NOTES ON A-HYPERGEOMETRIC FUNCTIONS

par

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1. Introduction

Hypergeometric functions of Gauss type are immediate generalisations of the classical elementary functions like sin, arcsin, arctan, log, etc. They were studied extensively in the 19th century by mathematicians like Kummer and Riemann. Towards the end of the 19th century and the beginning of the 20th century hypergeometric functions in several variables were introduced. For example Appell's functions, the Lauricella functions and the Horn series. Around 1990, in the series of papers [**19**], [**20**], [**21**], [**22**] it was realised by Gel'fand, Kapranov and Zelevinsky that all above types and their differential equations fit into a far more general but extremely elegant scheme of so-called A-hypergeometric functions, or GKZ-hypergeometric functions.
Nowadays hypergeometric functions of all types (including GKZ-type, but also many others not mentioned here) are ubiquitous throughout the mathematics and mathematical physics literature, ranging from orthogonal polynomials, modular forms to scattering theory and mirror symmetry.
The present notes form an introduction to A-hypergeometric functions. We describe their defining equations and explicit solutions in the form of power series expansions and so-called Euler integral representations. We also discuss the associated D-modules and their relation with the work of B.Dwork in [**14**]. The latter book describes a theory of generalised hypergeometric functions which runs for a large part in parallel with the theory of Gel'fand, Kapranov and Zelevinsky. However, the language is entirely different and a large part of [**14**] is also devoted to the p-adic theory of generalised hypergeometric functions.
Essentially the first book devoted entirely to A-hypergeometric functions is the one by Saito, Sturmfels and Takayama [**37**]. In addition, there are several introductory notes such as [**38**] and [**33**], discussing similar, and on the other hand, different aspects of the theory. The book [**40**] deals with a certain type of A-hypergeometric function, namely the Aomoto-systems $X(2,4)$ and $X(3,6)$. However, it does cover aspects such as monodromy calculations for this system and a moduli interpretation of the underlying geometry. These subjects are not addressed in this survey, simply because a general theory is still lacking. In a forthcoming publication we like to show how subgroups of the monodromy group of general A-hypergeometric systems can be computed.
Another aspect not dealt with in these notes is the question which hypergeometric equations have all of their solutions algebraic over the rational function field generated by their variables. This is a classical question. In 1873 H.A.Schwarz compiled his famous list of Gauss hypergeometric functions which are algebraic. This list was extended to general one variable hypergeometric functions in 1989 by G.Heckman and F.Beukers in [**7**]. In the several variable case Schwarz's list had also been extended to functions such as Appell's F1 (T.Sasaki, [**34**]), Appell F2 (M.Kato, [**27**]), Appell F4 (M.Kato, [**26**]), Lauricella's FD (Cohen-Wolfart, [**4**]) and the Aomoto system $X(3,6)$ (K.Matsumoto, T.Sasaki, N.Takayama, M.Yoshida, [**31**]).

In 2006 the present author found a combinatorial characterisation for algebraic A-hypergeometric functions (in the irreducible case) in [**9**]. It is perhaps interesting to note that, as an application, Esther Bod [**6**] succeeded in extending Schwarz's list to all irreducible Appell, Lauricella and Horn equations.
Finally, we should mention the book of Gel'fand, Kapranov and Zelevinsky, [**24**], which is not on A-hypergeometric functions proper, but on A-resultants and discriminants which arise in connection the singular loci of A-hypergeometric systems.

2. The one variable case

Let $\alpha_1,\ldots,\alpha_n;\beta_1,\ldots,\beta_n$ be any complex numbers and consider the generalised hypergeometric equation in one variable,

$$z(D+\alpha_1)\cdots(D+\alpha_n)F=(D+\beta_1-1)\cdots(D+\beta_n-1)F,\qquad D=z\frac{d}{dz} \tag{1}$$

This is a Fuchsian equation of order n with singularities at $0,1,\infty$. The local exponents read,

$$\begin{array}{ll} 1-\beta_1,\ldots,1-\beta_n & \text{at } z=0\\ \alpha_1,\ldots,\alpha_n & \text{at } z=\infty\\ 0,1,\ldots,n-2,\ -1+\sum_1^n(\beta_i-\alpha_i) & \text{at } z=1 \end{array}$$

When the β_i are distinct modulo 1 a basis of solutions at $z=0$ is given by the functions

$$z^{1-\beta_i}\ {}_nF_{n-1}\left(\left.\begin{array}{c}\alpha_1-\beta_i+1,\ldots,\alpha_n-\beta_i+1\\ \beta_1-\beta_i+1,..^{\vee}..,\beta_n-\beta_i+1\end{array}\right| z\right)\qquad (i=1,\ldots,n).$$

Here $..^{\vee}..$ denotes suppression of the term $\beta_i-\beta_i+1$ and ${}_nF_{n-1}$ stands for the generalised hypergeometric function in one variable

$${}_nF_{n-1}\left(\left.\begin{array}{c}\alpha_1,\ldots,\alpha_n\\ \beta_1,\ldots,\beta_{n-1}\end{array}\right| z\right)=\sum_{k=0}^{\infty}\frac{(\alpha_1)_k\cdots(\alpha_n)_k}{(\beta_1)_k\cdots(\beta_{n-1})_k k!}z^k.$$

Here $(x)_k$ is the Pochhammer symbol defined by $(x)_k=\Gamma(x+k)/\Gamma(x)=x(x+1)(x+2)\cdots(x+k-1)$. The function $\Gamma(z)$ is of course the Euler Γ-function. When the α_j are distinct modulo 1 we have the following n independent power series solutions in $1/z$,

$$z^{-\alpha_j}\ {}_nF_{n-1}\left(\left.\begin{array}{c}\alpha_j-\beta_1+1,\ldots,\alpha_j-\beta_n+1\\ \alpha_j-\alpha_1+1,..^{\vee}..,\alpha_j-\alpha_n+1\end{array}\right| \frac{1}{z}\right)\qquad (j=1,\ldots,n).$$

At $z=1$ we have the following interesting situation.

Theorem 2.1 **(Pochhammer)**. — *The equation* (1) *has $n-1$ independent holomorphic solutions near $z=1$.*

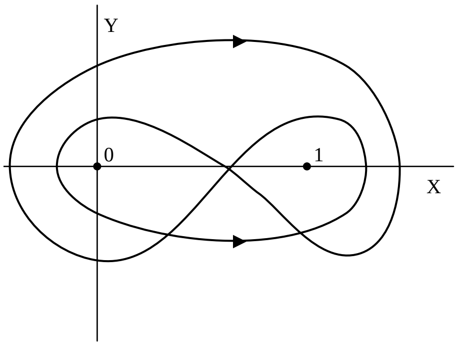
However, the solutions are not as easy to write down.
Finally we mention the Euler integral for ${}_nF_{n-1}(\alpha_1,\ldots,\alpha_n;\beta_1,\ldots,\beta_{n-1}|z)$,

$$\prod_{i=1}^{n-1}\frac{\Gamma(\beta_i)}{\Gamma(\alpha_i)\Gamma(\beta_i-\alpha_i)}\int_0^1\cdots\int_0^1\frac{\prod_{i=1}^{n-1}t_i^{\alpha_i-1}(1-t_i)^{\beta_i-\alpha_i-1}}{(1-zt_1\cdots t_{n-1})^{\alpha_n}}dt_1\cdots dt_{n-1}$$

for all $\Re\beta_i > \Re\alpha_i > 0$ $(i = 1, \ldots, n-1)$.
In the case $n = 2$ this becomes the famous Euler integral

$$ {}_2F_1(a,b,c|z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}dt \quad (\Re c > \Re b > 0) $$

The restriction $\Re c > \Re b > 0$ is included to ensure convergence of the integral at 0 and 1. We can drop this condition if we take the Pochhammer contour γ given by



as integration path. Notice that the integrand acquires the same value after analytic continuation along γ.
It is a straightforward exercise to show that for any $b, c-b \notin \mathbb{Z}$ we have

$$ {}_2F_1(a,b,c|z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \frac{1}{(1-e^{2\pi ib})(1-e^{2\pi i(c-b)})} \int_\gamma t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}dt. $$

In Section 20 we shall generalise the Pochhammer contour to higher dimensional versions.

3. Appell and Lauricella functions

There exist many generalisations of hypergeometric functions in several variables. The most well-known ones are the Appell functions in 2 variables, introduced by P.Appell in 1880, and Lauricella functions of n variables. The Appell functions read

$$ \begin{aligned} F_1(a,b,b',c,x,y) &= \sum \frac{(a)_{m+n}(b)_m(b')_n}{(c)_{m+n}m!n!}x^m y^n \\ F_2(a,b,b',c,c',x,y) &= \sum \frac{(a)_{m+n}(b)_m(b')_n}{(c)_m(c')_n m!n!}x^m y^n \\ F_3(a,a',b,b',c,x,y) &= \sum \frac{(a)_m(a')_n(b)_m(b')_n}{(c)_{m+n}m!n!}x^m y^n \\ F_4(a,b,c,c',x,y) &= \sum \frac{(a)_{m+n}(b)_{m+n}}{(c)_m(c)_n m!n!}x^m y^n \end{aligned} $$

The guiding principle for these functions is the following. Consider the product

$$ {}_2F_1\left(\left.\begin{matrix} a,b \\ c \end{matrix}\right| x\right) {}_2F_1\left(\left.\begin{matrix} a',b' \\ c' \end{matrix}\right| y\right) = \sum \frac{(a)_m(a')_n(b)_m(b')_n}{(c)_m(c')_n m!n!}x^m y^n. $$

Now replace one or two of the product pairs

$$(a)_m, (a')_n \quad (b)_m(b')_n \quad (c)_m(c')_n$$

by the corresponding

$$(a)_{m+n}, \quad (b)_{m+n}, \quad (c)_{m+n}.$$

Replacing all three pairs would give us

$$\sum \frac{(a)_{m+n}(b)_{m+n}}{(c)_{m+n}m!n!} = {}_2F_1\left(\begin{matrix} a,b \\ c \end{matrix} \middle| x+y\right)$$

which we omit for obvious reasons.
In 1893 G.Lauricella introduced the 3-variable versions of these functions in [**29**], but nowadays one considers the obvious n-variable analogues as well. We use the notations

$$\mathbf{x} = x_1,\ldots,x_n \qquad \mathbf{x}^{\mathbf{m}} = x_1^{m_1}\ldots x_n^{m_n}$$

$$(\mathbf{a})_{\mathbf{m}} = (a_1)_{m_1}\cdots(a_n)_{m_n} \qquad \mathbf{m}! = m_1!\cdots m_n! \qquad |\mathbf{m}| = m_1+\cdots+m_n$$

We have

$$\begin{aligned}
F_A(a,\mathbf{b},\mathbf{c}|\mathbf{x}) &= \sum_{\mathbf{m}\geq 0} \frac{(a)_{|\mathbf{m}|}(\mathbf{b})_{\mathbf{m}}}{(\mathbf{c})_{\mathbf{m}}\mathbf{m}!}\mathbf{x}^{\mathbf{m}} \quad |x_1|+\cdots+|x_n|<1 \\
F_B(\mathbf{a},\mathbf{b},c|\mathbf{x}) &= \sum_{\mathbf{m}\geq 0} \frac{(\mathbf{a})_{\mathbf{m}}(\mathbf{b})_{\mathbf{m}}}{(c)_{|\mathbf{m}|}\mathbf{m}!}\mathbf{x}^{\mathbf{m}} \quad \forall i: |x_i|<1 \\
F_C(a,b,\mathbf{c}|\mathbf{x}) &= \sum_{\mathbf{m}\geq 0} \frac{(a)_{|\mathbf{m}|}(b)_{|\mathbf{m}|}}{(\mathbf{c})_{\mathbf{m}}\mathbf{m}!}\mathbf{x}^{\mathbf{m}} \quad |\sqrt{x_1}|+\cdots+|\sqrt{x_n}|<1 \\
F_D(a,\mathbf{b},c|\mathbf{x}) &= \sum_{\mathbf{m}\geq 0} \frac{(a)_{|\mathbf{m}|}(\mathbf{b})_{\mathbf{m}}}{(c)_{|\mathbf{m}|}\mathbf{m}!}\mathbf{x}^{\mathbf{m}} \quad \forall i: |x_i|<1
\end{aligned}$$

When $n=2$ these functions coincide with Appell's F_2, F_3, F_4, F_1 respectively. When $n=1$, they all coincide with Gauss' ${}_2F_1$. Lauricella gave several transformation formulae, of which we mention a few. Many more can be found in Exton's book [**18**] on hypergeometric equations.

$$\begin{aligned}
& F_A(a,\mathbf{b},\mathbf{c}|\mathbf{x}) \\
= \ & (1-x_1)^{-a_1}F_A\left(a, c_1-b_1, b_2,\ldots,b_n,\mathbf{c} \ \middle| \ \frac{x_1}{x_1-1}, \frac{x_2}{1-x_1},\ldots,\frac{x_n}{1-x_1}\right) \\
= \ & (1-|\mathbf{x}|)^{-a}F_A\left(a,\mathbf{c}-\mathbf{b},\mathbf{c} \ \middle| \ \frac{\mathbf{x}}{|\mathbf{x}|-1}\right)
\end{aligned}$$

$$
\begin{aligned}
& F_D(a,\mathbf{b},c|\mathbf{x}) \\
= \; & (1-\mathbf{x})^{-\mathbf{b}} F_D\left(c-a,\mathbf{b},c \;\middle|\; \frac{\mathbf{x}}{\mathbf{x}-1}\right) \\
= \; & (1-x_1)^{-a} F_D\left(a, c-|\mathbf{b}|, b_2,\ldots,b_n,c \;\middle|\; \frac{x_1}{x_1-1}, \frac{x_1-x_2}{x_1-1},\ldots,\frac{x_1-x_n}{x_1-1}\right) \\
= \; & (1-x_1)^{c-a}(1-\mathbf{x})^{-\mathbf{b}} \times \\
& F_D\left(c-a, c-|\mathbf{b}|, b_2,\ldots,b_n,c \;\middle|\; x_1, \frac{x_2-x_1}{x_2-1},\ldots,\frac{x_n-x_1}{x_n-1}\right)
\end{aligned}
$$

Similar transformations for F_B, F_C have not been found. The following quadratic transformation was discovered in 1974 by Srivastava and Exton,

$$
(1+|\mathbf{x}|)^a F_C(a/2, a/2+1/2, \mathbf{c}|\ \mathbf{x}^2) = F_A\left(a, \mathbf{c}-1/2, 2\mathbf{c}-1 \;\middle|\; \frac{2\mathbf{x}}{1+|\mathbf{x}|}\right)
$$

4. Horn series

In 1889 J.Horn began the investigation of multiple power series

$$
\sum_{\mathbf{m}\ge 0} A(\mathbf{m})\mathbf{x}^{\mathbf{m}}
$$

having the property that $A(\mathbf{m}+\mathbf{e}_k)/A(\mathbf{m})$ is a rational function of $\mathbf{m} = m_1,\ldots,m_n$ for each $k=1,\ldots,n$. Here $\mathbf{e}_k$ denotes the k-th unit vector in $\mathbb{R}^n$. More precisely, Horn studied the cases $n=2,3$. Let us consider the formal Laurent series

$$
\sum_{\mathbf{m}\in\mathbb{Z}^n} A(\mathbf{m})\mathbf{x}^{\mathbf{m}}
$$

such that $f_k(\mathbf{m}) = A(\mathbf{m}+\mathbf{e}_k)/A(\mathbf{m}) \in \mathbb{C}(\mathbf{m})$. Of course we have the compatibility conditions

$$
\forall i,j: \qquad f_i(\mathbf{m}+\mathbf{e}_j)f_j(\mathbf{m}) = f_j(\mathbf{m}+\mathbf{e}_i)f_i(\mathbf{m}).
$$

In the 1930's Ore [**32**] suggested the following general result to hold.

