GENERALIZED KÄHLER AND HYPER-KÄHLER QUOTIENTS

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ABSTRACT. We develop a theory of reduction for generalized Kähler and hyper-Kähler structures which uses the generalized Riemannian metric in an essential way, and which is not described with reference solely to a single generalized complex structure. We show that our construction specializes to the usual theory of Kähler and hyper-Kähler reduction, and it gives a way to view usual hyper-Kähler quotients in terms of generalized Kähler reduction.

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INTRODUCTION

Generalized geometrical structures, such as Dirac structures [4] and generalized complex or Kähler structures [5, 7], are similar to their classical counterparts, i.e. integrable tangent distributions, complex or Kähler structures, except that they operate not on the tangent bundle but on the direct sum $TM \oplus T^*M$, or more generally, on an exact Courant algebroid E which is an extension of TM by T^*M.

In the presence of an action on the underlying manifold by a Lie group G, one may ask whether such generalized geometries *reduce* to a suitable quotient space. To address this question, we developed in [1] a theory of reduction for exact Courant algebroids and their associated generalized geometrical structures, based on the idea of *extending* the G-action on TM to an

action on the Courant algebroid E. This is particularly interesting since E has symmetries beyond the usual diffeomorphisms; there are also the *B-field* transformations, well known to physicists. See [10, 13, 18, 19] for related work.

In this article, we provide a streamlined approach to the reduction procedures described in [1], focusing on the useful special case where the extended G-action is defined by a bracket-preserving map $\widetilde{\psi} : \mathfrak{g} \to \Gamma(E)$ and an equivariant map $\mu : M \to \mathfrak{h}^*$ for $\mathfrak{h}$ a $\mathfrak{g}$-module. The map μ is called a *moment map*, and is not associated with any geometrical structure but rather to the extended action itself. We phrase the constructions in this paper explicitly in terms of the data $(\widetilde{\psi}, \mathfrak{h}, \mu)$, showing that specific choices lead to well-known reductions.

The main construction in this paper is a reduction procedure for generalized complex structures which goes beyond the natural generalization of holomorphic or symplectic quotients developed in [1]. For this, we require a G-invariant generalized Riemannian metric $\mathcal{G}$, compatible with the generalized complex structure $\mathcal{J}$, and such that the Hermitian structure $(\mathcal{J}, \mathcal{G})$ is compatible with the G-action in a suitable sense. In particular, we obtain new reduction procedures for generalized Kähler and generalized hyper-Kähler structures.

Finally, we prove that this generalized Hermitian reduction does specialize to the usual Kähler and hyper-Kähler reductions, and that it commutes with the forgetful functors taking hyper-Kähler geometry to generalized Kähler geometry.

In a sequel to this work [2], we apply the theory developed here to several infinite-dimensional quotients, including the generalized (hyper-) Kähler structure on the moduli space of instantons on a generalized (hyper-) Kähler 4-manifold first obtained by Hitchin [8] (see [16] for the hyper-Kähler case), as well as the generalized Kähler structures on certain Lie groups obtained in [5].

1. Generalized geometry and Courant algebroids

1.1. **Geometry of $TM \oplus T^*M$.** Given a manifold M of dimension m, the direct sum $TM \oplus T^*M$ is equipped with the following canonical structures: a fiberwise inner product of signature (m, m) given by

$$(1) \qquad \langle X + \xi, Y + \eta \rangle := \eta(X) + \xi(Y), \quad X + \xi, Y + \eta \in \Gamma(TM \oplus T^*M),$$

and the *Courant bracket* [4] on smooth sections of $TM \oplus T^*M$, defined by

$$(2) \qquad [\![X + \xi, Y + \eta]\!] := [X, Y] + \mathcal{L}_X \eta - i_Y d\xi.$$

A central idea in "generalized geometry" [5, 7] is that $TM \oplus T^*M$, together with the operations (1) and (2), should be thought of as a "generalized tangent bundle" to M. This leads to a unified view of various geometrical structures on M, as we now briefly recall.

A *Dirac structure* [4] on M is a Lagrangian subbundle $L \subset TM \oplus T^*M$ (i.e., $L = L^\perp$, where $L^\perp$ is the orthogonal complement of L with respect to the pairing (1)) whose space of sections is closed under the Courant bracket:

$$(3) \qquad [\![\Gamma(L), \Gamma(L)]\!] \subseteq \Gamma(L).$$

Examples of Lagrangian subbundles of $TM \oplus T^*M$ include the graphs of 2-forms $\omega : TM \to T^*M$ and bivector fields $\pi : T^*M \to TM$. In these cases the integrability condition (3) amounts to $d\omega = 0$ and π being a Poisson bivector field; other examples can be found in [4]. All these definitions clearly carry over to the complexified bundle $(TM \oplus T^*M) \otimes \mathbb{C}$, leading to *complex Dirac structures*.

