Differentiable manifolds – exercise sheet 3

Whenever necessary, you can assume that the Čech cohomology $\check{H}^k(M;\mathbb{R};\mathfrak{U})$ is independent of $\mathfrak{U}$ as long as $\mathfrak{U}$ is a good cover of M.

Exercise 1. Show that if $\mathfrak{U}$ is a good cover of M and $\mathfrak{V}$ isa good cover of N, then

$$\mathfrak{U}\times\mathfrak{V} = \{U\times V : U\in\mathfrak{U} \text{ and } V\in\mathfrak{V}\}$$

is a good cover of $M\times N$.

Exercise 2 (Homeomorphism invariance)**.** Let $f: M\longrightarrow N$ be continuous and surjective and let $\mathfrak{U} = \{U_\alpha : \alpha\in A\}$ be an open cover of N. Show that

$$f^{-1}(\mathfrak{U}) = \{f^{-1}(U_\alpha) : \alpha\in A\}$$

is an open cover of M and define a map

$$f^*: \check{C}^k(N;\mathbb{R};\mathfrak{U})\longrightarrow \check{C}^k(M;\mathbb{R};f^{-1}(\mathfrak{U})),\qquad (f^*c)_{\mathrm{a}} = c_{\mathrm{a}}\circ f,$$

where $c\in\check{C}^k(N;\mathbb{R};\mathfrak{U})$ and a is an ordered multiindex of size $k+1$, so that $c_{\mathrm{a}}: U_{\mathrm{a}}\longrightarrow G$.

Show that f^* defined above is an isomorphism of Abelian groups for every k and that it commutes with differentials, that is,

$$f^*\delta = \delta f^*.$$

Conclude that the Čech cohomologies of M and N with respect to the covers $\mathfrak{U}$ and $f^{-1}(\mathfrak{U})$ are isomorphic. Conclude further that if f is an homeomorphism, then M and N have isomorphic Čech cohomologies with respect to any good cover of these manifolds.

Remark: In fact the exercise above shows that if $f: M\longrightarrow N$ is smooth, surjective and $f^{-1}(\mathfrak{U})$ is a good cover of M for some good cover of N, then the cohomologies of M and N are isomorphic. An example where one can use this more general statement is with the map

$$f:\mathbb{C}^*\longrightarrow S^1,\qquad f(z) = \frac{z}{|z|}.$$

Exercise 3 (Euler characteristic)**.** Let $\{V^k : 0\leq k\leq n\}$ be a family of finite dimensional vector spaces where $n\in\mathbb{N}$ is some fixed number. Whenever necessary, let $V_{-1} = V_{n+1} = \{0\}$. Let $d_k: V^k\longrightarrow V^{k+1}$ be linear maps such that $d_{k+1}\circ d_k = 0$ for all i and define

$$H^k = \frac{\ker(d_k)}{\mathrm{Im}(d_{k-1})}.$$

Show that

$$\sum(-1)^k\dim(V^k) = \sum(-1)^k\dim(H^k).$$

Conclude that if $\mathfrak{U}$ is a finite open cover of a manifold then

$$\sum(-1)^k\dim(\check{C}^k(M;\mathbb{R};\mathfrak{U})) = \sum(-1)^k\dim(\check{H}^k(M;\mathbb{R};\mathfrak{U})).$$

Hint: Use the rank nullity theorem from linear algebra, namely, if $A: V\longrightarrow W$ is a linear map,

$$\dim(V) = \dim(\mathrm{Im}(A)) + \dim(\ker(A)).$$

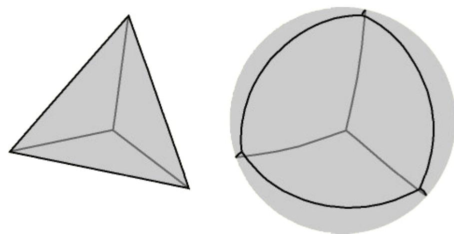

Figure 1: Tetrahedral decomposition of the sphere.

Definition 4. For a cover $\mathfrak{U}$ of M, the *Euler characteristic of M with respect to the cover* $\mathfrak{U}$ is the number

$$\chi(M;\mathfrak{U}) = \sum(-1)^k \dim(\check{H}^k(M;\mathbb{R};\mathfrak{U}))$$

The *Euler characteristic of M*, denoted by $\chi(M)$, is $\chi(M;\mathfrak{U})$ where $\mathfrak{U}$ is any finite good cover of M.

Exercise 5. Cover the sphere S^2 with four open sets obtained by slightly enlarging the tetrahedral triagulation of the sphere (see Figure 1). Compute the Euler characteristic of S^2 with respect to this open cover.

Exercise 6. Consider S^1 as the interval $[0,1]$ with the ends identified. Cover S^1 by the open sets $U_0 = (0, 2/3)$, $U_1 = (1/3, 1)$ and $U_2 = (2/3, 1) \cup (0, 1/3)$. Compute the Euler characteristic of S^1 from this cover.

Exercise 7. Assuming that the Čech cohomology $\check{H}^k(M;\mathbb{R};\mathfrak{U})$ is independent of $\mathfrak{U}$ as long as $\mathfrak{U}$ is a good cover of M show that S^2 is not diffeomorphic to $S^1 \times S^1$ (first you will need to find a good covr for $S^1 \times S^1$).