Exam Representations of Finite Groups, WISB324

With Solutions

June 25, 2019, 17:00-20:00

1. Let G be the finite group given by

$$G = \left\langle a, b, c | a^3 = b^3 = c^3 = e, ab = ba, ac = ca, c^{-1}bc = ab \right\rangle .$$

It has 27 elements and 11 conjugation classes. In the following we compute the irreducible characters of G without computing the conjugation classes.

(a) (1/2 pt) Determine the dimensions of the irreducible representations of G.

Solution Suppose there are A three-dimensional, B two-dimensional and C one-dimensional representations. Then, $9A+4B+C = 27$, the order of G and $A+B+C = 11$, the number of conjugation classes. Subtract the two to get $8A + 3B = 16$. So B is divisible by 8 which is only possible if $B = 0$. Hence $A = 2$ and $C = 11 - A - B = 9$.

(b) (1 pt) Determine the one-dimensional representations of G.

Solution This can be done independently of (a). Suppose we have a one-dimensional reprentation ρ and put $\rho(a) = \alpha, \rho(b) = \beta, \rho(c) = \gamma$. From the defining relations of G it follows that $\alpha^3 = \beta^3 = \gamma^3 = 1$ and $\gamma^{-1}\beta\gamma = \alpha\beta$, hence $\alpha = 1$. Let $\omega = e^{2\pi i/3}$, then we see that $\beta = \omega^k, \gamma = \omega^l$ for some $k, l = 0, 1, 2$. These are nine possibilities, corresponding to $C = 9$ we had in (a).

(c) (1/2 pt) Show that $\{e\}, \{a\}, \{a^2\}$ are conjugation classes of G.

Solution From the defining relations it follows that a commutes both with b and c. Hence a is in the center of G and so $\{a^k\}$ is a conjugation class for $k = 0, 1, 2$.

(d) (1/2 pt) Show that $\chi(g) = 0$ for every $g \notin \{e, a, a^2\}$ and every irreducible character χ with $\chi(e) > 1$.

Solution Choose a one-dimensional character ρ such that $\rho(g) = \omega$. Then ρ times χ and ρ^2 times χ are also irreducible characters. If $\chi(g) \neq 0$, the three characters $\chi, \chi\rho, \chi\rho^2$ would be inequivalent since their values at g would be distinct. This contradicts the fact that we have only two three-dimensional representations. Hence $\chi(g) = 0$.

(e) (1/2 pt) Show that $\chi(a^2) = \overline{\chi(a)}$ for every character χ.

Solution Notice that $a^2 = a^{-1}$. From the theory we know that $\chi(a^{-1}) = \overline{\chi(a)}$.

(f) (1/2 pt) Show that there is an irreducible character such that $\chi(a) \notin \mathbb{R}$. Define $\alpha = \chi(a)$ for this character.

Solution Suppose that $\chi(a) \in \mathbb{R}$ for all χ. Then $\chi(a^2) = \chi(a)$ for all χ. Since $\{a\}, \{a^2\}$ are distinct conjugation classes we have the column orthogonality relation $0 = \sum_\chi \chi(a)\chi(a^2) = \sum_\chi \chi(a)^2 > 0$, which is a contradiction.

(g) (1 pt) Determine the possible values of α.

Solution Choose the character χ from part (f). Then $\overline{\chi}$ given by $\overline{\chi}(g) = \overline{\chi)(g)}$ is also character and necessarily the character of the second three-dimensional representation. We have the absolute value of the column corresponding to $\{a\}$: $27 = 9 \times 1^2 + |\alpha|^2 + |\overline{\alpha}|^2$, hence $18 = 2|\alpha|^2$. So $|\alpha| = 3$. The inner product relation of the columns corresponding to $\{e\}$ and $\{a\}$ reads $0 = 9 + 3\alpha + 3\overline{\alpha}$ Hence $\alpha + \overline{\alpha} = -3$. So real part of $\alpha = -3/2$. Imaginary part is then $\sqrt{3^2 - (-3/2)^2} = \sqrt{27/4} = \pm 3\sqrt{3}/2$.

2. Consider the vector space of bilinear polynomials in $x_1, x_2, x_3, y_1, y_2, y_3$ given by

$$V = \left\{ \sum_{i,j=1}^{3} \lambda_{ij} x_i y_j \,\middle|\, \lambda_{ij} \in \mathbb{C} \right\}.$$

We give V a $\mathbb{C}S_3$-module structure by letting every $\sigma \in S_3$ action as $\sigma : x_i y_j \mapsto x_{\sigma(i)} y_{\sigma(j)}$.

(a) (1/2 pt) Write down the character table of S_3. Briefly motivate your answer.

Solution

	(1)	(12)	(123)
χ_{triv}	1	1	1
χ_{sign}	1	-1	1
χ_Δ	2	0	-1

(b) (1 pt) Determine the character of the $\mathbb{C}S_3$-module V and write it as sum of irreducible characters of S_3.

