Differentiable manifolds – exercise sheet 10

Exercise 1. Let M be a compact manifold. Show that every vector field on M is complete.

Exercise 2. Let X be a vector field on a manifold M and let $p \in M$ be a point where X vanishes (i.e., $X_p = 0$). Show that the path $\gamma : \mathbb{R} \longrightarrow M$ given by $\gamma(t) = p$ is an integral curve of X.

Exercise 3. Let $\varphi : M \longrightarrow N$ be a embedding whose image is a closed subset of N and let $X, Y \in \mathfrak{X}(M)$ be vector fields. Show that there are vector fields $\tilde{X}, \tilde{Y} \in \mathfrak{X}(N)$ such that for every $p \in M$ $\varphi_*|_p(X_p) = \tilde{X}_{\varphi(p)}$ and similarly for Y and $\tilde{Y}$. The vector fields $\tilde{X}$ and $\tilde{Y}$ are normally referred to as *extensions* of X and Y to N.

Show that for $q \in \mathrm{Im}(\varphi)$, $[\tilde{X}, \tilde{Y}]_q$ only depends on X and Y and not on the particular extensions chosen. Precisely, if $q = \varphi(p)$, show that $[\tilde{X}, \tilde{Y}]|_{\varphi(p)} = \varphi_*|_p([X, Y]|_p)$.

Exercise 4. Let $X \in \mathfrak{X}(M)$ be a vector field on M, let $f : M \longrightarrow \mathbb{R}$ be a smooth function and let $p \in M$. Let e^{tX} be the time t flow of X so that

$$e^{tX}(p)$$

is a path on M defined for t small enough. By letting t vary and keeping p fixed, consider the following path on $\mathbb{R}$:

$$\gamma : I \longrightarrow \mathbb{R}; \qquad \gamma(t) = f(e^{tX}(p)).$$

Show that $\frac{d}{dt}\gamma\big|_{t=0} = \mathcal{L}_X f|_p$.

Exercise 5. Let $X, Y \in \mathfrak{X}(M)$ be vector fields on M and let $p \in M$. Let e^{tX} be the time t flow of X so that

$$e^{tX}_* : T_pM \longrightarrow T_{e^{tX}p}M.$$

By letting t vary and keeping p fixed, consider the following path on T_pM:

$$\gamma : I \longrightarrow T_pM; \qquad \gamma(t) = (e^{tX}_*)^{-1} Y_{e^{tX}p}.$$

Show that $\frac{d}{dt}\gamma\big|_{t=0} = [X, Y]_p$.

Exercise 6. Let $X \in \mathfrak{X}(M)$ be a vector field on M, let $\xi \in \Omega^1(M)$ be a 1-form and let $p \in M$. Let e^{tX} be the time t flow of X so that

$$(e^{tX})^* : T^*_{e^{tX}p}M \longrightarrow T^*_pM.$$

By letting t vary and keeping p fixed, consider the following path on T^*_pM:

$$\gamma : I \longrightarrow T_pM; \qquad \gamma(t) = (e^{tX})^*\xi|_{e^{tX}p}.$$

Define the Lie derivative of ξ in the direction of X to be

$$\mathcal{L}_X\xi = \frac{d}{dt}\gamma|_{t=0}.$$

Show that for $f \in \Omega^0(M)$ then $\mathcal{L}_X df = d\mathcal{L}_X f$.