Differentiable manifolds – exercise sheet 7

Exercise 1. Let $f : S^2 \subset \mathbb{R}^3 \longrightarrow \mathbb{R}$ be defined by

$$f(x, y, z) = xyz.$$

Find the critical points and the critical values of f.

Exercise 2. Compute the transition functions of the bundle $TS^2 \longrightarrow S^2$ obtained by parametrizing S^2 using stereographic projection.

Exercise 3. Let $f : M \longrightarrow N$ be smooth and let $E \xrightarrow{\pi} N$ be a vector bundle. Show that if E is trivial, then f^*E is trivial.

Exercise 4. Recall from last exercise sheet that you produced a nontrivial line bundle over S^1. Show that there are only two line bundles over S^1.

Exercise 5. Let $E \longrightarrow S^1$ be the nontrivial line bundle over S^1.

1. Show that E has a section s_1 which vanishes only at $1 \in S^1$ and a section s_2 which vanishes only at $-1 \in S^1$.
2. Let $E \oplus E \longrightarrow S^1$ be the Whitney sum of E with itself and define a map

$$\varphi : S^1 \times \mathbb{R}^2 \longrightarrow E \oplus E, \qquad \varphi(x, a, b) = (as_1(x) + bs_2(x), as_2(x) - bs_1(x)).$$

   Show that φ is a vector bundle isomorphism. That is $E \oplus E$ is isomorphic to the trivial bundle of rank 2.

Exercise 6. Let $E \longrightarrow M$ be a vector bundle over M and $\mathfrak{U}$ be a good cover of M. Show that $\pi^{-1}(\mathfrak{U})$ is a good cover of E. What can you say about $\check{H}^i(E; \mathbb{R}; \pi^{-1}(\mathfrak{U}))$? Conclude that cohomology of the total space can not distinguish between different vector bundles over M.

Exercise 7 (Čech cohomology and lines bundles)**.** Let $\mathfrak{U} = \{U_\alpha : \alpha \in A\}$ be a good cover of a manifold M. For the rest of the exercise, we let $\pi : E \longrightarrow M$ be a rank 1 real vector bundle (i.e. a line bundle) over M.

1. Show that a choice of local nonvanishing sections s_α over U_α (for each α) gives isomorphisms

$$\Phi_\alpha : \pi^{-1}(U_\alpha) \longrightarrow U_\alpha \times \mathbb{R}.$$

2. Define the transition functions g_α^β for this collection of Φ_α by

$$\Phi_\beta \circ \Phi_\alpha^{-1} : U_{\alpha\beta} \times \mathbb{R} \longrightarrow U_{\alpha\beta} \times \mathbb{R}$$

$$\Phi_\beta \circ \Phi_\alpha^{-1}(x, v) = (x, g_\beta^\alpha(x)v) \qquad g_\beta^\alpha(x) \in Gl(1; \mathbb{R}) = \mathbb{R}^*.$$

   Show that the collection $\check{g} = \{g_\beta^\alpha : \alpha, \beta \in A\}$ forms a degree 1 Čech cochain with coefficients in the smooth functions with values in the abelian group $\mathbb{R}^*$.

3. Show that $\delta\check{g} = 0$.
4. Show that if we choose different nonvanishing sections σ_α of E over U_α and run the same argument above with s_α replaced by σ_α, the Čech cocyle $\check{g}$ changes by a coboundary: $\check{g} + \delta\check{f}$, with $\check{f} \in \check{C}^0(M, \mathbb{R}^*)$, hence the cohomology class $[\check{g}] \in \check{H}^1(M; C^\infty(M; \mathbb{R}^*); \mathfrak{U})$ does not depend on the choices made.
5. Conversely, argue that given a cohomology class $[\check{g}] \in \check{H}^1(M; C^\infty(M; \mathbb{R}^*); \mathfrak{U})$, any representative $\check{g} = \{g_\beta^\alpha : \alpha, \beta \in A\}$ of that class can be used to construct a line bundle for which the procedure above associates to the class $[\check{g}]$.