***Theorem 4.1* (Ore-Sato).** — *Let* $\mathrm{supp}(A)$ *be the subset of* $\mathbf{m}\in\mathbb{Z}^n$ *where* $A(\mathbf{m})\ne 0$. *Suppose that* $\mathrm{supp}(A)$ *is connected and Zariski-dense in* $\mathbb{C}^n$. *A subset* $S\subset\mathbb{Z}^n$ *is called connected if every point of* S *can be reached by unit steps* $\pm\mathbf{e}_i$ *inside* S *from any other point of* S.
Then there exist $R(\mathbf{m})\in\mathbb{C}(\mathbf{m})^*$, $\theta_1,\ldots,\theta_N\in\mathbb{C}$, $c_1,\ldots,c_N\in\mathbb{C}^*$, $s_1,\ldots,s_N\in\mathbb{Z}$ *and an integral matrix*

$$
\begin{matrix}
d_{11} & \ldots & d_{1n} \\
\vdots & & \vdots \\
d_{N1} & \ldots & d_{Nn}
\end{matrix}
$$

such that

$$
A(\mathbf{m}) = R(\mathbf{m})\mathbf{c}^{\mathbf{m}} \prod_{j=1}^{N} \Gamma(\theta_j + 1 + \sum_{p=1}^{n} d_{jp}m_p)^{s_i}.
$$

In the history of this theorem the conditions on the support of A were ignored. This was remedied in [**1**, Corollary 2] where it was derived as a Corollary of the work of M.Sato (see [**35**]). Erdelyi worked out the two-variable case in [**17**] in 1951.
Series with $R(\mathbf{m}) = 1, c_i = 1$ for all i will be called *Horn series*. Under the assumptions $n = 2$ and $\deg f_i \le 2$ Horn (1889 and 1931) found 34 such power series, among which 14 where the degrees of numerators and denominators of all f_i are 2 (the so-called *complete series*). Series derived from one variable functions or products of one variable functions are not included. Beside the Appell F_1, F_2, F_3, F_4 Horn's list of complete series consists of

$$\begin{aligned}
G_1(a,b,b',x,y) &= \sum \frac{(a)_{m+n}(b)_{n-m}(b')_{m-n}}{m!n!}x^m y^n \\
G_2(a,a',b,b',x,y) &= \sum \frac{(a)_m(a')_n(b)_{n-m}(b')_{m-n}}{m!n!}x^m y^n \\
G_3(a,a',x,y) &= \sum \frac{(a)_{2n-m}(a')_{2m-n}}{m!n!}x^m y^n \\
H_1(a,b,c,d,x,y) &= \sum \frac{(a)_{m-n}(b)_{m+n}(c)_n}{(d)_m m!n!}x^m y^n \\
H_2(a,b,c,d,e,x,y) &= \sum \frac{(a)_{m-n}(b)_m(c)_n(d)_n}{(e)_m m!n!}x^m y^n \\
H_3(a,b,c,x,y) &= \sum \frac{(a)_{2m+n}(b)_n}{(c)_{m+n} m!n!}x^m y^n \\
H_4(a,b,c,d,x,y) &= \sum \frac{(a)_{2m+n}(b)_n}{(c)_m(d)_n m!n!}x^m y^n \\
H_5(a,b,c,x,y) &= \sum \frac{(a)_{2m+n}(b)_{n-m}}{(c)_n m!n!}x^m y^n \\
H_6(a,b,c,x,y) &= \sum \frac{(a)_{2m-n}(b)_{n-m}(c)_n}{m!n!}x^m y^n \\
H_7(a,b,c,d,x,y) &= \sum \frac{(a)_{2m-n}(b)_n(c)_n}{(d)_m m!n!}x^m y^n
\end{aligned}$$

The Pochhammer symbol $(x)_n$ for any $n \in \mathbb{Z}$ should be interpreted as $\Gamma(x+n)/\Gamma(x)$.

5. Definitions, first properties

After some preliminary papers by Gel'fand concerning hypergeometric functions on Grassmannian manifolds, the general idea of an A-hypergeometric function was formulated by Gel'fand, Kapranov and Zelevinsky around 1988.
The definition of A-hypergeometric functions begins with a finite subset $A \subset \mathbb{Z}^r$ (hence their name) consisting of N vectors $\mathbf{a}_1, \ldots, \mathbf{a}_N$ such that
i) The $\mathbb{Z}$-span of $\mathbf{a}_1, \ldots, \mathbf{a}_N$ equals $\mathbb{Z}^r$.
ii) There exists a linear form h on $\mathbb{R}^r$ such that $h(\mathbf{a}_i) = 1$ for all i.

The second condition ensures that we shall be working in the case of so-called Fuchsian systems. In a number of papers, eg [**2**], this condition is dropped to include the case of so-called confluent hypergeometric equations.
We are also given a vector of parameters $\alpha = (\alpha_1, \ldots, \alpha_r)$ which could be chosen in $\mathbb{C}^r$, but we will usually take $\alpha \in \mathbb{Q}^r$. The lattice $L \subset \mathbb{Z}^N$ of relations consists of all $(l_1, \ldots, l_N) \in \mathbb{Z}^N$ such that $\sum_{i=1}^N l_i \mathbf{a}_i = 0$.
The A-hypergeometric equations are a set of partial differential equations with independent variables $v_1, \ldots, v_N$. This set consists of two groups. The first are the structure equations

$$\Box_{\mathbf{l}}\Phi := \prod_{l_i>0} \partial_i^{l_i}\Phi - \prod_{l_i<0} \partial_i^{|l_i|}\Phi = 0 \tag{A1}$$

for all $\mathbf{l} = (l_1, \ldots, l_N) \in L$. The operators $\Box_{\mathbf{l}}$ are called the *box-operators*. The second group consists of the homogeneity or Euler equations.

$$Z_i\Phi := (a_{i1}v_1\partial_1 + a_{i2}v_2\partial_2 + \cdots + a_{iN}v_N\partial_N - \alpha_i)\,\Phi = 0,\ \ i = 1, 2, \ldots, r \tag{A2}$$

where $a_{i,k}$ denotes the i-th coordinate of $\mathbf{a}_k$. The coefficients a_{ik} are simply the coefficients of the $r \times N$-matrix with columns given by $\mathbf{a}_1, \ldots, \mathbf{a}_N$. We call this the *A-matrix*. So every operator Z_i corresponds to the i-th row in the A-matrix. It is not hard to prove the following Proposition.

Proposition 5.1. — *Let Ψ be an analytic function in $v_1, \ldots, v_N$. Then Ψ is a solution of the system $Z_i\Psi = 0$ for $i = 1, \ldots, r$ if and only if*

$$\Psi(\mathbf{t}^{\mathbf{a}_1}v_1, \ldots, \mathbf{t}^{\mathbf{a}_N}v_N) = \mathbf{t}^{\alpha}\Psi(v_1, \ldots, v_N)$$

for all $\mathbf{t} \in (\mathbb{C}^*)^N$.

Proof. Choose $i, 1 \le i \le N$. We will show that Ψ satisfies $Z_i\Psi = 0$ if and only if

$$\Psi(t^{a_{i1}}v_1, \ldots, t^{a_{iN}}v_N) = t^{\alpha_i}\Psi(v_1, \ldots, v_N)$$

for all $t \in \mathbb{C}^*$. Note that the functional equation is equivalent to

$$\left(t\frac{d}{dt} - \alpha_i\right)\Psi(t^{a_{i1}}v_1, \ldots, t^{a_{iN}}v_N) = 0.$$

Use the chain rule to obtain the equivalent statement

$$\left(-\alpha_i + \sum_{j=1}^N a_{ij}v_j\partial_j\right)\Psi(t^{a_{1i}}v_1, \ldots, t^{a_{Ni}}v_N) = 0.$$

This is equivalent to $Z_i\Psi = 0$. $\Box$

We denote the system of equations (A1) and (A2) by $H_A(\alpha)$. In the next section we explain the concept of the rank of a system of partial differential equations. Roughly speaking it is the dimension of the space of analytic solutions around a generic point.

6. The A-hypergeometric D-module

Let K be a differential field with commuting derivations $\partial_i = \frac{\partial}{\partial v_i}$ for $i = 1, \ldots, N$. The field of constants, being the subfield of elements of K all of whose derivatives are zero, is denoted by C_K.
Let $\mathcal{L}$ be a finite set of linear partial differential operators with coefficients in K. Consider the differential ring $K[\partial_1, \ldots, \partial_N]$ and let $(\mathcal{L})$ be the left ideal generated by the differential operators of the system. The quotient $K[\partial_i]/(\mathcal{L})$ can be considered as a differential module over K with the natural left action of the operators ∂_i. Denote its K-rank by d. We call d the *rank* of the system $\mathcal{L}$. Let us assume that d is finite. Fix a monomial K-basis $\partial^{\mathbf{b}} = \partial_1^{b_1} \cdots \partial_N^{b_N}$ of $K[\partial_i]/(\mathcal{L})$ with $\mathbf{b} \in S$ and where S is a finite set of N-tuples in $\mathbb{Z}^N_{\geq 0}$ of cardinality d.
The following Proposition links the rank with the dimension of the k-vector space of solutions of the system of differential equations $L(f) = 0,\ L \in \mathcal{L}$.

Proposition 6.1. — *Let $\mathcal{K}$ be some differential extension of K with field of constants k. Let $f_1, \ldots, f_m \in \mathcal{K}$ be a set of k-linear independent solutions of the system $L(f) = 0,\ L \in \mathcal{L}$. Then $m \leq d$. Moreover, if $m = d$ the determinant*

$$W_S(f_1, \ldots, f_d) = \det(\partial^{\mathbf{b}} f_i)_{\mathbf{b} \in S; i=1,\ldots,d}$$

is nonzero.

For any d solutions $f_1, \ldots, f_d$ we call W_S the *Wronskian matrix* with respect to S and $f_1, \ldots, f_d$. Obviously, if $f_1, \ldots, f_d$ are k-linear dependent solutions then $W_S(f_1, \ldots, f_d) = 0$.

Proof. Suppose that either $m > d$ or $m = d$ and $W_S = 0$. In both cases there exists a $\mathcal{K}$-linear relation between the vectors $df_i := (\partial^{\mathbf{b}} f_i)_{\mathbf{b} \in S}$ for $i = 1, 2, \ldots, m$. Choose $\mu < m$ maximal such that $df_i,\ i = 1, \ldots, \mu$ are $\mathcal{K}$-linear independent. Then, up to a factor, the vectors $df_i,\ i = 1, \ldots, \mu + 1$ satisfy a unique dependence relation $\sum_{i=1}^{\mu+1} A_i df_i = 0$ with $A_i \in \mathcal{K}$ not all zero. For any j we can apply the operator ∂_j to this relation to obtain

$$\sum_{i=1}^{\mu+1} \partial_j(A_i) df_i + A_i \partial_j(df_i) = 0.$$

Since $\partial_j \partial^{\mathbf{b}}$ is a K-linear combination of the elements $\partial^{\mathbf{b}}, \mathbf{b} \in S$ in $K[\partial_i]/(\mathcal{L})$ there exists a $d \times d$-matrix M_j with elements in K such that $\partial_j(df_i) = df_i \cdot M_j$. Consequently $\sum_{i=1}^{\mu+1} A_i \partial_j(df_i) = \sum_{i=1}^{\mu+1} A_i df_i \cdot M_j = 0$ and so we are left with $\sum_{i=1}^{\mu+1} \partial_j(A_i) df_i = 0$. Since the relation between $df_i, i = 1, \ldots, \mu + 1$ is unique up to a scalar factor, there exists $\lambda_j \in \mathcal{K}$ such that $\partial_j(A_i) = \lambda_j A_i$ for all i. Suppose $A_1 \neq 0$. Then this implies that $\partial_j(A_i/A_1) = 0$ for all i and all j. We conclude that $A_i/A_1 \in C_K$ for all i. Hence there is a relation between the df_i with coefficients in C_K. A fortiori there is a C_K-linear relation between the f_i. This contradicts our assumption of independence of $f_1, \ldots, f_m$.
So we conclude that $m \leq d$ and if $m = d$ then $W_S \neq 0$.

□

Now let $\mathcal{L}$ be the system $H_A(\alpha)$ with $K = \mathbb{C}(v_1, \ldots, v_N) = \mathbb{C}(\mathbf{v})$. The corresponding differential module is called the *A-hypergeometric module.*
In general the A-hypergeometric system has rank equal to the $r-1$-dimensional volume of the so-called A-polytope $Q(A)$. This polytope is the convex hull of the endpoints of the $\mathbf{a}_i$. The volume-measure is normalised to 1 for a $(r-1)$-simplex of lattice-points in the plane $h(\mathbf{x}) = 1$ whose vertices are spanned by a set of r vectors with determinant 1. In the first days of the theory of A-hypergeometric systems there was some confusion as to what 'general' means, see the correction in [**23**] and [**2**]. To describe this consider the ideal I_A in $\mathbb{C}[\partial_1, \ldots, \partial_N]$ generated by the box operators $\Box_{\mathbf{l}}$. This is an ideal in the commutative polynomial ring $\mathbb{C}[\partial_1, \ldots, \partial_N]$, known as the toric ideal associated to A. It can be generated by a finite number of box-operators. We say that I_A has the Cohen-Macaulay property if the ring R_A/I_A is Cohen-Macaulay.

