Notes on Čech cohomology

1 Čech cochains and differential

Čech cohomology is obtained using an open cover of a topological space and is arises using purely combinatorial data. The idea being that if one has information about the open sets that make up a space as well as how these sets are glued together one can deduce global properties of the space from local data.

Let $\mathfrak{U} = \{U_\alpha : \alpha \in A\}$ be an open cover of a connected manifold M. For $\alpha_1, \cdots, \alpha_n \in A$, we denote

$$U_{\alpha_0 \cdots \alpha_k} = U_{\alpha_0} \cap \cdots \cap U_{\alpha_k},$$

or, equivalently, in multi-index notation, if $\mathbf{a} = \{\alpha_0, \cdots, \alpha_k\}$

$$U_{\mathbf{a}} = \bigcap_{\alpha_i \in \mathbf{a}} U_{\alpha_i}.$$

Definition 1. A degree k-*Čech cochain with real coeficients* for the cover $\mathfrak{U}$ is a collection of functions

$$\check{f} := \{f_{\mathbf{a}} | \mathbf{a} \text{ ordered subset of } A \text{ with } k+1 \text{ elements}\} \tag{1}$$

where each $f_{\mathbf{a}} \in \check{f}$ is a *constant real function* (coefficients in $\mathbb{R}$)

$$f_{\mathbf{a}} : U_{\mathbf{a}} \longrightarrow \mathbb{R}.$$

satisfying

$$f_{\alpha_0 \cdots \alpha_i \alpha_{i+1} \cdots \alpha_k} = -f_{\alpha_0 \cdots \alpha_{i+1} \alpha_i \cdots \alpha_k} \tag{skew symmetry}$$

We denote the set of all degree k-Čech cochains with real coefficients obtained from a cover $\mathfrak{U}$ of M by $\check{C}^k(M; \mathbb{R}; \mathfrak{U})$. Note that pointwise addition of real numbers makes $\check{C}^k(M; \mathbb{R}; \mathfrak{U})$ into and abelian group and scalar multiplication gives it the structure of a real vector space.

Notice that according to this definition, and element of $\check{C}^0(M; \mathbb{R}; \mathfrak{U})$ corresponds to the assigment of a constant function to each open set of $\mathfrak{U}$. In particular if the cover $\mathfrak{U}$ is finite, say $\#\mathfrak{U} = n$, then $\check{C}(M; \mathbb{R}; \mathfrak{U}) = \mathbb{R}^n$. Similarly, the elements in $\check{C}^1(M; \mathbb{R}; \mathfrak{U})$ correspond to constant functions defined on overlaps of two sets of $\mathfrak{U}$. Let's see this in a concrete example.

Example 2. Consider S^1 as the interval $[0, 1]$ with the ends identified. We can cover S^1 by the open sets $U_0 = (0, 2/3)$, $U_1 = (1/3, 1)$ and $U_2 = (2/3, 1) \cup (0, 1/3)$. Then $U_{0,1} = (1/3, 2/3)$, $U_{1,2} = (2/3, 1)$ and $U_{2,0} = (0, 1/3)$ and $U_{0,1,2} = \emptyset$. That is, for this open decomposition of S^1 there are only Čech cycles of degree zero and one. An element in $\check{C}^0(M; \mathbb{R}; \mathfrak{U})$ is given by three constants, hence $\check{C}^0(M; \mathbb{R}; \mathfrak{U}) = \mathbb{R}^3$. Similarly, since there are only three double overlaps, $\check{C}^1(M; \mathbb{R}; \mathfrak{U}) = \mathbb{R}^3$.

