Differentiable manifolds – hand-in sheet 5

Hand in by 18/Jan

For the exercise below you can assume as true all the results from Exercise 13 of chapter 2 regarding the Hodge star operator.

Exercise

In what follows M is a compact manifold of dimension n endowed with a metric and an orientation and hence the Hodge star operator is defined as a bundle map $\star : \wedge^k T^*M \longrightarrow \wedge^{n-k} T^*M$. This allows us to define a symmetric inner product on the space of smooth forms on M via

$$\langle \alpha, \beta \rangle = \int_M \alpha \wedge \star\beta.$$

1) Let d^* be the adjoint of d, i.e., d^* is defined by the equality

$$\langle d\alpha, \beta \rangle = \langle \alpha, d^*\beta \rangle.$$

Show that $d^* = (-1)^k \star^{-1} d\star$ when acting on k-forms.

2) We say that a form α is closed if $d\alpha = 0$ and exact if $\alpha = d\beta$ for some β. Similarly, α is *co-closed* if $d^*\alpha = 0$ and *co-exact* if $\alpha = d^*\beta$ for some β. Finally we say that α is *harmonic* if $(dd^* + d^*d)\alpha = 0$. Show that if α is harmonic, then α is closed and co-closed.

3) Show that if a form α is orthogonal to the space of all exact forms then α is co-closed and that if α is orthogonal to the space of co-exact forms, then α is closed. Similarly, if α is orthogonal to the space of closed forms, then α is co-exact and if α is orthogonal to the co-closed forms then α is exact. Conclude that the space of harmonic forms is orthogonal to the space of exact and to the space of co-exact forms.

4) Show that if two harmonic forms represent the same cohomology class, they must be the same form, i.e., each cohomology class has at most one harmonic form.

Remark: Hodge theorem states that in the situation of this exercise, every cohomology class can be represented by a unique harmonic form. In this exercise we have proved uniqueness. Existence is a little harder.