***Theorem 6.2* (GKZ).** — *Let notations be as above. If the ideal I_A has the Cohen-Macaulay property, then the system $H_A(\alpha)$ is holonomic of rank equal to the volume of the convex hull $Q(A)$ of A.*

A theorem of Hochster [**25**] ensures that the following condition is sufficient for the Cohen-Macaulay property of I_A.
iii) The $\mathbb{R}_{\ge 0}$-span of A intersected with $\mathbb{Z}^r$ equals the $\mathbb{Z}_{\ge 0}$-span of A.
When A satifies condition (iii) we say that A is *saturated.* When I_A is not Cohen-Macaulay the rank of the system $H_A(\alpha)$ may be larger than the volume of $Q(A)$ for specific values of α. We say that the system has a rank jump at α if this occurs. For a complete story on rank jumps we refer to [**30**], [**39**] and [**5**]. In [**2**, Theorem 5.15] it is shown that if $H_A(\alpha)$ is non-resonant (see Definition 8.1) then the rank also equals the volume of $Q(A)$.

From now on we assume that all conditions i),ii),iii) are satisfied.

7. Dwork modules (optional)

Although it is not relevant to the remainder of these lectures we like to point out that B.Dwork completely independently arrived at a description of A-hypergeometric functions in [**14**], although the language is entirely different. In this section we describe the isomorphism between the Dwork approach and the GKZ approach.
In [**14**] the following module is defined. Let $R_A = K[\mathbf{t}^{\mathbf{a}_1}, \ldots, \mathbf{t}^{\mathbf{a}_N}]$ where, as before, $K = \mathbb{C}(\mathbf{v})$ and where we denote $\mathbf{t}^{\mathbf{a}_i} = t_1^{a_{i1}} \cdots t_r^{a_{ir}}$. We call this the ring of A-polynomials. Define

$$f = \sum_{i=1}^{N} v_i \mathbf{t}^{\mathbf{a}_i}$$

and $f_j = t_j \frac{\partial}{\partial t_j} f$ for $j = 1, 2, \ldots, r$. We turn R_A into a D-module by defining a connection ∇ according to

$$\nabla(\partial_i) = \delta_i + \partial_i(f), \quad i = 1, 2, \ldots, N$$

where δ_i is defined by $\delta_i(a) = \partial_i(a)$ for all $a \in \mathbb{C}(\mathbf{v})$ and $\delta_i(t_j) = 0$ for all $j = 1, \ldots, r$. Notice by the way that $\partial_i(f) = \mathbf{t}^{\mathbf{a}_i}$.
For $i = 1, \ldots, r$ define the differential operators

$$D_{i,f,\alpha} = t_i \frac{\partial}{\partial t_i} + f_i - \alpha_i$$

where $\alpha_i \in \mathbb{C}$. Denote the sum $\sum_{i=1}^r D_{i,f,\alpha}(R_A)$ by $D_{f,\alpha}R_A$. The following Lemma is easy to prove.

Lemma 7.1. — *The operators $D_{i,f,\alpha}$ commute with the action of $\delta_j + \partial_j(f)$ for all i, j.*

Proof. Notice that formally,

$$t_i \frac{\partial}{\partial t_i} + f_i = \frac{1}{f} \circ t_i \frac{\partial}{\partial t_i} \circ f$$

and

$$\delta_j + \partial_j(f) = \frac{1}{f} \circ \partial_j \circ f.$$

So the operators are the same twist of the partial differential operators $t_i(\partial/\partial t_i)$ and ∂_j. Since they commute, their twists will also commute.

□

As a result we find that $D_{f,\alpha}R_A$ is stable under the action of $\delta_j + \partial_j(f)$ and hence, is a sub D-module of R_A.
In [**14**] B.Dwork defined the following hypergeometric D-module.

Definition 7.2. — *With the notations above the Dwork module is defined as the quotient D-module $R_A/D_{f,\alpha}R_A$. Notation : $W_{f,\alpha}$.*

Remark 7.3. — *In fact, Dwork uses more general differential base fields than $\mathbb{C}(\mathbf{v})$ in his definitions. However, these more general modules can be considered as restrictions of the Dwork modules we defined above.*

We now like to show that the A-hypergeometric module defined earlier is isomorphic (as D-module) to $W_{f,\alpha}$.
Let $\Box_1, \ldots, \Box_s$ be a finite set of box operators such that every box operator is contained in $\mathbb{C}[\partial] \cdot \Box_1 + \cdots + \mathbb{C}[\partial] \cdot \Box_s$. Then the A-hypergeometric D-module can be written as

$$K[\partial]/(K[\partial] \cdot \Box_1 + \cdots + K[\partial] \cdot \Box_s + K[\partial] \cdot Z_1 + \cdots + K[\partial] \cdot Z_r)$$

where $K[\partial]$ is shorthand for $K[\partial_1, \ldots, \partial_N]$ (and $K = \mathbb{C}(\mathbf{v})$). We map $K[\partial]$, considered as K-module, to the K-module $W_{f,\alpha}$ of Dwork by mapping $P(\partial_1, \ldots, \partial_N)$ to $P(\mathbf{t}^{\mathbf{a}_i}, \ldots, \mathbf{t}^{\mathbf{a}_N})(\mathrm{mod}\ D_{f,\alpha}R_A)$ for $i = 1, 2, \ldots, N$ for every $P \in K[\partial]$. Call this map σ. To show that σ is a D-module homomorphism we need to verify that $\sigma(\partial_i \circ P) = (\delta_i + \partial_i(f))\sigma(P)$ for every i and every $P \in K[\partial]$. This is obvious, as we can see,

$$\begin{aligned} \sigma(\partial_i \circ P) &= \sigma(\delta_i P + P\partial_i) \\ &= \delta_i \sigma(P) + \mathbf{t}^{\mathbf{a}_i}\sigma(P) \\ &= \delta_i \sigma(P) + \partial_i(f)\sigma(P) \end{aligned}$$

The map σ is of course surjective, so it suffices to determine the kernel of σ. To do this it suffices to determine the inverse image of $D_{f,\alpha}R_A$ of the map $\tau : K[\partial] \to R_A$ given by $P(\partial_1, \ldots, \partial_N) \to P(\mathbf{t}^{\mathbf{a}_1}, \ldots, \mathbf{t}^{\mathbf{a}_N})$.
First of all note that the kernel of τ consists precisely of the left ideal in $K[\partial]$ generated by the box operators $\Box_{\mathbf{l}}$, $\mathbf{l} \in L$.
Take the operator $D = \partial_1^{i_1} \cdots \partial_N^{i_N} \in K[\partial]$ and determine the image of $D \circ Z_i$ under τ.

$$\begin{aligned} D \circ Z_i &= \partial_1^{i_1} \cdots \partial_N^{i_N} \circ (a_{i1}v_1\partial_1 + \cdots + a_{iN}v_N\partial_N - \alpha_i) \\ &= (a_{i1}v_1\partial_1 + \cdots + a_{iN}v_N\partial_N)\partial_1^{i_1} \cdots \partial_N^{i_N} + \cdots \\ &\quad (i_1 a_{i1} + \cdots + i_N a_{iN})\partial_1^{i_1} \cdots \partial_N^{i_N} - \alpha_i \partial_1^{i_1} \cdots \partial_N^{i_N} \end{aligned}$$

Hence we get

$$\begin{aligned} \tau(D \circ Z_i) &= \left(-\alpha_i + t_i \frac{\partial}{\partial t_i} + \sum_{j=1}^{N} a_{ij} v_j \mathbf{t}^{\mathbf{a}_j}\right) \mathbf{t}^{i_1\mathbf{a}_1 + \cdots + i_N\mathbf{a}_N} \\ &= \left(-\alpha_i + t_i \frac{\partial}{\partial t_i} + f_i\right) \tau(D) \end{aligned}$$

By taking K-linear combinations we conclude that

$$\tau(P \circ Z_i) = \left(-\alpha_i + t_i \frac{\partial}{\partial t_i} + f_i\right) \tau(P)$$

for any $P \in K[\partial]$. Hence the Dwork submodule $D_{f,\alpha}$ is precisely the image of τ of the left ideal in $B[\partial]$ generated by $Z_1, \ldots, Z_r$. Together with the fact that the kernel of τ is the left ideal generated by the box-operators, we obtain

Proposition 7.4. — *The A-hypergeometric D-module is isomorphic to the Dwork module.*

8. Contiguity

Consider the system $H_A(\alpha)$,

$$\Box_{\mathbf{l}}\Phi = 0, \ \mathbf{l} \in L, \qquad Z_j \Phi = \alpha_j \Phi, \ j = 1, \ldots, r.$$

Apply the operator ∂_i from the left. We obtain,

$$\Box_{\mathbf{l}}\partial_i\Phi = 0, \ \mathbf{l} \in L, \qquad Z_j \partial_i \Phi = -a_{ji}\partial_i \Phi, \ j = 1, \ldots, r.$$

In other words, $F \mapsto \partial_i F$ maps the solution space of $H_A(\alpha)$ to the solution space of $H_A(\alpha - \mathbf{a}_i)$.
We can phrase this alternatively in terms of D-modules. Denote by $\mathcal{H}_A(\alpha)$ the left ideal in $K[\partial]$ generated by the hypergeometric operators $\Box_{\mathbf{l}}$ and Z_j. Then the map $P \mapsto \partial_i P$ gives a D-module homomorphism from $K[\partial]/\mathcal{H}_A(\alpha - \mathbf{a}_i)$ to $K[\partial]/\mathcal{H}_A(\alpha)$. We are interested in the cases when this is a D-module isomorphism.

Definition 8.1. — *The system $H_A(\alpha)$ is called non-resonant if $\alpha + \mathbb{Z}^r$ does not contain a point which is on one of the faces of $C(A)$, the positive real cone generated by the points of A.*

We then have

Theorem 8.2. — *When $H_A(\alpha)$ is non-resonant, the map $F \mapsto \partial_i F$ yields an isomorphism between the solution spaces of $H_A(\alpha)$ and $H_A(\alpha - \mathbf{a}_i)$.*

Proof. We basically follow Dwork's approach from [**14**]. It suffices to show that ∂_i does not have a kernel, as the dimension of the solution spaces are the same. To this end we will construct an operator $P \in K[\partial]$ (with $K = \mathbb{C}(\mathbf{v})$) such that $P\partial_i \equiv 1 (\mathrm{mod}\ \mathcal{H}_A(\alpha))$. In particular, $F \mapsto P(F)$ would be the inverse of ∂_i.
Suppose the positive cone $C(A)$ is given by a finite set of linear inequalities $l(\mathbf{x}) \geq 0,\ l \in \mathcal{F}$. Assume moreover that the linear forms l are integral valued on $\mathbb{Z}^r$ and normalise them so that the greatest common divisor of all values is 1. For any differential operator $\partial^{\mathbf{u}}$ we define the valuation $val_l(\partial^{\mathbf{u}}) = \sum_{j=1}^N u_j l(\mathbf{a}_j)$. More generally, for any differential operator $P \in K[\partial]$ we define $val_l(P)$ to be the minimal valuation of all terms in P.
Suppose $val_l(\partial^{\mathbf{u}}) \leq val_l(\partial^{\mathbf{w}})$ for every $l \in \mathcal{F}$. Hence $\sum_{j=1} l((w_j - u_j)\mathbf{a}_j) \geq 0$ for all $l \in \mathcal{F}$. So $\sum_{j=1}^N (w_j - u_j)\mathbf{a}_j$ is a lattice point in $C(A)$. By the assumption of saturatedness there exist non-negative integers w'_j such that $\sum_{j=1}^N w'_j \mathbf{a}_j = \sum_{j=1}^N (w_j - u_j)\mathbf{a}_j$. Hence $\partial^{\mathbf{w}}$ is equivalent modulo the box operator $\Box_{\mathbf{w}-\mathbf{w}'-\mathbf{u}}$ with $\partial^{\mathbf{w}'}\partial^{\mathbf{u}}$.
Let $l \in \mathcal{F}$ be given. We show that modulo the ideal $\mathcal{H}_A(\alpha)$, the operator $\partial^{\mathbf{u}}$ is equivalent to an operator P such that $val_l(P) > val_l(\partial^{\mathbf{u}})$ and $val_{l'}(P) \geq val_{l'}(\partial^{\mathbf{u}})$ for all $l' \in \mathcal{F},\ l' \neq l$. Let $Z_l = -l(\alpha) + \sum_{j=1}^N l(\mathbf{a}_j) v_j \partial_j$. Notice that $\partial^{\mathbf{u}} Z_l = Z_l \partial^{\mathbf{u}} + l(\mathbf{u})\partial^{\mathbf{u}}$. Hence,

$$\sum_{j=1}^N l(\mathbf{a}_j) v_j \partial_j \partial^{\mathbf{u}} \equiv l(\alpha - \mathbf{u})\partial^{\mathbf{u}} (\mathrm{mod}\ \mathcal{H}_A(\alpha)).$$

For each term on the left we have $l(\mathbf{a}_j) \neq 0 \Rightarrow val_l(\partial_j \partial^{\mathbf{u}}) > val_l(\partial^{\mathbf{u}})$. Since, by non-resonance, $l(\alpha - \mathbf{u}) \neq 0$ our assertion is proven. Choose $k_l \in \mathbb{Z}_{\geq 0}$ for every $l \in \mathcal{F}$. By repeated application of our principle we see that any monomial $\partial^{\mathbf{u}}$ is equivalent modulo $\mathcal{H}_A(\alpha)$ to an operator P with $val_l(P) \geq k_l + val_l(\partial^{\mathbf{u}})$ for all $l \in \mathcal{F}$.
In particular, there exists an operator P, equivalent to 1 and $val_l(P) \geq val_l(\partial_i)$ for every $l \in \mathcal{F}$. Then, P is equivalent to an operator $P'\partial_i$. Summarizing, $1 \equiv P'\partial_i (\mathrm{mod}\ \mathcal{H}_A(\alpha))$. So $F \mapsto \partial_i F$ is injective on the solution space of $H_A(\alpha)$. $\square$

Since the $\mathbb{Z}$-span of the points of A is $\mathbb{Z}^r$ itself, we see that in the case of non-resonance, any two systems $H_A(\alpha)$ and $H_A(\beta)$ with $\alpha \equiv \beta (\mathrm{mod}\ \mathbb{Z}^r)$ are equivalent. We call these systems *contiguous*. In particular, there exists a differential operator P such that $F \mapsto P(F)$ is an isomorphism from the solution space of $H_A(\alpha)$ to that of $H_A(\beta)$.

