Differentiable manifolds – exercise sheet 1

Exercise 1. Let X be the set of all points $(x, y) \in \mathbb{R}^2$ with $y = \pm 1$ and let M be the quotient of X by the equivalence relation generated by $(x, 1) \sim (x, -1)$ for $x \neq 0$. Show that with the induced topology, M is second countable and locally homeomorphic to $\mathbb{R}$ but not Hausdorff.

Exercise 2. Show that the disjoint union of an uncountable number of copies of $\mathbb{R}$ is locally Euclidean, Hausdorff but not second countable.

Exercise 3. The usual differentiable structure on $\mathbb{R}$ was obtained by taking the maximal atlas, $\mathcal{F}$, containing the identity map. Let $\mathcal{F}_1$ be the maximal atlas containing the map $t \mapsto t^3$. Prove that $\mathcal{F} \neq \mathcal{F}_1$ but that $(\mathbb{R}, \mathcal{F})$ is diffeomorphic to $(\mathbb{R}, \mathcal{F}_1)$.

Exercise 4. Show that

1. The composition of diffeomophisms is a diffeomorphism;
2. The inverse of a diffeomorphism is a diffeomorphism;
3. If $\varphi_i : M_i \longrightarrow N_i$ are diffeomorphisms (for $i = 1, \cdots, n$), we can define

$$\varphi : M_1 \times \cdots \times M_n \longrightarrow N_1 \times \cdots \times N_n, \qquad \varphi(p_1, \cdots, p_n) = (\varphi_1(p_1), \cdots, \varphi_n(p_n)).$$

Show that φ is a diffeomorphism.

Exercise 5. Using stereographic projection, show that the n-sphere

$$S^n = \{x \in \mathbb{R}^{n+1} : |x| = 1\}$$

is a smooth manifold.

Exercise 6. Identifying $\mathbb{R}^2 = \mathbb{C}$ use the following variation of stereographic projection, show that the 2-sphere is a smooth manifold:

$$\varphi_1 : S^2 \backslash \{(0,0,1)\} \longrightarrow \mathbb{C}, \qquad \varphi_1(x, y, z) = \frac{x + iy}{1 - z};$$

$$\varphi_2 : S^2 \backslash \{(0,0,-1)\} \longrightarrow \mathbb{C}, \qquad \varphi_1(x, y, z) = \frac{x - iy}{1 + z}.$$

Further show that the transition function $\varphi_1 \circ \varphi_2^{-1} : \mathbb{C}^* \longrightarrow \mathbb{C}^*$ is holomorphic.

Exercise 7 (Real projective space)**.** The real projective space $\mathbb{R}P^2$ is the set of all lines through the origin in $\mathbb{R}^3$. Argue that this is the same set as the sphere S^2 with antipodal points identified. Endow $\mathbb{R}P^2$ with the structure of a differentiable manifold. **Hint**: (useful parametrization) denote the line passing through (x_1, x_2, x_3) by $[x_1, x_2, x_3]$ and then consider the sets $U_i = \{[x_1, x_2, x_3] : x_i \neq 0\}$, $i = 1, 2, 3$. The three sets U_i cover $\mathbb{R}P^2$ and a point any point in, say, U_1 has a unique representative of the form $[1, x_2, x_3]$. Using this, compute the transition functions.

Exercise 8. The real projective space $\mathbb{R}P^n$ is the set of all lines through the origin in $\mathbb{R}^{n+1}$. Endow $\mathbb{R}P^n$ with the structure of a differentiable manifold. 1

Exercise 9 (Complex projective space)**.** The real projective space $\mathbb{C}P^n$ is the set of all complex lines through the origin in $\mathbb{C}^{n+1}$, i.e.,

$$\mathbb{C}P^n = \{(z_1, \cdots, z_{n+1}) \in \mathbb{C}^{n+1} \backslash \{0\}\} / \sim$$

where $(z_1, \cdots, z_{n+1}) \sim (w_1, \cdots, w_{n+1})$ if there is $\lambda \in \mathbb{C}^*$ such that

$$(z_1, \cdots, z_{n+1}) = \lambda(w_1, \cdots, w_{n+1}).$$

Endow $\mathbb{C}P^n$ with the structure of a differentiable manifold. **Hint**: (useful parametrization) denote the line passing through $(z_1, \cdots, z_{n+1})$ by $[z_1, \cdots, z_{n+1}]$ and then consider the sets $U_i = \{[z_1, \cdots, z_{n+1}] : z_i \neq 0\}$, $i = 1, \cdots, n+1$. The $n+1$ sets U_i cover $\mathbb{C}P^1$. Then compute the transition functions.

Exercise 10. Compare the parametrization of $\mathbb{C}P^1$ obtained in Exercise 9 with the parametrization of S^2 obtained in Exercise 6. What can you conclude?