Differentiable manifolds – hand-in sheet 2

hand in by: 02/Nov

The orthogonal group

Exercise 1. Let $M_{n\times n}$ be the set of $n \times n$ matrices. Recall that the *group of orthogonal matrices* is

$$O(n) = \{A \in M_{n\times n} | AA^t = Id\},$$

where $\cdot^t$ indicate matrix transposition. In this exercise we prove that $O(n)$ is a Lie group.

1. (1 pt) Let Sym_n be the set of $n \times n$ symmetric matrices, i.e.,

   $$Sym_n = \{A \in M_{n\times n} | A = A^t\}.$$

   Show that Sym_n can be given the structure of a manifold for which it is diffeomorphic to $\mathbb{R}^{\frac{n(n+1)}{2}}$.

2. (4 pt) Show that the map

   $$\varphi : M_{n\times n} \longrightarrow Sym_n; \qquad \varphi(A) = AA^t.$$

   is smooth and that $\mathrm{Id} \in Sym_n$ is a regular value of this map. Hence conclude that $O(n) \subset M_{n\times n}$ is an embedded submanifold. What is the dimension of $O(n)$?

3. (2 pt) Show that $O(n)$ endowed with matrix multiplication is a Lie group, i.e., show that multiplication and inversion are smooth maps on $O(n)$.

4. (3 pt) Describe the tangent space of $O(n)$ at Id.