9. Irreducibility

A system of partial linear differential equations is said to be *irreducible* if the corresponding D-module over its base field has only itself and the trivial module as sub D-modules. In the case of A-hypergeometric equations life often becomes easier under the assumption of irreducibility of $H_A(\alpha)$. Moreover, irreducibility also holds generically for A-hypergeometric systems. Although there exist interesting examples of reducible A-hypergeometric systems, we tend to concentrate on the irreducible ones. In [**22**], Theorem 2.11 we find the following criterion.

Theorem 9.1 **(GKZ)**. — *Any non-resonant A-hypergeometric system is irreducible.*

The proof is based on the analysis of perverse sheafs. In [**10**] we present a proof which is more elementary. The converse statement is almost true. We have to make the following assumption.
iv) For every $i \in \{1, 2, \ldots, N\}$ there exists $\mathbf{l} \in L$ such that $l_i \neq 0$.
An equivalent formulation, in terms of A, would be
iv') For every $i \in \{1, 2, \ldots, N\}$ the set A minus $\mathbf{a}_i$ has rank r.
To see what this condition implies for the hypergeometric system, suppose that there exists an index i such that $l_i = 0$ for all $\mathbf{l} \in L$. To fix ideas, assume that $i = N$. Then the derivation ∂_N will not occur in the box equations (A1). In that case it is clear that the set $\mathbf{a}_1, \ldots, \mathbf{a}_{N-1}$ has rank $r-1$. Choose a basis of $\mathbb{Z}^r$ such that $\mathbf{a}_1, \ldots, \mathbf{a}_{N-1}$ is in the space spanned by the basis vectors $\mathbf{e}_1, \ldots, \mathbf{e}_{r-1}$ and $\mathbf{e}_r = \mathbf{a}_N$. Write $A' = \{\mathbf{a}_1, \ldots, \mathbf{a}_{N-1}\}$. Let $\alpha = (\beta_1, \ldots, \beta_r)$ with respect to these new coordinates. We easily verify that the set of box equation is simply the set corresponding to A' and the set of homogeneity equations corresponds to those for A' and the parameters $(\beta_1, \ldots, \beta_{r-1})$ and the equation $v_N \partial_N \Phi = \beta_r \Phi$.

Theorem 9.2. — *Consider the resonant system $H_A(\alpha)$ and assume that condition (iv) holds. Then $H_A(\alpha)$ is reducible.*

Proof. Unfortunately we can give only part of the proof here. A complete proof can be found in [**10**]. The incompleteness of our presentation consists of the statement that for every index i the operator $F \mapsto \partial_i F$ is not the trivial operator on the solution space of $H_A(\alpha)$.
We now continue with the remainder of the proof. Since $F \mapsto \partial_i F$ is an isomorphism of solution spaces of $H_A(\alpha)$ and $H_A(\alpha - \mathbf{a}_i)$ and $\mathbb{Z}A = \mathbb{Z}^r$ we see that $H_A(\beta)$ is irreducible for any $\beta \in \mathbb{R}^r$ with $\beta \equiv \alpha (\mathrm{mod}\ \mathbb{Z}^r)$. Since the system is resonant there exists such a β in a face F of $C(A)$. Suppose $A \cap F = \{\mathbf{a}_1, \ldots, \mathbf{a}_t\}$. We assert that there exist non-trivial solutions of the form $f = f(v_1, \ldots, v_t)$. Suppose that $s = \mathrm{rank}(\mathbf{a}_1, \ldots, \mathbf{a}_t)$. By an $SL(r, \mathbb{Z})$ change of coordinates we can see to it that F is given by $x_{s+1} = \cdots = x_r = 0$. Then the coordinate a_{rj} of $\mathbf{a}_j$ is zero for $i = s+1, \ldots, r$ and $j = 1, \ldots, t$. Also, $\beta_{s+1} = \cdots = \beta_r = 0$. A solution $f = f(v_1, \ldots, v_t)$ satisfies the homogeneity equations

$$\left(-\beta_i + \sum_{j=1}^{t} a_{ij} v_j \partial_j \right) f = 0, \ i = 1, \ldots, s.$$

Notice that the homogeneity equation with $i = s+1, \dots, r$ are trivial.
Consider the box-operator $\Box_\lambda$ with $\lambda \in L$. Write $\lambda = (\lambda_1, \dots, \lambda_N)$. The positive support is the set of indices i where $\lambda_i > 0$, the negative support is the set of indices i where $\lambda_i < 0$.
Suppose the positive support is contained in $1, 2, \dots, t$. Then $\sum_{\lambda_i > 0} \lambda_i \mathbf{a}_i$ is in $\mathcal{F}$. Hence $-\sum_{\lambda_i<0} \lambda_i \mathbf{a}_i$ is also in F. Since F is a face, all non-zero terms of the latter have index $\le t$. So the negative support is also in $1, 2, \dots, t$. Hence

$$\text{negative support} \subset \{1, \dots, t\} \iff \text{positive support} \subset \{1, \dots, t\}.$$

If the positive and negative support of λ contain indices $> t$ then $f(v_1, \dots, v_t)$ satisfies $\Box_\lambda f = 0$ trivially.
Define a new set $\tilde{A} = \{\tilde{\mathbf{a}}_1, \dots, \tilde{\mathbf{a}}_t\} \subset \mathbb{Z}^s$ where $\tilde{\mathbf{a}}_j$ is the projection of $\mathbf{a}_j$ on its first s coordinates. Define a new parameter $\tilde{\beta}$ similarly. The solutions of the form $f(v_1, \dots, v_t)$ of the original GKZ-system satisfy the new GKZ-system corresponding to $H_{\tilde{A}}(\tilde{\beta})$. They all satisfy the additional equations $\partial_i F = 0$ for $i > t$, so they form a proper subspace. Hence the system is reducible, contradicting our initial assumption of irreducibility. $\Box$

10. Formal solutions

Choose a point $\gamma = (\gamma_1, \dots, \gamma_N) \in \mathbb{R}^N$ such that $\alpha = \gamma_1 \mathbf{a}_1 + \cdots + \gamma_N \mathbf{a}_N$. Then a formal solution of the A-hypergeometric system can be given by

$$\Phi_{L,\gamma}(v_1, \dots, v_N) = \sum_{\mathbf{l} \in L} \frac{\mathbf{v}^{\mathbf{l}+\gamma}}{\Gamma(\mathbf{l} + \gamma + \mathbf{1})}$$

where we use the short-hand notation

$$\frac{\mathbf{v}^{\mathbf{l}+\gamma}}{\Gamma(\mathbf{l}+\gamma+\mathbf{1})} = \frac{v_1^{l_1+\gamma_1} \cdots v_N^{l_N+\gamma_N}}{\Gamma(l_1+\gamma_1+1) \cdots \Gamma(l_N+\gamma_N+1)}.$$

The reader easily verifies that $\Phi_{L,\gamma}$ indeed satisfies the system of A-hypergeometric equations. In general these solutions are formal, there is no convergence. However, there are exceptions which turn out to be general enough. To see this, notice that the choice of the parameters γ is only determined up elements of $L(\mathbb{R})$. So there is some freedom there. By a proper choice of γ this formal solution gives rise to actual power series solutions with a non-trivial region of convergence.

11. Gauss hypergeometric function

Consider the set $A \subset \mathbb{Z}^3$ given by the A-matrix

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

and the parameter triple $(-a, -b, c-1)$.
The lattice of relations L is generated by $(-1, -1, 1, 1)$. Choose $\gamma = (-a, -b, c-1, 0)$. Formal solution :

$$\Phi = \sum_{n \in \mathbb{Z}} \frac{v_1^{-n-a} v_2^{-n-b} v_3^{n+c-1} v_4^n}{\Gamma(-n-a+1)\Gamma(-n-b+1)\Gamma(n+c)\Gamma(n+1)}$$

Notice that $n \geq 0$ because $1/\Gamma(n+1) = 0$ whenever n is a negative integer. Application of Euler's standard identity $\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)$ yields

$$\Phi \sim v_1^{-a} v_2^{-b} v_3^{c-1} \sum_{n \geq 0} \frac{\Gamma(n+a)\Gamma(n+b)}{\Gamma(n+c)\Gamma(n+1)} \left(\frac{v_3 v_4}{v_1 v_2}\right)^n$$

This is proportional to ${}_2F_1\left(\begin{matrix} a\ b \\ c \end{matrix}\middle| z\right)$, when we put $v_1 = v_2 = v_3 = 1, v_4 = z$.
The polytope $Q(A)$ is a square and the cone is given by the inequalities $x_1 \geq 0, x_2 \geq 0, x_1 + x_3 \geq 0, x_2 + x_3 \geq 0$ and the faces given by $x_1 = 0, x_2 = 0, x_1 + x_3 = 0, x_2 + x_3 = 0$. So the non-resonance conditions read $-a, -b, c-a, c-b \notin \mathbb{Z}$. Note that these are precisely the well known irreducibility conditions for the Gauss hypergeometric equation.

12. Appell F_1

We reverse the procedure by starting from the series expansion and then deduce the data A and α. Appell $F_1(a, b, b', c|x, y)$ is proportional to

$$\sum_{m,n \geq 0} \frac{\Gamma(m+n+a)\Gamma(m+b)\Gamma(n+b')}{\Gamma(m+n+c)\Gamma(m+1)\Gamma(n+1)}\, x^m y^n.$$

Application of Γ-identities gives, up to a constant factor,

$$\sum_{m,n \geq 0} \frac{x^m y^n}{\Gamma(-m-n+1-a)\Gamma(-m+1-b)\Gamma(-n+1-b')\Gamma(m+n+c)\Gamma(m+1)\Gamma(n+1)}$$

The lattice L has the form $(-m-n, -m, -n, m+n, m, n)$ with $m, n \in \mathbb{Z}$ and so is spanned by

$$(-1,-1,0,1,1,0) \quad \text{and} \quad (-1,0,-1,1,0,1).$$

A corresponding set A can be given by the A-matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 & -1 \end{pmatrix}$$

The parameter vector can also be read off, $-a\mathbf{a}_1 - b\mathbf{a}_2 - b'\mathbf{a}_3 + (c-1)\mathbf{a}_4 = (-a, -b, -b', c-1)$.
Notice that our Appell series is precisely the series which we get from

$$\sum_{m,n \geq 0} \frac{v_1^{-m-n-a} v_2^{-m-b} v_3^{-n-b'} v_4^{m+n+c} v_5^m v_6^n}{\Gamma(-m-n+1-a)\Gamma(-m+1-b)\Gamma(-n+1-b')\Gamma(m+n+c)\Gamma(m+1)\Gamma(n+1)}$$

if we set $v_1 = v_2 = v_3 = v_4 = 1$ and $v_5 = x, v_6 = y$.

Denote $\partial_i = \frac{\partial}{\partial v_i}$. The GKZ-equations read

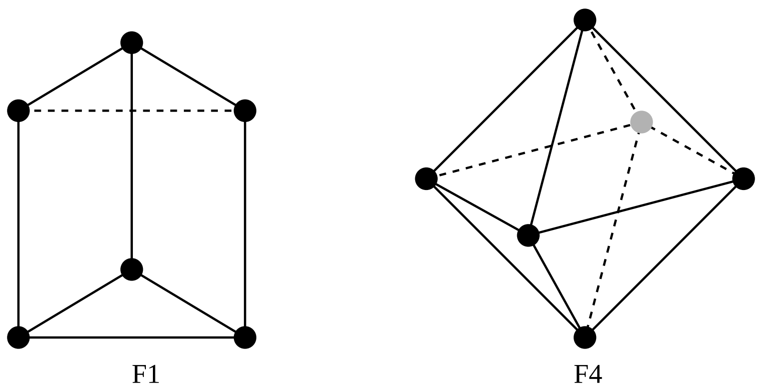
$$\partial_1\partial_2\Phi - \partial_4\partial_5\Phi = 0, \quad \partial_1\partial_3\Phi - \partial_4\partial_6\Phi = 0$$

$$\begin{array}{rcl} (v_1\partial_1 + v_5\partial_5 + v_6\partial_6 + a)\Phi & = & 0 \\ (v_2\partial_2 + v_5\partial_5 + b)\Phi & = & 0 \\ (v_3\partial_3 + v_6\partial_6 + b')\Phi & = & 0 \\ (v_4\partial_4 - v_5\partial_5 - v_6\partial_6 + 1 - c)\Phi & = & 0 \end{array}$$

Let $\mathcal{Z}$ be the left ideal in $K[\partial_1, \ldots, \partial_6]$ generated by the operators $Z_1, \ldots, Z_4$ (here $K = \mathbb{C}(v_1, \ldots, v_6)$). Consider for each box operator $\Box$ the intersection of the class $\Box(\text{mod } \mathcal{Z})$ with $K[\partial_5, \partial_6]$ and set $v_1 = \cdots = v_4 = 1, v_5 = x, v_6 = y$. We obtain the classical differential equations

$$\begin{array}{rcl} x(1-x)F_{xx} + y(1-x)F_{xy} + (c-(a+b+1)x)F_x - byF_y - abF & = & 0 \\ y(1-y)F_{yy} + x(1-y)F_{xy} + (c-(a+b'+1)y)F_y - b'xF_x - ab'F & = & 0 \end{array}$$

Here we display a picture of the A-polytopes corresponding to F_1 and F_4,



The polytope $Q(A)$ is actually a triangular prism given by the inequalities

$$x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,\ x_2 + x_3 + x_4 \geq 0,\ x_1 + x_4 \geq 0.$$

From this we see that the non-resonance conditions read $a, b, b', c-b-b', c-a \notin \mathbb{Z}$. These are the irreducibility conditions for Appell's F_1.

13. Horn G_3

Horn's $G_3(a, b, x, y)$ is proportional to

$$\sum_{m,n\geq 0} \frac{\Gamma(2m-n+a)\Gamma(2n-m+b)}{\Gamma(m+1)\Gamma(n+1)}\, x^m y^n.$$

Using Γ-identities gives us, up to a constant factor,

$$\sum_{m,n\geq 0} \frac{(-1)^{m+n} x^m y^n}{\Gamma(-2m+n+1-a)\Gamma(-2n+m+1-b)\Gamma(m+1)\Gamma(n+1)}.$$

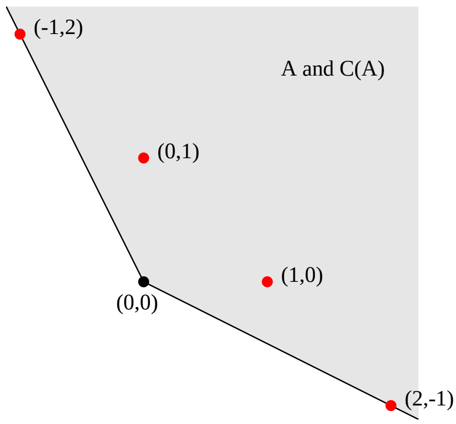
The lattice L is spanned by

$$(-2,1,1,0), \qquad (1,-2,0,1).$$

A set A is given by the A-matrix

$$\begin{pmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & -1 & 2 \end{pmatrix}$$

The parameters read $\alpha = (-a, -b)$. Here is the cone $C(A)$ associated to G_3 together with the points of A.



