Differentiable manifolds – homework 4

Solve exercises 9 and 10 from Warner.

Exercise 1. Let $E \xrightarrow{\pi} M$ be a rank k vector bundle over M and let $\sigma_1, \cdots, \sigma_k : M \longrightarrow E$ be sections such that $\{\sigma_1(p), \cdots, \sigma_k(p)\}$ is a linearly independent set of E_p. Show that E is isomorphic to the trivial bundle $M \times \mathbb{R}^k$, i.e., there is a diffeomorphism

$$\Phi : E \longrightarrow M \times \mathbb{R}^k$$

such that $\Phi : E_p \longrightarrow p \times \mathbb{R}^k$ and this map is linear.

Definition 2. Let $E \xrightarrow{\pi} M$ be a vector bundle over M. A degree k *Čech cochain with coefficients in* $\Gamma(E)$ for the cover $\mathfrak{U}$ is a collection of functions

$$\check{f} := \{f_{\mathbf{a}} | \mathbf{a} \text{ ordered subset of } A \text{ with } k+1 \text{ elements}\} \tag{1}$$

where each $f_{\mathbf{a}} \in \check{f}$ is a *smooth section of E over U_{a}* (coefficients in $\Gamma(E)$) satisfying

$$f_{\alpha_0 \cdots \alpha_i \alpha_{i+1} \cdots \alpha_k} = -f_{\alpha_0 \cdots \alpha_{i+1} \alpha_i \cdots \alpha_k} \qquad \text{(skew symmetry)}$$

We denote the set of all degree k Čech cochains with coefficients in $\Gamma(E)$ obtained from a cover $\mathfrak{U}$ of M by $\check{C}^k(M; \Gamma(E); \mathfrak{U})$. We defined the Čech differential using the same way expression we used for Čech cohomology with real coefficients.

Exercise 3. Show that

1. $\check{H}^0(M; \Gamma(E); \mathfrak{U}) = \Gamma(E)$.
2. $\check{H}^i(M; \Gamma(E); \mathfrak{U}) = \{0\}$ for $i > 0$.

Exercise 4. Let (U, φ) and (V, ψ) be two charts on a manifold such that $U \cap V \neq \emptyset$. Let $(x_1, \cdots, x_n)$ be the coordinates relative to φ and $(y_1, \cdots, y_n)$ be the coordinates relative to ψ. Show that

$$dx_i = \sum_j \frac{\partial x_i}{\partial y_j} dy_j.$$

Exercise 5. Let $\varphi : M \longrightarrow N$ be a smooth map and $f : N \longrightarrow R$ be a smooth function. Show that

$$\varphi^*(df) = d(\varphi^* f) = d(f \circ \varphi).$$

Exercise 6. Given $\alpha \in \Omega^1(M)$ and $p \in M$, show that there is a function $f \in \Omega^0(M)$ such that $df|_p = \alpha|_p$. Show that one may not be able to find f such that $df = \alpha$ in a neighborhoodof p.

New exercises regarding the material from Lecture 1:

Exercise 7. The (real) projective space, $\mathbb{R}P^n$ is the set of all lines in $\mathbb{R}^{n+1}$ passing through the origin. This can be equivalently defined as the quotient of $\mathbb{R}^{n+1} \backslash \{0\}$ by the equivalence relation $x \equiv y$ if and only if there is $\lambda \in \mathbb{R}^*$ such that $x = \lambda y$.

Give $\mathbb{R}P^n$ the structure of a manifold.

Exercise 8. The (complex) projective space, $\mathbb{C}P^n$ is the set of all (complex) lines in $\mathbb{C}^{n+1}$ passing through the origin. This can be equivalently defined as the quotient of $\mathbb{C}^{n+1} \backslash \{0\}$ by the equivalence relation $x \equiv y$ if and only if there is $\lambda \in \mathbb{C}^*$ such that $x = \lambda y$.

Give $\mathbb{C}P^n$ the structure of a complex manifold.