Differentiable manifolds – hand-in sheet 4

Hand in by 21/Dec

Exercise

1) Let M^n be a manifold and $D_1, D_2 \subset TM$ be two distributions of complimentary rank, say $rank(D_1) = k$ and $rank(D_2) = n-k$ and such that $D_1 \oplus D_2 = TM$. Let $D_1^0, D_2^0 \subset T^*M$ be their annihilators, that is, over a point $p \in M$

$$D_i^0|_p = \{\xi \in T_p^*M : \xi(X) = 0 \text{ for all } X \in D_i|_p\}.$$

1. Show that D_i^0 are subbundles of T^*M with $rank(D_1^0) = n-k$ and $rank(D_2^0) = k$. Also $D_1^0 \oplus D_2^0 \cong T^*M$ and that $D_1^0 \cong D_2^*$ and $D_2^0 \cong D_1^*$.

2. Let $\pi_1 : TM \longrightarrow TM$ be the projection onto D_1 along D_2, that is, $\pi_1|_{D_1} = \mathrm{Id}$ and $\pi_1|_{D_2} = 0$. Similarly, let π_2 be the projection onto D_2 along D_1. Define the Nijenhuis operators of the pair (D_1, D_2) by the expressions

$$N_1 : \Gamma(D_1) \times \Gamma(D_1) \longrightarrow \Gamma(D_2)$$
$$N_1(X, Y) = \pi_2([X, Y]).$$

   and similarly

$$N_2 : \Gamma(D_2) \times \Gamma(D_2) \longrightarrow \Gamma(D_1)$$
$$N_2(X, Y) = \pi_1([X, Y]).$$

   Show that N_i is a tensor, i.e., it is $C^\infty(M)$-linear on its entries. Conclude that $N_1 \in (\wedge^2 D_2^0) \otimes D_2 = (\wedge^2 D_1^*) \otimes D_2$ and $N_2 \in (\wedge^2 D_1^0) \otimes D_1 = (\wedge^2 D_2^*) \otimes D_1$

3. Since $T^*M = D_1^0 \oplus D_2^0$, we get

$$\wedge^k T^*M = \oplus_{p+q=k} (\wedge^p D_2^0) \otimes (\wedge^q D_1^0) = \oplus_{p+q=k} (\wedge^p D_1^*) \otimes (\wedge^q D_2^*).$$

   We denote the space $(\wedge^p D_1^*) \otimes (\wedge^q D_2^*)$ by $\wedge^{p,q} T^*M$. Show that

$$d : \Gamma(\wedge^{p,q} T^*M) \longrightarrow \Gamma((\wedge^{p+2,q-1} T^*M) \oplus (\wedge^{p+1,q} T^*M) \oplus (\wedge^{p,q+1} T^*M) \oplus (\wedge^{p-1,q+2} T^*M))$$

   (Hint: use induction on the degree of the form and Leibniz rule)

4. Show that the composition of d with the projections

$$\mathcal{N}_1 = \pi_{p+2,q-1} \circ d : \Gamma(\wedge^{p,q} T^*M) \longrightarrow \Gamma(\wedge^{p+2,q-1} T^*M)$$
$$\mathcal{N}_2 = \pi_{p-1,q+2} \circ d : \Gamma(\wedge^{p,q} T^*M) \longrightarrow \Gamma(\wedge^{p-1,q+2} T^*M)$$

   are $C^\infty(M)$-linear. Relate $\mathcal{N}_i$ with N_i.

5. Conclude that the distribution D_1 is integrable if and only if

$$d : \Gamma(\wedge^{p,q} T^*M) \longrightarrow \Gamma((\wedge^{p+1,q} T^*M) \oplus (\wedge^{p,q+1} T^*M) \oplus (\wedge^{p-1,q+2} T^*M)).$$