Note that the power series for G_3 arises from

$$\sum_{m,n\geq 0} \frac{v_1^{-2m+n-a} v_2^{-2n+m-b} v_3^m v_4^n}{\Gamma(-2m+n+1-a)\Gamma(-2n+m+1-b)\Gamma(m+1)\Gamma(n+1)}$$

by setting $v_1 = v_2 = 1$ and $v_3 = -x, v_4 = -y$.
It is an exercise to show that all box operators $\Box_l$ are contained in the ideal in $\mathbb{C}[\partial_1, \ldots, \partial_4]$ generated by $\partial_1^2 - \partial_2\partial_3, \partial_2^2 - \partial_1\partial_4, \partial_1\partial_2 - \partial_3\partial_4$. So the GKZ-equations read

$$\partial_1^2\Phi - \partial_2\partial_3\Phi = 0, \ \partial_2^2\Phi - \partial_1\partial_4\Phi = 0, \ \partial_1\partial_2\Phi - \partial_3\partial_4\Phi = 0$$

$$\begin{aligned} (v_1\partial_1 + 2v_3\partial_3 - v_4\partial_4 + a)\Phi &= 0 \\ (v_2\partial_2 - v_3\partial_3 + 2v_4\partial_4 + b)\Phi &= 0 \end{aligned}$$

Let again $\mathcal{Z}$ be the left ideal in $K[\partial_1, \ldots, \partial_4]$ generated by the operators Z_1, Z_2 (here $K = \mathbb{C}(v_1, \ldots, v_4)$). Consider for each box operator $\Box$ the intersection of the class $\Box(\text{mod } \mathcal{Z})$ with $K[\partial_3, \partial_4]$ and set $v_1 = v_2 = 1, v_3 = -x, v_4 = -y$. We obtain the differential equations

$$\begin{aligned} x(4x+1)F_{xx}-(4x+2)yF_{xy}+y^2F_{yy}+((4a+6)x+1-b)F_x & \\ -2ayF_y+a(a+1)F &= 0\\ x^2F_{xx}-(4y+2)xF_{xy}+y(4y+1)F_{yy}+((4b+6)y+1-a)F_y & \\ -2bxF_x+b(b+1)F &= 0\\ 2x^2F_{xx}+(1-5xy)F_{xy}+2y^2F_{yy}+(2-2b+a)xF_x & \\ +(2-2a+b)yF_y-abF &= 0 \end{aligned}$$

In some of the older literature only the first two equations were mentioned. This system of two equations allows for the spurious solution $x^\rho y^\sigma$ where $\rho = -(2a+b)/3, \sigma = -(2b+a)/3$ (see [**16**]). This monomial doesn't satisfy the third equation however (I learnt this from Alicia Dickenstein, see also [**12**] and [**13**]). So, in this sense the A-hypergeometric equations are more elegant.
Finally, note that $C(A)$ is given by the inequalities $x_1+2x_2 \ge 0, 2x_1+x_2 \ge 0$. The non-resonance conditions read $a+2b, 2a+b \notin \mathbb{Z}$.

14. Power series solutions

Consider the system $H_A(\alpha)$ and a formal solution

$$\Phi = \sum_{\mathbf{l}\in L} \frac{v_1^{l_1+\gamma_1}\cdots v_N^{l_N+\gamma_N}}{\Gamma(l_1+\gamma_1+1)\cdots\Gamma(l_N+\gamma_N+1)}.$$

Choose a subset $\mathcal{I} \subset \{1,2,\ldots,N\}$ with $|\mathcal{I}| = N-r$ such that $\mathbf{a}_i$ with $i \notin \mathcal{I}$ are linearly independent.

Proposition 14.1. — *Define $\pi_{\mathcal{I}} : L \to \mathbb{Z}^{N-r}$ by $\mathbf{l} \mapsto (l_i)_{i\in\mathcal{I}}$. Then $\pi_{\mathcal{I}}$ is injective and its image is a sublattice of $\mathbb{Z}^{N-r}$ of index $|\det(\mathbf{a}_i)_{i\notin\mathcal{I}}|$.*

Proof. Let $\mathbf{b}_1,\ldots,\mathbf{b}_{N-r}$ be a $\mathbb{Z}$-basis of L. Denote $\mathbf{b}_i = (b_{i1},\ldots,b_{iN})$. We must prove that

$$\det(b_{ij})_{i=1,\ldots,N-r;j\in\mathcal{I}} = \pm\det(\mathbf{a}_i)_{i\notin\mathcal{I}}.$$

This follows from the following Lemma with $J = \mathcal{I}^c$ and the fact that the gcd of all $(N-r)\times(N-r)$ subdeterminants of (b_{ij}) is one and that the gcd of all $r\times r$-subdeterminants of (a_{ij}) is one.

□

Lemma 14.2. — *Let $\mathbf{w}_1,\ldots,\mathbf{w}_{N-r}$ and $\mathbf{u}_1,\ldots,\mathbf{u}_r$ be two sets of vectors in $\mathbb{R}^N$ such that $\mathbf{w}_i\cdot\mathbf{u}_j = 0$ for all i,j and such that the $\mathbf{w}_i, \mathbf{u}_j$ span $\mathbb{R}^N$. We denote the coordinates of $\mathbf{w}_i, \mathbf{u}_j$ by w_{ik} and u_{jk} respectively. Then there exists a number $c \ne 0$ such that for any $J \subset \{1,\ldots,N\}$ with $|J| = r$ we have*

$$\det(w_{ij})_{i=1,\ldots,N-r,j\in J^c} = \pm c\det(u_{ij})_{i=1,\ldots,r,j\in J}.$$

Proof. Consider two linear forms H_1, H_2 on $\wedge^r\mathbb{R}^N$ given by

$$H_1 : \mathbf{x}_1\wedge\cdots\wedge\mathbf{x}_r \mapsto \det(\mathbf{w}_1,\ldots,w_{N-r},\mathbf{x}_1,\ldots,\mathbf{x}_r)$$

and

$$H_2 : \mathbf{x}_1\wedge\cdots\wedge\mathbf{x}_r \mapsto \det(\mathbf{u}_i\cdot\mathbf{x}_j)_{i,j=1,\ldots,r}.$$

For any $\mathbf{w}$ in the span of $\mathbf{w}_1, \ldots, \mathbf{w}_{N-r}$ and any $j \in \{1, \ldots, r\}$ we have $H_1(\mathbf{x}_1 \wedge \ldots \wedge (\mathbf{x}_j + \mathbf{w}) \wedge \ldots \wedge \mathbf{x}_r) = H_1(\mathbf{x}_1 \wedge \ldots \wedge \mathbf{x}_r)$ and similarly for H_2. So we see that H_1, H_2 are uniquely determined by their value in $\mathbf{u}_1 \wedge \ldots \wedge \mathbf{u}_r$. More precisely,

$$\frac{H_1(\mathbf{x}_1, \ldots, \mathbf{x}_r)}{\det(\mathbf{w}_1, \ldots, \mathbf{w}_{N-r}, \mathbf{u}_1, \ldots, \mathbf{u}_r)} = \frac{H_2(\mathbf{x}_1, \ldots, \mathbf{x}_r)}{\det(\mathbf{u}_i \cdot \mathbf{u}_j)_{i,j=1,\ldots,r}}.$$

Define

$$c = \frac{\det(\mathbf{w}_1, \ldots, \mathbf{w}_{N-r}, \mathbf{u}_1, \ldots, \mathbf{u}_r)}{\det(\mathbf{u}_i \cdot \mathbf{u}_j)_{i,j=1,\ldots,r}}$$

and evaluate at $\mathbf{e}_{j_1} \wedge \cdots \wedge \mathbf{e}_{j_r}$ where $J = \{j_1, \ldots, j_r\}$ to obtain our assertion. Here $\mathbf{e}_1, \ldots, \mathbf{e}_N$ is the standard basis of $\mathbb{R}^N$.

□

We denote $\Delta_{\mathcal{I}} = |\det(\mathbf{a}_i)_{i \notin \mathcal{I}}|$. Choose γ such that $\gamma_i \in \mathbb{Z}$ for $i \in \mathcal{I}$. The formal solution series

$$\Phi = \sum_{\mathbf{l} \in L} \prod_{i \in \mathcal{I}} \frac{v_i^{l_i + \gamma_i}}{\Gamma(l_i + \gamma_i + 1)} \prod_{i \notin \mathcal{I}} \frac{v_i^{l_i + \gamma_i}}{\Gamma(l_i + \gamma_i + 1)}$$

is now a power series because the summation runs over the polyhedron $l_i + \gamma_i \ge 0$ for $i \in \mathcal{I}$ and the other l_j are dependent on $l_i, i \in \mathcal{I}$. By abuse of language we will call the corresponding simplicial cone $l_i \ge 0$ for $i \in \mathcal{I}$ the *sector of summation* with index $\mathcal{I}$.
Put $m_i = l_i + \gamma_i$ for $i \in \mathcal{I}$ (remember $\gamma_i \in \mathbb{Z}$ for all $i \in \mathcal{I}$). Then all $l_i + \gamma_i$ are linear functions in $\mathbf{m} = (m_i)_{i \in \mathcal{I}}$, possibly with rational coefficients. Denote them by $d_i(\mathbf{m}) + \beta_i$ for all i. Consider

$$\sum_{\mathbf{m} \ge \mathbf{0}} \prod_{i \in \mathcal{I}} \frac{v_i^{m_i}}{m_i!} \prod_{i \notin \mathcal{I}} \frac{v_i^{d_i(\mathbf{m}) + \beta_i}}{\Gamma(d_i(\mathbf{m}) + \beta_i + 1)}.$$

Note that this is a Horn series, in the sense that now the linear forms d_i may have rational coefficients instead of integer coefficients. As $\mathbf{m}$ runs over $\mathbb{Z}^{N-r}$ the vector

$$(d_1(\mathbf{m}), \cdots, d_N(\mathbf{m}))$$

runs over a lattice containing L of index $\Delta_{\mathcal{I}}$. Therefore the power series above in fact represents $\Delta_{\mathcal{I}}$ independent solutions of the GKZ-system.
There is one important assumption we need in order to make this approach work. Namely the guarantee that none of the arguments $d_i(\mathbf{m}) + \beta_i$ with $i \notin \mathcal{I}$ is a negative integer. The best way to do is to impose the condition $\beta_i \notin \mathbb{Z}$ for $i \notin \mathcal{I}$. Geometrically, since $\alpha = \sum_{i \notin \mathcal{I}} \beta_i \mathbf{a}_i$, this condition comes down to the requirement that $\alpha + \mathbb{Z}^r$ does not contain points in a face of the simplex spanned by $\mathbf{a}_i$ with $i \notin \mathcal{I}$. This is slightly stronger than the requirement of non-resonance, as faces of individual simplices are involved. We shall come back to this requirement in the Definition 19.2 of T-nonresonance.

15. An example, G_3 again

Consider the set A given by the columns of

$$A = \begin{pmatrix} -1 & 0 & 1 & 2 \\ 2 & 1 & 0 & -1 \end{pmatrix}$$

and parameter vector $(-a,-b)^t$. The lattice of relations is generated by

$$(1,-2,1,0), \quad (0,1,-2,1)$$

The formal solution series reads

$$\sum_{p,q} \frac{v_1^{p+\gamma_1} v_2^{-2p+q+\gamma_2} v_3^{p-2q+\gamma_3} v_4^{q+\gamma_4}}{\Gamma(p+\gamma_1+1)\Gamma(-2p+q+\gamma_2+1)\Gamma(p-2q+\gamma_3+1)\Gamma(q+\gamma_4+1)}$$

where $A(\gamma_1,\ldots,\gamma_4)^t = (-a,-b)^t$. Let us take $\mathcal{I} = \{2,3\}$ and notice that

$$\Delta_{\mathcal{I}} = |\det(\mathbf{a}_1, \mathbf{a}_4)| = 3.$$

Choose $\gamma_2, \gamma_3 \in \mathbb{Z}$ and put $m = -2p+q+\gamma_2, n = p-2q+\gamma_3$. Then

$$p+\gamma_1 = -(2m+n)/3 - (a+2b)/3, \quad q+\gamma_4 = -(2n+m)/3 - (2a+b)/3.$$

The Horn type solution reads

$$v_1^{-(a+2b)/3} v_4^{-(2a+b)/3} \sum_{m,n\ge 0} \frac{(v_1^{-2}v_2^3 v_4^{-1})^{m/3} (v_1^{-1}v_3^3 v_4^{-2})^{n/3}}{\Gamma(-(2m+n+a+2b)/3)\Gamma(-(m+2n+2a+b)/3)m!n!}$$

Splitting this sum over the three residue classes of $m+2n$ modulo 3 gives us 3 independent solutions.

16. The domain of convergence

Choose a summation sector with index $\mathcal{I}$. The domain of convergence of

$$\Phi_{\mathcal{I}} = \sum_{\mathbf{l}\in L} \prod_{i=1}^{N} \frac{v_i^{l_i+\gamma_i}}{\Gamma(l_i+\gamma_i+1)}$$

is in general the complement of an amoeba-like domain. We simplify by considering the complement of its skeleton. Any N-tuple $\rho_1,\ldots,\rho_N \in \mathbb{R}$ is called a *convergence direction* of Φ if there exists $\epsilon > 0$ such that Φ converges in the region

$$|v_1| = t^{\sigma_1}, \ldots, |v_N| = t^{\sigma_N}$$

for all σ_i with $\max_i |\sigma_i - \rho_i| < \epsilon$ and sufficiently small $t \in \mathbb{R}_{>0}$.
The subtlety with the ϵ is there to ensure that the convergence directions form an open set, so to exclude possible convergence directions which are on the boundary of domains of convergence.

Definition 16.1. — *The union of all convergence directions is called the convergence domain of* $\Phi_{\mathcal{I}}$.

Proposition 16.2. — *The vector* $(\rho_1,\ldots,\rho_N) \in \mathbb{R}^N$ *is a convergence direction with respect to* $\mathcal{I}$ *if and only if* $\rho_1 l_1 + \cdots + \rho_N l_N > 0$ *for every* $\mathbf{l} \neq \mathbf{0}$ *in the summation sector with index* $\mathcal{I}$.

For the proof of this Proposition we require two Lemmas on the growth behaviour of the Γ-function.