Definition 3. The *Čech differential* is a linear map $\delta^{k-1} : \check{C}^{k-1}(M; \mathbb{R}; \mathfrak{U}) \longrightarrow \check{C}^k(M; \mathbb{R}; \mathfrak{U})$,

$$\delta^{k-1}(\check{f})_{\alpha_0 \cdots \alpha_k} = \sum_i (-1)^i f_{\alpha_0 \cdots \alpha_{i-1} \alpha_{i+1} \cdots \alpha_k}.$$

In what follows we will denote all maps δ^k defined above simply by δ. The main property of δ is given in the following proposition:

Proposition 4. *The Čech differential satisfies*

$$\delta^2 = 0.$$

Proof. Let $\check{f}$ be a k-cochain. Then

$$(\delta \check{f})_{\alpha_0 \cdots \alpha_{k+1}} = \sum_{i=0}^{k+2} (-1)^i (\check{f})_{\alpha_0 \cdots \alpha_{i-1} \alpha_{i+1} \alpha_{k+1}}$$

Hence

$$\begin{aligned}
(\delta^2 f)_{\alpha_0, \cdots \alpha_{k+2}} &= \sum_{i=0}^{k+2} (-1)^i (\delta \check{f})_{\alpha_0 \cdots \alpha_{i-1} \alpha_{i+1} \alpha_{k+2}} \\
&= \sum_{j<i} (-1)^{i+j} (\check{f})_{\alpha_0 \cdots \alpha_{j-1} \alpha_{j+1} \cdots \alpha_{i-1} \alpha_{i+1} \alpha_{k+2}} \\
&+ \sum_{i<j} (-1)^{i+j-1} (\check{f})_{\alpha_0 \cdots \alpha_{i-1} \alpha_{i+1} \cdots \alpha_{j-1} \alpha_{j+1} \cdots \alpha_{k+2}} \\
= 0
\end{aligned}$$

□

It is standard practice in mathematics that whenever one finds a sequence of linear maps between vector spaces

$$\delta^k : V^{k-1} \longrightarrow V^k$$

with $\delta^k \circ \delta^{k-1} = 0$ one defines *cohomology spaces*:

$$H^k := \frac{\ker(\delta^k)}{\operatorname{Im}(\delta^{k-1})}.$$

In our case, these spaces depend on M and the open cover $\mathfrak{U}$, so we write:

$$\check{H}^k(M; \mathbb{R}; \mathfrak{U}) = \frac{\ker(\delta : \check{C}^k \longrightarrow \check{C}^{k+1})}{\operatorname{Im}(\delta : \check{C}^{k-1} \longrightarrow \check{C}^k)}.$$

Further we say that an element $\check{f} \in \check{C}^k$ is *closed* or a *cocycle* if $\delta \check{f} = 0$. An element $\check{f} \in \check{C}^k$ is *exact* or a *coboundary* if $\check{f}$ is in the image of δ, i.e., there is $\check{g} \in \check{C}^{k-1}$ for which $\delta \check{g} = \check{f}$.

Example 5 (Degree zero Čech cocycles)**.** Let M be a connected manifold and $\mathfrak{U}$ be a locally finite open cover. Next we see that degree zero Čech cohomology is particularly easy to describe. Since $\check{C}^{-1} = \{0\}$, we have

$$\check{H}^0 = \ker(\delta : \check{C}^0 \longrightarrow \check{C}^1)$$

Further, if $\check{f} \in \ker(\delta : \check{C}^0 \longrightarrow \check{C}^1)$, then if U_α intersects U_β we have

$$0 = (\delta \check{f})_{\alpha\beta} = \check{f}_\beta - \check{f}_\alpha,$$

that is $\check{f}_\alpha = \check{f}_\beta$ whenever U_α intersects U_β. Now, for such an $\check{f}$, let $c = \check{f}_\alpha(x)$ for a fixed x in a fixed U_α. Now, if we let $V \subset M$ be the set of points defined by

$$V = \{p \in M : \text{ if } p \in U_\alpha \text{ then } \check{f}_\alpha(p) = c\}.$$

