Differentiable manifolds – Mock Exam 1

Notes:

1. **Write your name and student number *****clearly***** on each page of written solutions you hand in.**
2. You can give solutions in English or Dutch.
3. You are expected to explain your answers.
4. You are **not** allowed to consult any text book, class notes, colleagues, calculators, computers etc.
5. Advice: read all questions first, then start solving the ones you already know how to solve or have good idea on the steps to find a solution. After you have finished the ones you found easier, tackle the harder ones.

1) Let M be the subset of $\mathbb{R}^3$ defined by the equation

$$M = \{(x_1, x_2, x_3) : x_1x_2^2 + x_2x_3^2 + x_3x_1^2 = 1\}.$$

a) Show that M is a smooth submanifold of $\mathbb{R}^3$;

b) Define $\pi : M \longrightarrow \mathbb{R}$; $\pi(x_1, x_2, x_3) = x_1$. Find the critical points and critical values of π.

2) Show that a smooth map $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ can not be injective.

3) Let $M \overset{\varphi}{\longrightarrow} N$ be an embedded submanifold for which $\varphi(M)$ is a closed subset of N. Show that if $X \in \mathfrak{X}(M)$, then there exists a vector field $\tilde{X} \in \mathfrak{X}(N)$ which is φ-related to X. Such $\tilde{X}$ is normally called an *extension* of X to N. Given $X, Y \in \mathfrak{X}(M)$, let $\tilde{X}, \tilde{Y}$ be extensions of X and Y to N. Show that for $p \in \varphi(M)$, $[\tilde{X}, \tilde{Y}](p)$ is tangent to $\varphi(M)$ and depends only on X and Y and not on the particular extensions $\tilde{X}$ and $\tilde{Y}$ chosen.

4) Show that $\mathbb{C}\backslash\{0\}$ with complex multiplication is a Lie group. Show that S^1, the set of complex numbers of norm 1, is also a Lie group.

5) Let $(U_\alpha : \alpha \in A)$ be an open cover of a manifold M and let $f_\alpha : U_\alpha \longrightarrow \mathbb{R}$ be a family of smooth functions such that on $U_\alpha \cap U_\beta$, $f_\alpha - f_\beta$ is constant, for all $\alpha, \beta \in A$. Show that if we define a 1-form ξ on M by declaring that, on U_α, $\xi = df_\alpha$, then ξ is a globally defined 1-form.