Lemma 16.3. — *Let $s \in \mathbb{C}$ and $n \in \mathbb{Z}_{\geq 0}$. Then there exists constants $c_1, c_2 > 0$ such that*

$$\frac{1}{\Gamma(s+n)} \leq c_1 \frac{|n^{-(s-1)}|}{n!}$$

and

$$\frac{1}{\Gamma(s-n)} \leq c_2 |n^s| n!$$

Proof. When $s \in \mathbb{Z}_{\leq 0}$ there are only finitely many non-zero values of $1/\Gamma(s+n)$. Hence the first statement of our Lemma is trivial. Now assume that $s \notin \mathbb{Z}_{\leq 0}$. Then

$$\Gamma(s+n) = n! \prod_{k=1}^{n} \left(1 + \frac{s-1}{k}\right) \Gamma(s).$$

It is an exercise to show that there exists a constant $c_s > 0$ such that

$$\lim_{n\to\infty} |n^{-(s-1)}| \prod_{k=1}^{n} \left(1 + \frac{s-1}{k}\right) = c_s$$

Since all terms in this limit are non-zero we conclude that there exists $c'_s > 0$ such that

$$|n^{-(s-1)}| \prod_{k=1}^{n} \left|1 + \frac{s-1}{k}\right| \geq c'_s$$

for all $n \in \mathbb{Z}_{\geq 0}$. The first statement of our Lemma follows directly.
When $s \in \mathbb{Z}_{\geq 0}$ the second statement is again trivial. So we assume $s \notin \mathbb{Z}_{\geq 0}$. Then

$$\Gamma(s-n) = \frac{1}{n!} \prod_{k=1}^{n} \left(-1 + \frac{s}{k}\right)^{-1} \Gamma(s).$$

Again, there exists a constant $c_s > 0$ such that

$$\lim_{n\to\infty} |n^{-s}| \prod_{k=1}^{n} \left|1 - \frac{s}{k}\right| = c_s.$$

Hence there exists $c'_s > 0$ such that

$$|n^s| \prod_{k=1}^{n} \left|1 - \frac{s}{k}\right|^{-1} \geq c'_s$$

for all $n \in \mathbb{Z}_{\geq 0}$. Our second inequality now follows directly.

□

Lemma 16.4. — *Denote $||\mathbf{l}|| = |l_1| + \cdots + |l_N|$ for any $\mathbf{l} \in \mathbb{Z}^N$. Then, for any $\mathbf{l} \in \mathbb{Z}^N$ with $l_1 + l_2 + \cdots + l_N = 0$,*

$$\frac{\prod_{l_i<0} |l_i|!}{\prod_{l_i>0} l_i!} \leq N^{||\mathbf{l}||/2}.$$

Proof. Note that $\sum_{l_i>0} |l_i| = \sum_{l_i<0} |l_i| = ||\mathbf{l}||/2$. The quotient can now be estimated by

$$\frac{(||\mathbf{l}||/2)!}{\prod_{l_i>0} |l_i|!} \frac{\prod_{l_i<0} |l_i|}{(||\mathbf{l}||/2)!} \le \frac{(||\mathbf{l}||/2)!}{\prod_{l_i>0} |l_i|!}.$$

From the multinomial expansion of $N^{||\mathbf{l}||/2} = (1 + \cdots + 1)^{||\mathbf{l}||/2}$ we now deduce the desired estimate $N^{||\mathbf{l}||/2}$.

□

Proof of Proposition 16.2. Consider the series solution

$$\Phi_{\mathcal{I}} = \sum_{\mathbf{l}\in L} \prod_{i=1}^{N} \frac{v_i^{l_i+\gamma_i}}{\Gamma(l_i + \gamma_i + 1)}$$

which is summed over all $\mathbf{l} \in L$ with $l_i \ge 0$ for all $i \in \mathcal{I}$. Choose $v_1, \ldots, v_N$ such that $|v_i| = t^{\rho_i}$ for $i = 1, \ldots, N$. Using the previous Lemmas each term can be estimated by

$$\prod_{i=1}^{N} \frac{|v_i^{l_i+\gamma_i}|}{|\Gamma(l_i + \gamma_i + 1)|} \le c_1 \cdot t^{\rho_1 l_1+\cdots+\rho_N l_N} ||\mathbf{l}||^{c_2} N^{||\mathbf{l}||/2}$$

where c_1, c_2 are suitable constants.

Suppose $\rho_1 l_1 + \cdots + \rho_n l_N > 0$ for every $\mathbf{l} \ne \mathbf{0}$ in the summation sector. Then there exists a constant $\sigma > 0$ such that $\rho_1 l_1 + \cdots + \rho_N l_N > \sigma||\mathbf{l}||$. Our estimate becomes

$$c_1 \cdot t^{\sigma||\mathbf{l}||} ||\mathbf{l}||^{c_2} N^{||\mathbf{l}||/2}.$$

We now see that the series converges for sufficiently small t.

If $\rho_1 l_1 + \cdots + \rho_N l_N \le 0$ for some $\mathbf{l} \ne \mathbf{0}$, then there exists a neighbouring direction such that strict inequality holds. Then the summation contains infinitely many terms where $t^{\rho_1 l_1+\cdots+l_N\rho_N}$ is exponentially increasing for any $t < 1$. Since the coefficients are exponentially bounded from below we see that the series does not converge. Hence $(\rho_1, \ldots, \rho_N)$ is not a convergence direction.

□

17. Explicit bases of solutions

Let $\rho = (\rho_1, \ldots, \rho_N) \in \mathbb{R}^N$ and suppose the form $\rho_1 l_1 + \cdots + \rho_N l_N$ is non-trivial on L. We like to write down a basis of power series solutions of $H_A(\alpha)$ which converge for $v_1 = t^{\rho_1}, \ldots, v_N = t^{\rho_N}$ when $|t|$ is sufficiently small.

Let $\mathcal{I}$ be an index set of a summation sector. Then ρ is a convergence direction for this sector if $\rho_1 l_1 + \cdots + \rho_N l_N > 0$ for all non-zero $(l_1, \ldots, l_N) \in L$ with $l_i \ge 0$ whenever $i \in \mathcal{I}$. In that case we say that ρ is a convergence direction for $\mathcal{I}$. Clearly, if ρ is a convergence direction for $\mathcal{I}$ then the same holds for $\rho + \mathbf{y}$ where $\mathbf{y}$ is any element in $\mathbb{R}^N$ perpendicular to $L(\mathbb{R})$. A basis of $L^\perp$ is given by the rows of the A-matrix consisting of the columns $\mathbf{a}_1, \ldots, \mathbf{a}_N$. So any convergence direction for $\mathcal{I}$ can be changed modulo linear combinations of the rows of the A-matrix. Put differently, if $(\rho_1, \ldots, \rho_N)$ is a convergence direction for $\mathcal{I}$, then $(\rho_1 - m(\mathbf{a}_1), \ldots, \rho_N - m(\mathbf{a}_N))$ is also a direction of convergence for $\mathcal{I}$ for any linear form m on $\mathbb{R}^r$.

Let us choose the linear form m such that $m(\mathbf{a}_i) = \rho_i$ for all $i \notin \mathcal{I}$ and write $\rho'_i = \rho_i - m(\mathbf{a}_i)$ for $i = 1, \ldots, N$. Then $\rho'_i = 0$ for all $i \notin \mathcal{I}$. Furthermore, $\rho'_1 l_1 + \cdots + \rho'_N l_N > 0$ for all $\mathbf{l} \in L \setminus \mathbf{0}$ such that $l_i \geq 0$ whenever $i \in \mathcal{I}$. Hence $\rho'_i > 0$ for all $i \in \mathcal{I}$. Thus we arrive at the following Proposition.

Proposition 17.1. — *Let $\rho = (\rho_1, \ldots, \rho_N) \in \mathbb{R}^N$ and $\mathcal{I}$ the index of a summation sector. Let m be the linear form on $\mathbb{R}^r$ such that $m(\mathbf{a}_i) = \rho_i$ for all $i \notin \mathcal{I}$. Then ρ is a convergence direction for I if and only if $m(\mathbf{a}_i) < \rho_i$ for all $i \in \mathcal{I}$.*

18. Triangulations

Let A be as always and $Q(A)$ the convex hull of A. For any subset J of $\{1, 2, \ldots, N\}$ we denote $A_J = \{\mathbf{a}_j | j \in J\}$ and by $Q(A_J)$ its convex hull. When $|J| = r$ and $Q(A_J)$ is an $(r-1)$-simplex we often refer to the set J as an $(r-1)$-simplex as well.

Definition 18.1. — *A triangulation of A is a subset*

$$T \subset \{J \subset \{1, 2, \ldots, N\} |\ |J| = r \text{ and } \mathrm{rank}(A_J) = r\}$$

such that

$$Q(A) = \cup_{J \in T} Q(A_J)$$

and for all $J, J' \in T$

$$Q(A_J) \cap Q(A_{J'}) = Q(A_{J \cap J'}).$$

By $\mathcal{T}(A)$ we denote the set of all triangulations of A.
With a triangulation one associates a vector

$$\chi_T = \sum_{J \in T} \mathrm{Vol}(Q(A_J)) \sum_{j \in J} (\mathbf{e}_j)$$

in $\mathbb{R}^N$.

Definition 18.2. — *The secondary polytope $\Sigma(A)$ associated with A is defined by*

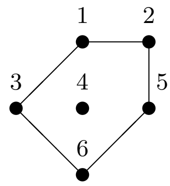
$$\Sigma(A) = \text{convex hull of } \{\chi_T | T \in \mathcal{T}(A)\}.$$

A triangulation T is called regular if χ_T is a vertex of $\Sigma(A)$.

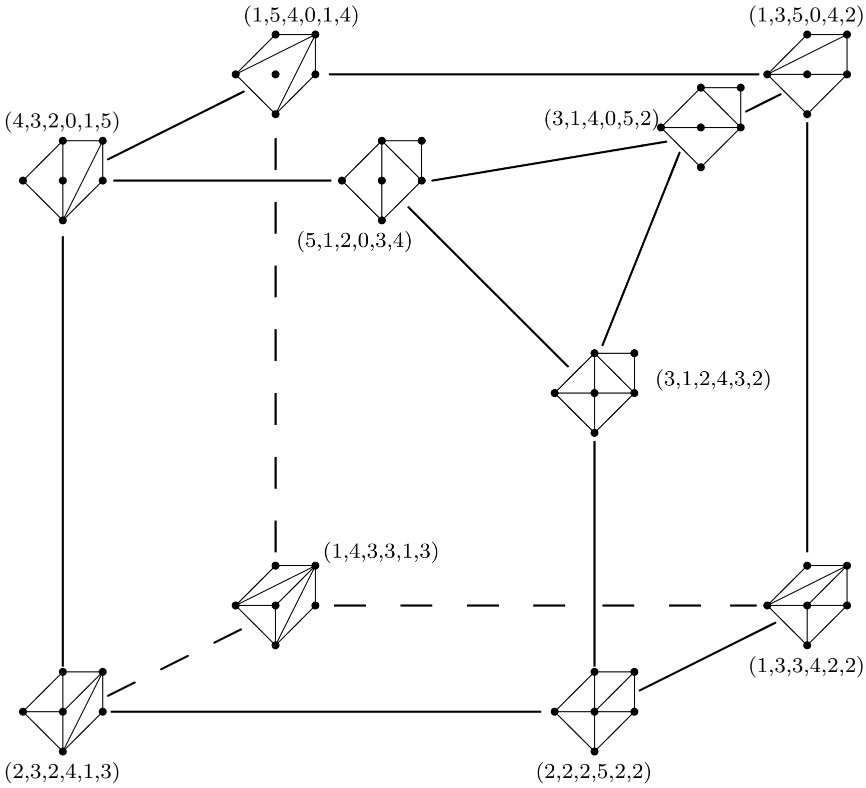
Later on we will show that different regular triangulations correspond to different vertices on $\Sigma(A)$.
Note that a priory $\Sigma(A) \subset \mathbb{R}^N$ but the image of χ_T under the projection $\psi : \mathbb{R}^N \to \mathbb{R}^r$ (given by $\mathbf{e}_i \mapsto \mathbf{a}_i$) is independent of T. Hence $\Sigma(A)$ is contained in a translation of the subspace $L(\mathbb{R})$. The secondary polytope was introduced by Gel'fand, Kapranov and Zelevinsky.
An example, which I reproduce with kind permission from Jan Stienstra [**38**], is the following. Consider the A-matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The primary polytope is two-dimensional, here is a picture of $Q(A)$, as it lies in the plane $x_1 = 1$.



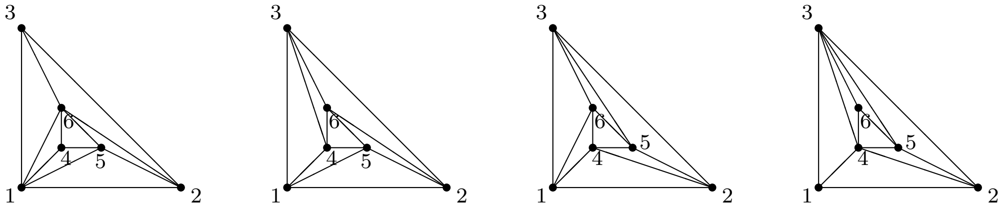
The secondary polytope $\Sigma(A)$ is three dimensional. Here is a picture of $\Sigma(A)$ with the triangulation at every vertex together with its coordinates.



Here we reproduce an example of nonregular triangulations from [**11**] (with thanks again to Jan Stienstra for the picture). Consider the A-matrix

$$\begin{pmatrix} 4 & 0 & 0 & 2 & 1 & 1 \\ 0 & 4 & 0 & 1 & 2 & 1 \\ 0 & 0 & 4 & 1 & 1 & 2 \end{pmatrix}$$

Here are four triangulations of A,



T_1 T_2 T_3 T_4

We compute $\chi_{T_1} = (40, 36, 32, 12, 28, 44)$, $\chi_{T_4} = (32, 36, 40, 44, 28, 12)$ and $\chi_{T_2} = \chi_{T_3} = (36, 36, 36, 28, 28, 28)$. So $\chi_{T_2} = \chi_{T_3} = \frac{1}{2}(\chi_{T_1} + \chi_{T_2})$. Hence $\chi_{T_2} = \chi_{T_3}$ are not vertices of the secondary polytope and T_2, T_3 are not regular triangulations.
We now show how to construct regular triangulations. Let $\rho = (\rho_1, \ldots, \rho_N) \in \mathbb{R}^N$. For every $(r-1)$-simplex J we define the linear form m_J on $\mathbb{R}^r$ such that $m_J(\mathbf{a}_j) = \rho_j$ for all $j \in J$. We assume that ρ is chosen so that all forms m_J are distinct. By $Q(A)^o$ we denote the convex hull minus the sets $Q(A_J)$ with $|J| < r$. Our triangulation arises as follows. For every $\mathbf{x} \in Q(A)^o$ we determine the $(r-1)$-simplex J such that $m_J(\mathbf{x})$ is minimal among the set

$$\{m_I(\mathbf{x}) | I \text{ is an } (r-1)-\text{simplex and } \mathbf{x} \in Q(A_I)\}.$$

We claim that J is uniquely determined and that the association $\mathbf{x} \mapsto J$ gives a triangulation of A. This claim is based on the following Proposition.