By the cocycle condition and the choice of c we see that $x \in V$, hence $V \neq \emptyset$. Further V is defined by a closed condition, so it is a closed subset of M. Finally, if $p \in V$, let $U_\alpha \in \mathfrak{U}$ be an open set containing p (U_α exists because $\mathfrak{U}$ is a cover). Then $\check{f}_\alpha(p) = c$ and hence, again by the cocycle condition $\check{f}_\beta(p) = c$ whenever $p \in U_\beta$. Hence V is open (by locally finiteness) and since M is connected, $V = M$. That is for all α, $\check{f}_\alpha = c$ and each $\check{f}_\alpha$ is just the restriction of the globally defined function

$$f : M \longrightarrow \mathbb{R}; \qquad f \equiv c$$

to U_α. Or said another way, $\check{f}$ corresponds to the restriction of a globally defined function to the open sets of the cover $\mathfrak{U}$:

$$\check{H}^0 = \{\text{Globally defined constant functions}\}$$

Exercise 6. For the cover of S^1 obtained in Example 2, compute $\check{H}^0$ and $\check{H}^1$.

Now notice that we used very little of the structure of the real numbers and in fact all the argument used above can be carried out for constant functions with values in any abelian group, such as $\mathbb{Z}$, $\mathbb{Z}_n$, S^1, $\mathbb{C}^*$, etc. This way we obtain cohomology groups $\check{H}\bullet(M; G; \mathfrak{U})$ (which are not necessarily vector spaces) for any abelian group G.

We can also relax the condition that the functions $f_{\mathbf{a}}$ are constant. For example we have

Definition 7. A degree *k-Čech cochain with coeficients in the smooth functions* for the cover $\mathfrak{U}$ is a collection of functions

$$\check{f} := \{f_{\mathbf{a}} | \mathbf{a} \text{ ordered subset of } A \text{ with } k+1 \text{ elements}\} \tag{2}$$

where each $f_{\mathbf{a}} \in \check{f}$ is a *smooth real function*

$$f_{\mathbf{a}} : U_{\mathbf{a}} \longrightarrow \mathbb{R}.$$

satisfying

$$f_{\alpha_0 \cdots \alpha_i \alpha_{i+1} \cdots \alpha_k} = -f_{\alpha_0 \cdots \alpha_{i+1} \alpha_i \cdots \alpha_k} \tag{skew symmetry}$$

We denote the set of all degree k-Čech cochains with smooth functions as coefficients obtained from a cover $\mathfrak{U}$ of M by $\check{C}^k(M; C^\infty(M); \mathfrak{U})$. Note that pointwise addition of functions makes $\check{C}^k(M; C^\infty(M); \mathfrak{U})$ into and abelian group and scalar multiplication gives it the structure of a real vector space.

The Čech differential is defined in the same way as before and the same proof still yields $\delta^2 = 0$ hence we also have Čech cohomology with coefficients in the smooth functions.

Exercise 8 (Čech cohomology with coefficients in $C^\infty(M)$)**.** Repeat the argument from Example 5 and conclude that $\check{H}^0(M; C^\infty(M); \mathfrak{U})$ can be identified with the space

$$C^\infty(M) = \{f : M \longrightarrow \mathbb{R} : f \text{ is smooth}\}.$$

Differently from the case of real coefficients, when we consider smooth functions, there is no cohomology in degree higher than zero:

Theorem 9. *For $k > 0$,*

$$\check{H}^k(M; C^\infty(M); \mathfrak{U}) = \{0\}.$$

Equivalently, every closed Čech cochain is a coboundary.