Solution The group S_3 permutes the nine products $x_i y_j$. The character value of $\sigma \in S_3$ is simply the number of monomials that are fixed under σ. Hence $\chi_V((1)) = 9, \chi_V((12)) = 1, \chi_V((123)) = 0$. By linear algebra it follows that $\chi_V = 2\chi_{\text{triv}} + \chi_{\text{sign}} + 3\chi_\Delta$.

(c) (1 pt) Write down generators of the subspaces of V that correspond to one-dimensional $\mathbb{C}S_3$ submodules of V.

Solution It is clear that the sum of all monomials $x_i y_j$ is fixed under every σ, as well as the sum $x_1 y_1 + x_2 y_2 + x_3 y_3$. This is a basis of the space with trivial action. For χ_{sign} simply try $x_1y_2 - x_2y_1$ and add its images under $(123), (123)^2$. That is

$$x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3.$$

This turns out to be an eigenvector with eigenvalue -1 for $\sigma = (12)$.

(d) (1/2 pt) Show that the $\mathbb{C}S_3$-module V is isomorphic to $W \otimes W$, where W is the $\mathbb{C}S_3$-module given by the permutation representation $\sigma : \mathbf{e}_i \mapsto \mathbf{e}_{\sigma(i)}$ for all $\sigma \in S_3$ and $i = 1, 2, 3$.

Solution The trace values of the permutation representation are $\chi_W = 3, \chi_W((12)) = 1, \chi_W((123)) = 0$. Notice that $\chi_V(\sigma) = \chi_W(\sigma)^2$ for all σ. Hence V is isomorphic to $W \otimes W$.

3. Let χ be a character of a finite group G.

(a) (1 pt) Show that if $\chi(g) = 0$ for all $g \neq e$, then χ is a multiple of χ_{reg}, the character of the regular $\mathbb{C}G$-module.

Solution We have $\chi_{\text{reg}}(e) = |G|$ and $\chi_{\text{reg}}(g) = 0$ for all $g \neq e$. Hence $\chi = (\chi(e)/|G|)\chi_{\text{reg}}$. If $\chi(e)/|G|$ is an integer we are done since representations are uniquely determined by their characters and so our representation would be the sum of $\chi(e)/|G|$ copies of the regular represenation. Notice also that the number of trivial representations in χ is given by the inner product $\frac{1}{|G|}\sum_{g \in G} \chi(g) = \chi(e)/|G|$.

(b) (1 pt) Suppose that $\chi(g) \in \mathbb{R}_{\geq 0}$ for all $g \in G$. Show that χ is either the trivial character, or reducible.

Solution The number of copies of the trivial character in the decomposition of χ is equals to the inner product $\frac{1}{|G|}\sum_{g \in G} \chi(g)$, which is positive because $\chi(g) \geq 0$ and $\chi(e) > 0$. Hence χ contains at least one copy of the trivial character. If $\chi(e) = 1$ then it equals the trivial character, if $\chi(e) > 1$ it is reducible because it contains a copy of the trivial character.

4. (1 bonus point) The regular representation of a finite group G consists of the vector space $\mathbb{C}G$ together with an action of G given by $\rho_1(g) : r \mapsto gr$ for all $g \in G, r \in \mathbb{C}G$. Denote this $\mathbb{C}G$-module by V_1. We define a second action of G by $\rho_2(g) : r \mapsto rg^{-1}$ for all $g \in G, r \in \mathbb{C}G$.

(a) (1/2) Show that $\mathbb{C}G$ with the action ρ_2 is a $\mathbb{C}G$-module. Denote it by V_2.

Solution It is clear that $\rho_2(g)$ is a linear map. It remains to show that $\rho_2(gh) = \rho_2(g)\rho_2(h)$. Notice that

$$\rho_2(g)(\rho_2(h)r) = \rho_2(g)(rh^{-1}) = rh^{-1}g^{-1} = r(gh)^{-1} = \rho_2(gh)(r).$$

(b) (1/2) Show that V_1 and V_2 are isomorphic $\mathbb{C}G$-modules by exhibiting a $\mathbb{C}G$-isomorphism between them.

Solution The isomorphism is given by $\phi : \sum_g \lambda_g g \mapsto \sum_{g \in G} \lambda_g g^{-1}$. Notice that for every $h \in G$,

$$\phi\left(\rho_1(h)\left(\sum_g \lambda_g g\right)\right) = \phi\left(\sum_g \lambda_g hg\right) = \sum_g \lambda_g (hg)^{-1}$$

$$= \left(\sum_g \lambda_g g^{-1}\right) h^{-1} = \rho_2(h)\left(\phi\left(\sum_g \lambda_g g\right)\right).$$

So $\phi \circ \rho_1(g) = \rho_2(g) \circ \phi$.