Proposition 18.3. — *Let $\mathbf{x}$ and J be as above. Then, for any $\mathbf{y} \in Q(A)$ and any $(r-1)$-simplex $Q(A_I)$ which contains $\mathbf{y}$ we have that $m_J(\mathbf{y}) \le m_I(\mathbf{y})$.*

A first consequence is the following. Suppose there exist two J, J' such that $m_J(\mathbf{x}) = m_{J'}(\mathbf{x})$ is minimal. Then $m_J(\mathbf{y}) \le m_{J'}(\mathbf{y})$ and $m_{J'}(\mathbf{y}) \le m_J(\mathbf{y})$ for all $\mathbf{y}$. Hence m_J and $m_{J'}$ are the same, which implies that $J = J'$. In other words, the $r-1$-simplex J for which $m_J(\mathbf{x})$ is minimal, is uniquely determined. Another consequence, once J is chosen for $\mathbf{x}$, it gets chosen for every point $\mathbf{y} \in Q(A_J) \cap Q(A)^o$. This explains why we get a triangulation.

Proof of the Proposition. The statement to be proven clearly implies the statement for $\mathbf{y} = \mathbf{a}_i$ for any i. In other words, $m_J(\mathbf{a}_i) \le m_I(\mathbf{a}_i)$ for every i, I with $i \in I$. This can be restated as

$$m_J(\mathbf{a}_i) \le \rho_i \text{ for every } i \in \{1, 2, \ldots, N\}.$$

Conversely, the latter statement implies the statement to be proven. Namely, let $\mathbf{y} = \sum_{i \in I} \tau_i \mathbf{a}_i$ where τ_i are the (non-negative) barycentric coordinates of $Q(A_I)$. Then $m_J(\mathbf{a}_i) \le \rho_i = m_I(\mathbf{a}_i)$ implies $m_J(\mathbf{y}) \le m_I(\mathbf{y})$ by linearity and $\tau_i \ge 0$ for $i \in I$.
To prove our Proposition let us now assume that there exists i such that $m_J(\mathbf{a}_i) > \rho_i$. Now choose a face of $Q(A_J)$ which, together with $\mathbf{a}_i$, forms an $(r-1)$-simplex $Q(A_{J'})$ which contains $\mathbf{x}$. Then $\mathbf{x} = \sum_{j \in J'} \tau_j \mathbf{a}_j$ with $\tau_j > 0$ for

all $j \in J'$. Then

$$\begin{aligned} m_{J'}(\mathbf{x}) &= \sum_{j\in J'} \tau_j m_{J'}(\mathbf{a}_j) \\ &= \sum_{j\in J'} \tau_j \rho_j \\ &= \tau_i\rho_i + \sum_{j\in J', j\ne i} \tau_j\rho_j \\ &< \tau_i m_J(\mathbf{a}_i) + \sum_{j\in J', j\ne i} \tau_j m_J(\mathbf{a}_j) \\ &= m_J(\mathbf{x}) \end{aligned}$$

This contradicts the minimality of $m_J(\mathbf{x})$, which concludes our proof.

□

We denote by T_ρ the triangulation we just found.

Proposition 18.4. — *We have that $\rho\cdot\chi_{T_\rho} < \rho\cdot\chi_T$ for every triangulation $T \ne T_\rho$. Here $\cdot$ denotes the Euclidean inner product. In particular, the triangulation T_ρ is regular.*
Conversely, to any regular triangulation T' there exists $\rho \in \mathbb{R}^N$ such that $T' = T_\rho$.

Proof For any triangulation T and $\rho \in \mathbb{R}^N$ we define $g_{T,\rho} : Q(A) \to \mathbb{R}$ as follows. To a point $\mathbf{x} \in Q(A)$ find the $(r-1)$-simplex J to which it belongs in the triangulation T, and determine $m_J(\mathbf{x})$. This will be the function value. Notice that

$$\rho\cdot\chi_T = \sum_{J\in T} \mathrm{Vol}(Q(A_J)) \sum_{j\in J} \rho_j = \int_{Q(A)} g_{T,\rho}(\mathbf{x})d\mu(\mathbf{x})$$

for a suitably chosen Euclidean measure $d\mu$ on $Q(A)$. Let $T \ne T_\rho$ By the preceding Proposition we have that $g_{T_\rho,\rho}(\mathbf{x}) \le g_{T,\rho}(\mathbf{x})$ for all $\mathbf{x} \in Q(A)$ where the inequality is strict on a set of positive measure. Hence $\rho\cdot\chi_{T_\rho} < \rho\cdot\chi_T$.
Conversely let T' be a regular triangulation, i.e. $\chi_{T'}$ is a vertex of the secondary polytope. Hence there exists $\rho \in \mathbb{R}^N$ such that $\rho\cdot\chi_{T'} < \rho\cdot\chi_T$ all triangulations $T \ne T'$. To ρ we construct T_ρ. If $T' \ne T_\rho$ we would have, according to the preceding, $\rho\cdot\chi_{T_\rho} < \rho\cdot\chi_{T'}$. This contradicts the minimality of $\rho\cdot\chi_{T'}$. Hence we conclude that $T' = T_\rho$.

□

An immediate consequence is the following.

Corollary 18.5. — *Let T, T' be two distinct regular triangulations. Then $\chi_T \ne \chi_{T'}$.*

In [**24**, Chapter 7], the book by Gel'fand, Kapranov and Zelevinsky on discriminants and resultants, we find an extensive discussion of triangulations of A and secondary polytopes. There, regular triangulations are called *coherent triangulations*.

19. A basis of solutions

From the preceding two sections we can deduce the following conclusion.

Theorem 19.1. — *Let ρ be a convergence direction which is sufficiently general (in the sense that all linear forms m_J are distinct). Then there exists a regular triangulation T of $Q(A)$ such that the summation sectors, for which ρ is a convergence direction, are given by J^c where J runs through the $(r-1)$-simplices in T.*

Let $\mathcal{I}$ be a summation sector corresponding to ρ. We then construct the series solution $\Phi_{\mathcal{I}}$ as in Section 14. In order to ensure non-trivial $\Phi_{\mathcal{I}}$ we had to assume that $\alpha + \mathbb{Z}^r$ does not contain a point on a face of the simplex spanned by the $\mathbf{a}_i$ with $i \notin \mathcal{I}$.

Definition 19.2. — *Let T be a regular triangulation of $Q(A)$. The parameters α will be called T-nonresonant if $\alpha+\mathbb{Z}^r$ does not contain a point on the boundary of any $(r-1)$-simplex Σ_J with $J \in T$.*

Notice that the T-nonresonance condition implies the nonresonance condition. Let us assume that α is T-nonresonant. For any $\mathcal{I} = J^c$ with $J \in T$ we get the series $\Phi_{\mathcal{I}}$ which represents $\Delta_{\mathcal{I}} = \mathrm{Vol}(\Sigma_J)$ independent solutions.

Proposition 19.3. — *Under the T-nonresonance condition the power series solutions just constructed form a basis of solutions of $H_A(\alpha)$.*

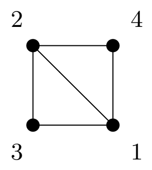
Proof. To show that the solutions are independent it suffices to show that for any two distinct summation sectors $\mathcal{I}$ and $\mathcal{I}'$ the values of $\gamma_1, \ldots, \gamma_N$, as chosen in $\Phi_{\mathcal{I}}$ and $\Phi_{\mathcal{I}'}$, are distinct modulo the lattice L. Suppose they are not distinct modulo L. Then there exists an index $i \in \mathcal{I}'$, but $i \notin \mathcal{I}$ such that $\gamma_i \in \mathbb{Z}$. But this is contradicted by our T-nonresonance assumption.
For every $J \in T$ we get $\mathrm{Vol}(\Sigma_J)$ solutions by writing down Φ_{J^c}. Summing over $J \in T$ shows that we obtain $\sum_{J \in T} \mathrm{Vol}(\Sigma_J) = \mathrm{Vol}(Q(A))$ solutions.

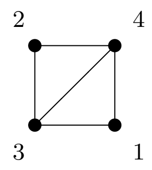
$\square$

Here is an example of Gauss' hypergeometric function with A-matrix

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

and parameters $(-a, -b, c-1)$. Here are the triangulations of $Q(A)$,

T_1 : 

T_2 : 

The secondary polytope is one-dimensional. We see that nonresonance means that $(-a, -b, c)$ modulo $\mathbb{Z}^3$ does not lie in the faces $14, 24, 23, 13$, in other words, $c-b, c-a, a, b \notin \mathbb{Z}$. In addition, T-nonresonance for the first triangulation means that $c \notin \mathbb{Z}$ and for the second triangulation $a - b \notin \mathbb{Z}$. It is well-known that if

$c \in \mathbb{Z}$ the local solutions around $z = 0$ will also contain $\log z$. Similarly around $z = \infty$ when $a - b \in \mathbb{Z}$. In general, the occurrence of T-resonance implies the appearance of logarithmic terms in the solutions, and we have to consider solutions in the larger space of power series tensored with $\mathbb{C}(\mathbf{v}^\gamma, \log(\mathbf{v}))$. This is completely elaborated in [**37**, Chapter 3].

20. Pochhammer cycles

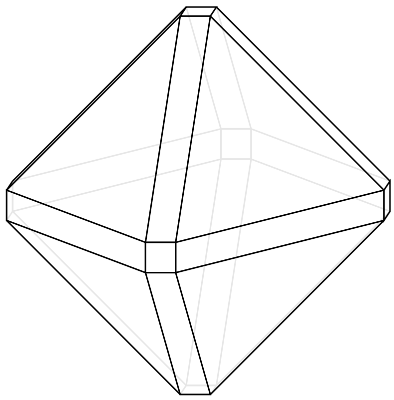
In the construction of Euler integrals for A-hypergeometric functions one often uses so-called twisted homology cycles. In [**22**] this is done on an abstract level, in [**28**] and [**40**] it is done more explicitly. In this paper we prefer to follow a more concrete approach by constructing a closed cycle of integration such that the (multivalued) integrand can be chosen in a continuous manner and the resulting integral is non-zero. For the ordinary Euler-Gauss function this is realised by integration over the so-called Pochhammer contour, as given in Section 2. Here we construct its n-dimensional generalisation. In Section 21.1 we use it to define an Euler integral for A-hypergeometric functions.

Consider the hyperplane H given by $t_0 + t_1 + \cdots + t_n = 1$ in $\mathbb{C}^{n+1}$ and the affine subspaces H_i given by $t_i = 0$ for $(i = 0, 1, 2, \ldots, n)$. Let H^o be the complement in H of all H_i. We construct an n-dimensional real cycle P_n in H^o which is a generalisation of the ordinary 1-dimensional Pochhammer cycle (the case $n = 1$). When $n > 1$ it has the property that its homotopy class in H^o is non-trivial, but that its fundamental group is trivial. One can find a sketchy discussion of such cycles in [**40**, Section 3.5].

Let ϵ be a positive but sufficiently small real number. We start with a polytope F in $\mathbb{R}^{n+1}$ given by the inequalities

$$|x_{i_1}| + |x_{i_2}| + \cdots + |x_{i_k}| \leq 1 - (n + 1 - k)\epsilon$$

for all $k = 1, \ldots, n + 1$ and all $0 \leq i_1 < i_2 < \cdots < i_k \leq n$. Geometrically this is an $n + 1$-dimensional octahedron with the faces of codimension ≥ 2 sheared off. For example in the case $n = 2$ it looks like



The faces of F can be enumerated by vectors $\mu = (\mu_0, \mu_1, \ldots, \mu_n) \in \{0, \pm 1\}^{n+1}$, not all μ_i equal to 0, as follows. Denote $|\mu| = \sum_{i=0}^{n} |\mu_i|$. The face corresponding to μ is defined by

$$F_\mu \quad : \quad \mu_0x_0+\mu_1x_1+\cdots+\mu_nx_n = 1-(n+1-|\mu|)\epsilon, \quad \mu_jx_j\ge\epsilon \text{ whenever } \mu_j\ne 0$$
$$|x_j|\le\epsilon \text{ whenever } \mu_j=0.$$

Notice that as a polytope F_μ is isomorphic to $\Delta_{|\mu|-1}\times I^{n+1-|\mu|}$ where Δ_r is the standard r-dimensional simplex and I the unit real interval. Notice in particular that we have 3^n-1 faces.
The $n-1$-dimensional side-cells of F_μ are easily described. Choose an index j with $0\le j\le n$. If $\mu_j\ne 0$ we set $\mu_jx_j=\epsilon$, if $\mu_j=0$ we set either $x_j=\epsilon$ or $x_j=-\epsilon$. As a corollary we see that two faces F_μ and $F_{\mu'}$ meet in an $n-1$-cell if and only if there exists an index j such that $|\mu_j|\ne|\mu'_j|$ and $\mu_i=\mu'_i$ for all $i\ne j$.
The vertices of F are the points with one coordinate equal to $\pm(1-n\epsilon)$ and all other coordinates $\pm\epsilon$.
We now define a continuous and piecewise smooth map $P:\cup_\mu F_\mu\to H$ by

$$P:(x_0,x_1,\ldots,x_n)\mapsto \frac{1}{y_0+y_1+\cdots+y_n}(y_0,y_1,\ldots,y_n) \tag{2}$$

where $y_j=|x_j|$ if $|x_j|\ge\epsilon$ and $y_j=\epsilon e^{\pi i(1-x_j/\epsilon)}$ if $|x_j|\le\epsilon$. When ϵ is sufficiently small we easily check that P is injective. We define our n-dimensional Pochhammer cycle P_n to be its image.

