Differentiable manifolds – exercise sheet 2

Exercise 1. Let $m \geq n$. Show that the set of $m \times n$ matrices of maximal rank is a manifold.

Exercise 2. Let M and N be smooth manifolds and $p \in M$ and $q \in N$. Show that $M \times N$ has a natural structure of manifold for which the following maps are smooth

$$\pi_1 : M \times N \longrightarrow M, \qquad \pi_1(x, y) = x;$$

$$\pi_2 : M \times N \longrightarrow N, \qquad \pi_2(x, y) = y;$$

$$\iota_q : M \longrightarrow M \times N; \quad \iota_q(x) = (x, q);$$

$$\iota_p : N \longrightarrow M \times N; \quad \iota_p(y) = (p, y).$$

Exercise 3. Identifying the circle S^1 with the complex numbers of length 1, show that the 2- torus $T^2 = S^1 \times S^1$ is a manifold and that the map

$$\pi : \mathbb{R}^2 \longrightarrow T^2, \qquad \pi(x, y) = (e^{2\pi i x}, e^{2\pi i y})$$

is a smooth surjection which is a local diffeomorphism.

Exercise 4. Identify the circle S^1 with the complex numbers of length 1 and let $n \in \mathbb{Z}$. Show that the map $z \mapsto z^n$ is smooth.

Exercise 5. Let S^n be the n-sphere. Show that the map

$$\varphi : S^n \longrightarrow S^n, \qquad \varphi(x) = -x$$

is smooth.

Exercise 6. Read the section of the book that proves the existence of partitions of unity on smooth manifolds .

Exercise 7. Show that $C^\infty(M)$ is an infinite dimensional vector space.