Proof. This theorem is a consequence of the existence of partitions of unity. Indeed, let $\check{f} \in \check{C}^k(M; C^\infty(M); \mathfrak{U})$ be a cocycle and $(\varphi_\alpha : \alpha \in A)$ be a partition of unity subbordinated to $\mathfrak{U}$. Spelling out the cocycle condition we have

$$0 = (\delta \check{f})_{\alpha_0, \cdots \alpha_{k+1}} = \sum_{i=0}^{k+1} (-1)^i \check{f}_{\alpha_0 \cdots \alpha_{i-1} \alpha_{i+1} \cdots \alpha_{k+1}} \qquad \forall \alpha_i \in A.$$

Equivalently,

$$\check{f}_{\alpha_1 \cdots \alpha_{k+1}} = \sum_{i=1}^{k+1} (-1)^{i+1} \check{f}_{\alpha_0 \cdots \alpha_{i-1} \alpha_{i+1} \cdots \alpha_{k+1}} \qquad \forall \alpha_i \in A. \tag{3}$$

Define $\check{g} \in \check{C}^{k-1}(M; C^\infty(M); \mathfrak{U})$ by

$$\check{g}_{\alpha_1, \cdots \alpha_k} = \sum_{\alpha \in A} \varphi_\alpha f_{\alpha \alpha_1 \cdots \alpha_k}.$$

Notice that even though $f_{\alpha\alpha_1\cdots\alpha_k}$ is only defined on $U_{\alpha\alpha_1\cdots\alpha_k}$, since φ_α has compact support in U_α, $\varphi_\alpha f_{\alpha\alpha_1\cdots\alpha_k}$ can be extended to $U_{\alpha_1\cdots\alpha_k}$ by declaring that it vanishes on $U_{\alpha_1\cdots\alpha_k}\backslash U_{\alpha\alpha_1\cdots\alpha_k}$ so $\check{g}_{\alpha_1\cdots\alpha_k}$ defined above is indeed a smooth function on $U_{\alpha_1\cdots\alpha_k}$.

Now we compute

$$
\begin{aligned}
(\delta\check{g})_{\alpha_1,\cdots,\alpha_{k+1}} &= \sum_{i=1}^{k+1}(-1)^{i+1}\check{g}_{\alpha_1\cdots\alpha_{i-1}\alpha_{i+1}\cdots\alpha_{k+1}} \\
&= \sum_{i=1}^{k+1}(-1)^{i+1}\sum_{\alpha\in A}\varphi_\alpha\check{f}_{\alpha\alpha_1\cdots\alpha_{i-1}\alpha_{i+1}\cdots\alpha_{k+1}} \\
&= \sum_{\alpha\in A}\varphi_\alpha\sum_{i=1}^{k+1}(-1)^{i+1}\check{f}_{\alpha\alpha_1\cdots\alpha_{i-1}\alpha_{i+1}\cdots\alpha_{k+1}} \\
&= \sum_{\alpha\in A}\varphi_\alpha\check{f}_{\alpha_1\cdots\alpha_{k+1}} \\
&= (\sum_{\alpha\in A}\varphi_\alpha)\check{f}_{\alpha_1\cdots\alpha_{k+1}} \\
&= \check{f}_{\alpha_1\cdots\alpha_{k+1}},
\end{aligned}
$$

where in the first equality we wrote the definition of Čech differential, in the second we used the definition of $\check{g}$, in the third we commuted the sums, in the fourth we used equation (3), in the fifth we notice that the term $\check{f}_{\alpha_1\cdots\alpha_{k+1}}$ does not depend on the index of summation, hence can be put in evidence and in the last equation we used again that α_α is a partition of unity. □

Exercise 10. If the multi-indices are mind boggling, repeat this argument above in the case $f \in \check{C}^2(M;C^\infty(M),\mathfrak{U})$ to convince yourself that everything is fine.

As a consequence of this theorem, we see that the Čech cohomology $\check{H}^\bullet(M;C^\infty(M),\mathfrak{U})$ are rather simple to describe. Indeed, according to Example 5 and Exercise 8, $\check{H}^0(M;C^\infty(M),\mathfrak{U})$ corresponds to the vector space of globally defined functions and the remaning groups $\check{H}^k(M;C^\infty(M),\mathfrak{U})$ are all trivial for $k>0$. Note that these equalities hold for any locally finite cover, that is these groups are independent of $\mathfrak{U}$, hence in this case it makes sense to write simply $\check{H}^\bullet(M;C^\infty(M))$.