Proposition 20.1. — *Let $\beta_0,\beta_1,\ldots,\beta_n$ be complex numbers. Consider the integral*

$$B(\beta_0,\beta_1,\ldots,\beta_n)=\int_{P_n}\omega(\beta_0,\ldots,\beta_n)$$

where

$$\omega(\beta_0,\ldots,\beta_n)=t_0^{\beta_0-1}t_1^{\beta_1-1}\cdots t_n^{\beta_n-1}\,dt_1\wedge dt_2\wedge\cdots\wedge dt_n.$$

Then, for a suitable choice of the multivalued integrand, we have

$$B(\beta_0,\ldots,\beta_n)=\frac{1}{\Gamma(\beta_0+\beta_1+\cdots+\beta_n)}\prod_{j=0}^n(1-e^{-2\pi i\beta_j})\Gamma(\beta_j).$$

Proof. The problem with ω is its multivaluedness. This is precisely the reason for constructing the Pochhammer cycle P_n. Now that we have our cycle we solve the problem by making a choice for the pulled back differential form $P^*\omega$ and integrating it over ∂F. Furthermore, the integral will not depend on the choice of ϵ. Therefore we let $\epsilon\to 0$. In doing so we assume that the real parts of all β_i are positive. The Proposition then follows by analytic continuation of the β_j.
On the face F_μ we define $T:F_\mu\to\mathbb{C}$ by

$$T:(x_0,x_1,\ldots,x_n)=\prod_{\mu_j\ne 0}|x_j|^{\beta_j-1}e^{\pi i(\mu_j-1)\beta_j}\prod_{\mu_k=0}\epsilon^{\beta_j-1}e^{\pi i(x_j/\epsilon-1)(\beta_j-1)}.$$

This gives us a continuous function on ∂F. For real positive λ we define the complex power λ^z by $\exp(z\log\lambda)$. With the notations as in (2) we have $t_i=$

$y_i/(y_0+\cdots+y_n)$ and, as a result,

$$dt_1\wedge dt_2\wedge\cdots\wedge dt_n=\sum_{j=0}^n(-1)^j y_j dy_0\wedge\cdots\wedge \check{dy_j}\wedge\cdots dy_n$$

where $\check{dy_j}$ denotes suppression of dy_j. It is straightforward to see that integration of $T(x_0,\ldots,x_n)$ over F_μ with $|\mu|<n+1$ gives us an integral of order $O(\epsilon^\beta)$ where β is the minimum of the real parts of all β_j. Hence they tend to 0 as $\epsilon\to 0$. It remains to consider the cases $|\mu|=n+1$. Notice that T restricted to such an F_μ has the form

$$T(x_0,\ldots,x_n)=\prod_{j=0}^n e^{\pi i(\mu_j-1)\beta_j}|x_j|^{\beta_j-1}.$$

Furthermore, restricted to F_μ we have

$$\sum_{j=0}^n(-1)^j y_j dy_0\wedge\cdots\wedge \check{dy_j}\wedge\cdots dy_n=dy_1\wedge dy_2\wedge\cdots\wedge dy_n$$

and $y_0+y_1+\cdots+y_n=1$. Our integral over F_μ now reads

$$\prod_{j=0}^n \mu_j e^{\pi i(\mu_j-1)\beta_j}\int_\Delta (1-y_1-\ldots-y_n)^{\beta_0-1}y_1^{\beta_1-1}\cdots y_n^{\beta_n-1}dy_1\wedge\cdots\wedge dy_n$$

where Δ is the domain given by the inequalities $y_i\ge\epsilon$ for $i=1,2,\ldots,n$ and $y_1+\cdots+y_n\le 1-\epsilon$. The extra factor $\prod_j\mu_j$ accounts for the orientation of the integration domains. The latter integral is a generalisation of the Euler beta-function integral. Its value is $\Gamma(\beta_0)\cdots\Gamma(\beta_n)/\Gamma(\beta_0+\cdots+\beta_n)$. Adding these evaluation over all F_μ gives us our assertion. $\square$

For the next section we notice that if $\beta_0=0$ the subfactor $(1-e^{-2\pi i\beta_0})\Gamma(\beta_0)$ becomes $2\pi i$.

21. Euler integrals

We now adopt the usual notation from A-hypergeometric functions. Define

$$I(A,\alpha,v_1,\ldots,v_N)=\int_{\mathcal{C}}\frac{\mathbf{t}^{-\alpha}}{1-\sum_{i=1}^N v_i\mathbf{t}^{\mathbf{a}_i}}\frac{dt_1}{t_1}\wedge\frac{dt_2}{t_2}\wedge\cdots\wedge\frac{dt_r}{t_r},$$

where $\mathcal{C}$ is an r-cycle which doesn't intersect the hyperplane $1-\sum_{i=1}^N v_i\mathbf{t}^{\mathbf{a}_i}=0$ for an open subset of $\mathbf{v}\in\mathbb{C}^N$ and such that the multivalued integrand can be defined on $\mathcal{C}$ continuously and such that the integral is not identically zero. We shall specify $\mathcal{C}$ in the course of this section.
First note that an integral such as this satisfies the A-hypergeometric equations easily. The substitution $t_i\to\lambda_i t_i$ shows that

$$I(A,\alpha,\lambda^{\mathbf{a}_1}v_1,\ldots,\lambda^{\mathbf{a}_n}v_N)=\lambda^\alpha I(A,\alpha,v_1,\ldots,v_N).$$

This accounts for the homogeneity equations. For the "box"-equations, write $\mathbf{l}\in L$ as $\mathbf{u}-\mathbf{w}$ where $\mathbf{u},\mathbf{w}\in\mathbb{Z}_{\ge 0}^N$ have disjoint supports. Then

$$\Box_{\mathbf{l}} I(A,\alpha,\mathbf{v}) = |\mathbf{u}|! \int_{\mathcal{C}} \frac{\mathbf{t}^{-\alpha+\sum_i u_i \mathbf{a}_i} - t^{-\alpha+\sum_i w_i \mathbf{a}_i}}{(1-\sum_{i=1}^N v_i \mathbf{t}^{\mathbf{a}_i})^{|\mathbf{u}|+1}} \frac{dt_1}{t_1}\wedge\frac{dt_2}{t_2}\wedge\cdots\wedge\frac{dt_r}{t_r}$$

where $|\mathbf{u}|$ is the sum of the coordinates of $\mathbf{u}$, which is equal to $|\mathbf{w}|$ since $|\mathbf{u}| - |\mathbf{w}| = |\mathbf{l}| = \sum_{i=1}^N l_i h(\mathbf{a}_i) = h(\sum_i l_i \mathbf{a}_i) = 0$. Notice that the numerator in the last integrand vanishes because $\sum_i u_i \mathbf{a}_i = \sum_i w_i \mathbf{a}_i$. So $\Box_{\mathbf{l}} I(A,\alpha,\mathbf{v})$ vanishes.

We now specify our cycle of integration $\mathcal{C}$. Choose r vectors in A such that their determinent is 1. In the rare instances where such an r-tuple does not exist we have to work with fractional powers of the t_i, but we shall not consider this complication. After permutation of indices, and change of coordinates if necessary, we can assume that $\mathbf{a}_i = \mathbf{e}_i$ for $i = 1,\ldots,r$ (the standard basis of $\mathbb{R}^r$). Our integral now acquires the form

$$\int_{\mathcal{C}} \frac{\mathbf{t}^{-\alpha}}{1-v_1t_1-\cdots-v_rt_r-\sum_{i=r+1}^N v_i\mathbf{t}^{\mathbf{a}_i}} \frac{dt_1}{t_1}\wedge\frac{dt_2}{t_2}\wedge\cdots\wedge\frac{dt_r}{t_r}.$$

Perform the change of variables $t_i \to t_i/v_i$ for $i = 1,\ldots,r$. Up to a factor $v_1^{\alpha_1}\cdots v_r^{\alpha_r}$ we get the integral

$$\int_{\mathcal{C}} \frac{\mathbf{t}^{-\alpha}}{1-t_1-\cdots-t_r-\sum_{i=r+1}^N u_i\mathbf{t}^{\mathbf{a}_i}} \frac{dt_1}{t_1}\wedge\frac{dt_2}{t_2}\wedge\cdots\wedge\frac{dt_r}{t_r},$$

where the u_i are Laurent monomials in $v_1,\ldots,v_N$. Without loss of generality we might as well assume that $v_1 = \ldots = v_r = 1$ so that we get the integral

$$\int_{\mathcal{C}} \frac{\mathbf{t}^{-\alpha}}{1-t_1-\cdots-t_r-\sum_{i=r+1}^N v_i\mathbf{t}^{\mathbf{a}_i}} \frac{dt_1}{t_1}\wedge\frac{dt_2}{t_2}\wedge\cdots\wedge\frac{dt_r}{t_r}.$$

For the r-cycle $\mathcal{C}$ we choose the projection of the Pochhammer r-cycle on $t_0 + t_1+\cdots+t_r = 1$ to $t_1,\ldots,t_r$ space. Denote it by $\mathcal{C}_r$. By keeping the v_i sufficiently small the hypersurface $1-t_1-\cdots-t_r-\sum_{i=r+1}^N v_i\mathbf{t}^{\mathbf{a}_i} = 0$ does not intersect $\mathcal{C}_r$.

To show that we get a non-zero integral we set $\mathbf{v} = \mathbf{0}$ and use the evaluation in Proposition 20.1. We see that it is non-zero if all α_i have non-integral values. When one of the α_i is integral we need to proceed with more care.

We develop the integrand in a geometric series and integrate it over $\mathcal{C}_r$. We have

$$\begin{aligned}&\frac{\mathbf{t}^{-\alpha}}{1-t_1-\cdots-t_r-\sum_{i=r+1}^N v_i\mathbf{t}^{\mathbf{a}_i}}\\ &= \sum_{m_{r+1},\ldots,m_N\ge 0}\binom{|m|}{m_{r+1},\ldots,m_N}\frac{\mathbf{t}^{-\alpha+m_{r+1}\mathbf{a}_{r+1}+\cdots+m_N\mathbf{a}_N}}{(1-t_1-\cdots-t_r)^{|m|+1}}\, v_{r+1}^{m_{r+1}}\cdots v_N^{m_N}\end{aligned}$$

where $|m| = m_{r+1}+\cdots+m_N$. We now integrate over $\mathcal{C}_r$ term by term. For this we use Proposition 20.1. We infer that all terms are zero if and only if there exists i such that the i-th coordinate of α is integral and positive and the i-th coordinate of each of $\mathbf{a}_{r+1},\ldots,\mathbf{a}_N$ is non-negative. In particular this means that the cone $C(A)$ is contained in the halfspace $x_i \ge 0$. Moreover, the points $\mathbf{a}_j = \mathbf{e}_j$ with $j \ne i$ and $1 \le j \le r$ are contained in the subspace $x_i = 0$, so they span (part of) a face of $C(A)$. The set $\alpha + \mathbb{Z}^r$ has non-trivial intersection with this face because $\alpha_i \in \mathbb{Z}$. From Theorem 9.1 it follows that our system is resonant. Thus we conclude

Theorem 21.1. — *When $H_A(\alpha)$ is nonresonant, the integral $I(A, \alpha, \mathbf{v})$ represents a non-trivial solution of $H_A(\alpha)$.*

Furthermore, nonresonance implies irreducibility of $H_A(\alpha)$. This means that analytic continuation of $I(A, \alpha, \mathbf{v})$ gives us a basis of solutions of $H_A(\alpha)$. A fortiori all solutions of the hypergeometric system can be given by linear combinations of period integrals of the type $I(A, \alpha, \mathbf{v})$ (but with different integration cycle).

22. Some examples of Euler integrals

Recall the classical Euler integral for ${}_2F_1(a, b, c|z)$ from Section 2. This is a one-dimensional integral, whereas the Euler integrals in Theorem 21.1 would be 3-dimensional, since $A \subset \mathbb{Z}^3$ in this case. We like to show how the 3-dimensional integral can be reduced to the classical integral.
The integrand of the A-hypergeometric Euler integral reads

$$\frac{t_1^a t_2^b t_3^{1-c}}{1 - t_1 - t_2 - t_3 - z t_1 t_2 / t_3} \frac{dt_1}{t_1} \wedge \frac{dt_2}{t_2} \wedge \frac{dt_3}{t_3}.$$

Replace t_2 by $t_2 t_3$ to get

$$\frac{t_1^a t_2^b t_3^{b+1-c}}{1 - t_1 - t_2 t_3 - t_3 - z t_1 t_2} \frac{dt_1}{t_1} \wedge \frac{dt_2}{t_2} \wedge \frac{dt_3}{t_3}.$$

Now replace t_1 by $t_1/(1 + z t_2)$ and t_3 by $t_3/(1 + t_2)$ to get

$$\frac{t_1^a t_3^{b+1-c} (1 + z t_2)^{-a} (1 + t_2)^{c-b-1} t_2^b}{1 - t_1 - t_3} \frac{dt_1}{t_1} \wedge \frac{dt_2}{t_2} \wedge \frac{dt_3}{t_3}.$$

After integration over t_1, t_3 and the replacement $t_2 \to -u$ we get up to a constant factor,

$$(1 - zu)^{-a} u^{b-1} (1 - u)^{c-b-1} du,$$

the classical Euler integrand.
In a slightly more general vein we consider the Aomoto-Gel'fand hypergeometric functions, which are the precursor of A-hypergeometric functions. Let $f_1, \ldots, f_n$ be n linear forms in k variables $x_1, \ldots, x_k$ where $k < n$. For any parameters $\lambda_1, \ldots, \lambda_n$ consider the integral

$$I(k, n, \lambda) = \int f_1^{\lambda_1} \cdots f_n^{\lambda_n} dx_1 \wedge \cdots \wedge dx_k$$

as a function of the coefficients of the forms $f_1, \ldots, f_n$. The integral is taken over a suitable k-cycle. After a linear change of coordinates we may assume $f_1 = x_1, \ldots, f_k = x_k$. Consider the n-fold Euler integral

$$\int \frac{x_1^{\lambda_1} \cdots x_k^{\lambda_k} t_1^{-\lambda_{k+1}-1} \cdots t_{n-k}^{-\lambda_n - 1}}{1 - t_1 f_{k+1} - \cdots - t_{n-k} f_n} dx_1 \wedge \cdots \wedge dx_k \wedge dt_1 \wedge \cdots \wedge dt_{n-k}$$

Replace t_i by t_i / f_{k+i} to get

$$\int \frac{t_1^{-\lambda_{k+1}} \cdots t_{n-k}^{-\lambda_n}}{1 - t_1 - \cdots - t_{n-k}} \, x_1^{\lambda_1} \cdots x_k^{\lambda_k} f_{k+1}^{\lambda_{k+1}} \cdots f_n^{\lambda_n}$$
$$dt_1 \wedge \cdots \wedge dt_{n-k} \wedge dx_1 \wedge \cdots \wedge dx_k$$

Integrate with respect to $t_1, \ldots, t_{n-k}$ to recover Aomoto's integral $I(k, n, \lambda)$ with $f_1 = x_1, \ldots, f_k = x_k$.

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August 17, 2010

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