Differentiable manifolds – exercise sheet 8

Exercise 1. Consider the manifold $M = \mathbb{R}$. Show that every 1-form $\alpha \in \Omega^1(\mathbb{R})$ is exact, i.e., is in the image of the map $d : \Omega^0(\mathbb{R}) \longrightarrow \Omega^1(\mathbb{R})$.

Exercise 2. Is there a 1-form $\alpha \in \Omega^1(S^1)$ which is not exact?

Exercise 3. Given a function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$, compute $df \in \Omega^1(\mathbb{R}^2)$. Using the identity,

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

give an example of a 1-form $\alpha \in \Omega^1(\mathbb{R}^2)$ which is not exact (even locally).

Exercise 4. Let M be a manifold, $E \xrightarrow{\pi} M$ be a vector bundle over M and $\langle \cdot, \cdot \rangle$ be a Riemannian inner product on E (see definition 7 and exercise 8 in sheet 6). Then, for each $v \in E_p$, we have that $\langle v, \cdot \rangle$ is an element of E_p^*, defined by the property

$$\langle v, \cdot \rangle : w \in E_p \mapsto \langle v, w \rangle.$$

This gives a map $E \longrightarrow E^*$, $v \in E_p \mapsto \langle v, \cdot \rangle \in E_p^*$. Show that this map is a bundle isomorphism.

Exercise 5. Let $\varphi : M \longrightarrow N$ and $f : N \longrightarrow \mathbb{R}$ be smooth functions. Show that $\varphi^*(df) = d(\varphi^* f)$.

Exercise 6. Read the sections on line integrals and conservative covector fields.