Differentiable manifolds – homework 5

Solve exercise 11 from Warner.

Exercise 1. Consider the circle S^1 as the interval $[0, 1]$ with ends identified. Let $\mathfrak{U}$ be the cover of S^1 given by the sets $U_0 = (0, 2/3)$, $U_1 = (1/3, 1)$ and $U_2 = (2/3, 1] \cup [0, 1/3)$. Compute the Čech cohomology of S^1 with coefficients in smooth functions with values in $\mathbb{R}^*$, $\check{H}^\bullet(S^1; C^\infty(M; \mathbb{R}^*); \mathfrak{U})$, for this cover. (Hint read Example 2.7 in the notes on Čech cohomology)

Exercise 2. Let M and N be manifolds, $f : M \longrightarrow N$ be smooth and let $E = N \times \mathbb{R}^k$ be the trivial rank k vector bundle over N. Show that f^*E is the trivial rank k vector bundle over M. Conversely, conclude that if $E \xrightarrow{\pi} N$ is a vector bundle and $f^*E \xrightarrow{\tilde{\pi}} M$ is not trivial, then E is not isomorphic to the trivial bundle over N.

Exercise 3. Last exercise sheet, we showed that $\mathbb{R}P^n$, the set of lines through the origin in $\mathbb{R}^{n+1}$ can be given a manifold structure.

- Show that $\mathbb{R}P^1$ is diffeomorphic to the circle S^1.
- Consider the subset
$$\tau \subset \mathbb{R}P^1 \times \mathbb{R}^2; \qquad \tau = \{(l, v) : v \in l\}.$$
Show that τ is a manifold by giving coordinate charts.
- Let $\pi : \tau \longrightarrow \mathbb{R}P^1$ be the projection onto the first factor, $\pi(l, v) = l$. Show that the map π makes τ into a line bundle over $\mathbb{R}P^1$.
- Show that τ is not isomorphic to the trivial line bundle.

Exercise 4. Now we repeat the previous exercise in higher dimensions. Consider
$$\tau_n \subset \mathbb{R}P^n \times R^{n+1}; \qquad \tau_n = \{(l, v) : v \in l\}.$$

- Show that τ_n is a manifold by giving coordinate charts.;
- Let $\pi : \tau_n \longrightarrow \mathbb{R}P^1$ be the projection onto the first factor, $\pi(l, v) = l$. Show that the map π makes τ_n into a line bundle over $\mathbb{R}P^n$.
- Show that τ_n is not isomorphic to the trivial line bundle.

Hint: For the last part, use the map $f : \mathbb{R}P^1 \longrightarrow \mathbb{R}P^n$, $f([x, y]) = [x, y, 0, \cdots, 0]$ and show that $\tau = f^*\tau_n$. Then use the last two exercises to arrive at the conclusion.