$U(\mathfrak{g})^K \cap U(\mathfrak{g})\mathfrak{k}.$$

Thus,

<u>Lemma</u> The map $u \mapsto R_u$ induces an algebra isomorphism

$$r : U(\mathfrak{g})^K / U(\mathfrak{g})^K \cap U(\mathfrak{g})\mathfrak{k} \longrightarrow \mathbb{D}(G/K).$$

10 Via the isomorphism r of Lemma 9, we may view χ_λ as a character of $\mathbb{D}(G/K)$.

<u>Def</u>. For $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ we define

$$\mathcal{E}_\lambda(G/K) := \{ \varphi \in C^\infty(G/K) \mid \forall_{D \in \mathbb{D}(G/K)} : D\varphi = \chi_\lambda(D)\varphi \}$$

<u>Lemma</u> The Poisson transform $\mathcal{P}_\lambda$ maps $C^\infty(P : -\lambda)$ into $\mathcal{E}_\lambda(G/K)$.

<u>Proof</u> This follows from Lemma 9 and Eqn (7.1). $\boxtimes$

<u>11 Helgason's conjecture</u>

We denote by $C^\omega(P:\lambda)$ the subspace of real analytic functions in $C^\infty(P:\lambda)$. Restriction to K induces a bijection

$$C^\omega(P:\lambda) \to C^\omega(K/M). \qquad (11.1)$$

We may view $C^\omega(K/M)$ as the direct limit

$$\lim_{\substack{\longrightarrow \\ U \supset K/M}} \mathcal{O}(U)$$

with U running over open neighborhoods in a suitable complexification $(K/M)_{\mathbb{C}}$ of K/M, and with $\mathcal{O}(U)$ the Fréchet space of holomorphic functions. Accordingly, $C^\omega(K/M)$ is equipped with the direct limit locally convex topology. This topology is transferred to $C^\omega(P:\lambda)$ so that (11.1) becomes a topological linear isomorphism. The associated continuous linear dual, equipped with the strong dual topology, is denoted $\mathcal{B}(P:-\lambda)$. We note that the bilinear pairing

$$C^\infty(P:-\lambda) \times C^\infty(P:\lambda) \to \mathbb{C}$$

induces an embedding $C^\infty(P:-\lambda) \hookrightarrow \mathcal{B}(P:-\lambda)$. Accordingly, $\mathcal{B}(P:-\lambda)$ can be viewed as the space of <u>hyperfunction</u> sections of the line bundle $G \times_P \mathbb{C}_{-\lambda+\rho}$.

From Lemma 4 we see that $\mathcal{P}_\lambda$ extends to a continuous linear G-equivariant map

$$\mathcal{P}_\lambda : \mathcal{B}(P:-\lambda) \longrightarrow \mathcal{E}_\lambda(G/K) \qquad 11.1$$

<u>Theorem</u> (Helgason's conjecture, proved in 1982 by KKMOOT, Annals of Math. (Kashiwara, Kowata, Minemura, Okamoto, Oshima, Tanaka)

Let $\mathrm{Re}\,\lambda$ be P-dominant, i.e. $\langle \mathrm{Re}\,\lambda, \alpha\rangle \geq 0$ for all $\alpha \in \Sigma^+$ $(= \Sigma(P))$. Then (11.1) is a topological linear isomorphism.

12 <u>Example</u> $G = SU(1,1)$, $K = S(U(1)\times U(1))$, so that $G/K \simeq D$, the Poincaré disk, see Lecture 2, page 29! It can be shown that $\mathbb{D}(D) \simeq \mathbb{C}[\Delta_D]$ (polynomials in the hyperbolic Laplacian. The character χ_λ is given by $\chi_\lambda(\Delta_D) = |\lambda|^2 - |\rho|^2$. For $\lambda = \rho$ we see that

$$\begin{aligned}\mathcal{E}_\rho(D) &= \{\varphi \in C^\infty(D) \mid \Delta_D \varphi = 0\} \\ &= \{\varphi \in C^\infty(D) \mid \Delta \varphi = 0\} \\ &= \mathcal{H}(D)\end{aligned}$$

as calculated. Moreover, the classical Poisson transform extends to a topological linear isomorphism

$$\mathcal{P} : \mathcal{B}(\partial D) \xrightarrow{\simeq} \mathcal{H}(D).$$

13 Description of χ_λ

The character $\chi_\lambda : U(\mathfrak{g})^K \longrightarrow \mathbb{C}$ can be described in terms of the Iwasawa decomposition, as follows.

From $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ (as vector spaces) it follows by Poincaré - Birkhoff - Witt that

$$U(\mathfrak{g}) = U(\mathfrak{a}) \oplus (\mathfrak{n}\, U(\mathfrak{g}) + U(\mathfrak{g})\, \mathfrak{k}) \tag{13.1}$$

We note that, since $\mathfrak{a}$ is abelian,

$$U(\mathfrak{a}) \simeq S(\mathfrak{a}) \simeq P(\mathfrak{a}^*) \text{ canonically.}$$

We define $T_\rho : P(\mathfrak{a}^*) \to P(\mathfrak{a}^*)$ by

$$T_\rho(P)(\lambda) = P(\lambda + \rho).$$

Def Define $\tilde{\gamma} : U(\mathfrak{g})^K \longrightarrow P(\mathfrak{a}^*)$ by $\tilde{\gamma}(u) = T_\rho(u_0)$ where

$u_0 \in U(\mathfrak{a})$ is determined by
$u - u_0 \in \mathfrak{n}U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{k}$, see (13.1)

<u>Theorem</u> (Harish-Chandra).
$\tilde{\gamma}$ factors through an isomorphism
from $\mathbb{D}(G/K) \simeq U(\mathfrak{g})^K / U(\mathfrak{g})^K \cap U(\mathfrak{g})\mathfrak{k}$
onto $P(\mathfrak{a}^*)^W$ (the Weyl group invariant
polynomials).

$$\mathbb{D}(G/K) \xrightarrow[\sim]{\gamma} P(\mathfrak{a}^*)^W.$$

<u>Lemma</u> Let $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$, $D \in \mathbb{D}(G/K)$.
Then

$$\chi_\lambda(D) = \gamma(D)(\lambda)$$

<u>Proof.</u> Let $u \in U(\mathfrak{g})^K$. Then

$$\pi_\lambda(u)\mathbb{1}_\lambda = \chi_\lambda(u)\mathbb{1}_\lambda$$

where $\chi_\lambda(u) = (\pi_\lambda(u)\mathbb{1}_\lambda)(e)$

We note that $\mathbb{1}_\lambda$ is left K-hence
left $\mathfrak{k}$-invariant. Therefore,

$$\pi_\lambda(U(\mathfrak{g})\mathfrak{k})\mathbb{1}_\lambda = 0.$$

Since $\mathbb{1}_\lambda$ is right $\mathfrak{k}$ invariant, we have, for $v \in U(\mathfrak{g})$, $Y \in \mathfrak{n}$:

$$(\pi_\lambda(Yv)\mathbb{1}_\lambda)(e) = L_Y L_v \mathbb{1}_\lambda(e)$$

$$= -R_{v^\vee} R_Y \mathbb{1}_\lambda(e) = 0$$

where $v \mapsto v^\vee$ denotes the canonical anti-automorphism of $U(\mathfrak{g})$ induced by $X \mapsto -X$, $\mathfrak{g} \to \mathfrak{g}$.

It follows that $(\pi_\lambda(u)\mathbb{1}_\lambda)(e) = \pi_\lambda(u_0)\mathbb{1}_\lambda(e)$. On A, $\mathbb{1}_\lambda$ is given by

$$\mathbb{1}_\lambda(a) = a^{-\lambda-\rho} = e^{(-\lambda-\rho)\log a}$$

Hence

$$(\pi_\lambda(u_0)\mathbb{1}_\lambda)(e) = (R_{u_0^\vee}\mathbb{1}_\lambda)(e)$$

$$= u_0^\vee(-\lambda-\rho) = u_0(\lambda+\rho) =$$

$$= \gamma(u)(\lambda) \qquad \boxtimes$$

<u>Remark</u> It follows that $\chi_{w\lambda} = \chi_\lambda$ for all $\lambda \in \mathfrak{a}_\mathbb{C}^*$, $w \in W$. Hence

$$\mathcal{E}_\lambda(G/K) = \mathcal{E}_{w\lambda}(G/K).$$

14 Elementary spherical functions

Lemma Let $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$. There exists a unique K-invariant function $\varphi_\lambda \in \mathcal{E}_\lambda(G/K)$ such that $\varphi_\lambda(e) = 1$. This function is given by

$$\varphi_\lambda(x) = \int_K e^{(\lambda-\rho)H(xk)}\, dk$$

Proof. We establish existence by noting that $\varphi_\lambda := \mathcal{P}_\lambda(1_{-\lambda})$ is left K-invariant (use that $\mathcal{P}_\lambda : C^\infty(P:-\lambda) \to C^\infty(G/K)$ is intertwining) and belongs to $\mathcal{E}_\lambda(G/K)$. Hence

$$\varphi_\lambda(x) = \int_{K/M} 1_{-\lambda}(xk)\, dkM$$

$$= \int_{K/M} e^{(\lambda-\rho)H(xk)}\, dkM$$

$$= \int_K e^{(\lambda-\rho)H(xk)}\, dk$$

(since H is right M-invariant, K cpt and dk and dkM are normalized)

<u>Remark</u> We note that $\varphi_\lambda = \varphi_{w\lambda}$ for all $w \in W$, in view of Remark 13.

15 The c-function

Let $\lambda \in i\mathfrak{a}^*$ be regular ($\langle \lambda, \alpha \rangle \neq 0$ $\forall_{\alpha \in \Sigma}$). Then

$$\varphi_\lambda(a) \sim \sum_{w \in W} c(w\lambda)\, a^{w\lambda - \rho} \quad (a \xrightarrow{A^+} \infty)$$

with $c : i\mathfrak{a}^{*\mathrm{reg}} \longrightarrow \mathbb{C}$ a real analytic function.

<u>Thm</u> c extends to a meromorphic function $\mathfrak{a}^*_{\mathbb{C}} \to \mathbb{C}$, $\lambda \longmapsto |c(\lambda)|^{-2}$ is C^ω on $i\mathfrak{a}^*$.

16 The spherical Plancherel theorem

Define

$$C_c^\infty(K\backslash G/K) := C_c^\infty(G/K)^K.$$

left action

For $f \in C_c^\infty(K\backslash G/K)$, define $\mathcal{F}f \in C^\infty(i\mathfrak{a}^*)$ by

$$\mathcal{F}f(\lambda) = \int_{G/K} f(x)\, \overline{\varphi_\lambda(x)}\, dx =$$

$$= \int_{G/K} f(x)\, \varphi_{-\lambda}(x)\, dx$$

Then $\mathcal{F}$ has a unique extension to an isometry

$$L^2(G/K)^K \xrightarrow{\simeq} L^2\left(i\mathfrak{a}^*, \frac{d\lambda}{|c(\lambda)|^2}\right)^W$$

(fact: $|c(w\lambda)| = |c(\lambda)|$).

17 **Remark** In particular, on $C_c^\infty(A)^W$, $\mathcal{F}^{-1}$ is given by

$$\mathcal{F}^{-1}\psi(x) = \int_{i\mathfrak{a}^*/W} \varphi_\lambda(x)\,\psi(\lambda)\,\frac{d\lambda}{|c(\lambda)|^2}.$$

18 The Plancherel decomposition of $L^2(G/K)$.

The left regular representation L of G on $L^2(G/K)$ is unitary (since dx is G-invariant).
The above Plancherel decomposition may be viewed as the K-fixed part of the Representation theoretic Plancherel decomposition which we will describe now.

We consider the Hilbert space

$$\mathfrak{H} = L^2\left(i\mathfrak{a}^{*+} \times K/M, \frac{d\lambda}{|c(\lambda)|^2} \otimes dkM\right),$$

and equip it with the unitary representation π of G defined by

$$(\pi(x)\varphi)(\lambda, kM) = (\pi_{-\lambda}(x)\varphi(\lambda,\cdot))(k)$$

$$= e^{(+\lambda-\rho)H(x^{-1}k)} \varphi(\lambda, K(x^{-1}k))$$

This representation is known as the direct integral of the representations π_λ, $\lambda \in i\mathfrak{a}^{*+}$, relative to the measure $\frac{d\lambda}{|c(\lambda)|^2}$

Notation

$$\pi = \int_{i\mathfrak{a}^{*+}}^{\oplus} \pi_{-\lambda} \frac{d\lambda}{|c(\lambda)|^2}$$

Given $f \in C_c^\infty(G/K)$ we define its Fourier transform $\hat{f} : i\mathfrak{a}^{*+} \times K/M \to \mathbb{C}$ by

$$\hat{f}(\lambda, kM) := (\pi_{-\lambda}(f)\mathbb{1}_{-\lambda})(kM)$$

where

$$\pi_{-\lambda}(f)\,\mathbb{1}_{-\lambda} = \int_{G/K} f(x)\,\pi_{-\lambda}(x)\,\mathbb{1}_{-\lambda}\,dx.$$

It is clear that the map $f \mapsto \hat{f}$ is G-intertwining

- $$C_c^\infty(G/K) \longrightarrow \mathfrak{h}$$

with respect to L and π.

<u>Theorem</u> $f \mapsto \hat{f}$ extends to an isometric isomorphism

$$L^2(G/K) \xrightarrow{\simeq} \mathfrak{h}$$

- We also say that $f \mapsto \hat{f}$ induces a unitary isomorphism

$$\left(L, L^2(G/K)\right) \xrightarrow{\simeq} \int_{i\mathfrak{a}^{*+}}^{\oplus} \pi_\lambda \frac{d\lambda}{|c(\lambda)|^2}$$

<u>Remark</u> The spherical Plancherel formula is obtained by restriction of the above to $L^2(G/K)^K$ $\left(\mathfrak{h}^K \simeq L^2(i\mathfrak{a}^{*+}, \frac{d\lambda}{|c(\lambda)|^2}) \right)$

The name Plancherel formula is justified by the following observations:

1) $\lambda, \mu \in i\mathfrak{a}^{*+}$, $\lambda \neq \mu \Rightarrow \pi_\lambda \not\sim \pi_\mu$

2) for almost every $\lambda \in i\mathfrak{a}^{*+}$, the representation π_λ is irreducible.

We will come back to this in the next lecture.