We now consider a special class of complex Dirac structures. A *generalized complex structure* [5, 7] on M is a complex structure $\mathcal{J}$ on the bundle $TM \oplus T^*M$ which is orthogonal with respect to (1), and such that the $+i$-eigenbundle $L \subset (TM \oplus T^*M) \otimes \mathbb{C}$ satisfies the integrability condition (3). The orthogonality of $\mathcal{J}$ implies that $L = L^\perp$, so L is a complex Dirac structure also satisfying

$$(4) \qquad\qquad L \cap \overline{L} = \{0\}.$$

Conversely, any complex Dirac structure on M satisfying (4) uniquely determines a generalized complex structure on M. Examples of generalized complex structures include those determined by complex structures $I : TM \to TM$ and symplectic structures $\omega : TM \to T^*M$, namely

$$\mathcal{J}_I = \begin{pmatrix} -I & 0 \\ 0 & I^* \end{pmatrix}, \quad \mathcal{J}_\omega = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}.$$

One can similarly define generalized versions of Riemannian metrics, Hermitian structures, as well as Kähler and hyper-Kähler structures [5], see Sections 4.2 and 4.3.

Interpreting geometries on M as structures on $TM \oplus T^*M$ introduces two important features. First of all, one can easily adapt definitions in order to incorporate geometrical structures which are "twisted" by a closed 3-form on M: given $H \in \Omega^3_{cl}(M)$, one replaces the Courant bracket (2) by the *H-twisted Courant bracket* [17]

$$(5) \qquad [\![X + \xi, Y + \eta]\!]_H = [X, Y] + \mathcal{L}_X \eta - i_Y d\xi + i_Y i_X H,$$

and changes the integrability condition (3) accordingly; this leads to *H-twisted Dirac structures*, *H-twisted generalized complex structures* and so on. Second, there is an action of the additive group of closed 2-forms $\Omega^2_{cl}(M)$ on $TM \oplus T^*M$ preserving the operations (1), (2), (5), given by

$$X + \xi \mapsto X + \xi + i_X B,$$

for $B \in \Omega^2_{cl}(M)$. As a result, structures on $TM \oplus T^*M$, such as generalized complex structures, inherit these extra symmetries, known as *B-field* transformations.

1.2. **Exact Courant algebroids.** An axiomatization of the properties of the Courant bracket on $TM \oplus T^*M$ leads to the general notion of a Courant algebroid, introduced in [14]. This more intrinsic approach to the Courant bracket will play an important role in the context of reduction.

A *Courant algebroid* over a manifold M is a vector bundle $E \to M$ equipped with a fibrewise nondegenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$, a bilinear bracket $[\![\cdot, \cdot]\!]$ on the smooth sections $\Gamma(E)$, and a bundle map $\pi : E \to TM$ (called the *anchor*), such that, for all $e_1, e_2, e_3 \in \Gamma(E)$ and $f \in C^\infty(M)$, the following properties are satisfied:

C1) $[\![e_1, [\![e_2, e_3]\!]]\!] = [\![[\![e_1, e_2]\!], e_3]\!] + [\![e_2, [\![e_1, e_3]\!]]\!]$,
C2) $[\![e_1, f e_2]\!] = f[\![e_1, e_2]\!] + (\mathcal{L}_{\pi(e_1)} f) e_2$,
C3) $\mathcal{L}_{\pi(e_1)} \langle e_2, e_3 \rangle = \langle [\![e_1, e_2]\!], e_3 \rangle + \langle e_2, [\![e_1, e_3]\!] \rangle$,
C4) $\pi([\![e_1, e_2]\!]) = [\pi(e_1), \pi(e_2)]$,
C5) $[\![e_1, e_1]\!] = \frac{1}{2} \pi^* d \langle e_1, e_1 \rangle$,

where in (C5) we identify $E \cong E^*$ via $\langle \cdot, \cdot \rangle$ in order to view π^* as taking values in E. The model example of a Courant algebroid is $TM \oplus T^*M$, with pairing (1), anchor given by the canonical projection $TM \oplus T^*M \to TM$, and bracket given by the H-twisted Courant bracket (5). It is straightforward to extend the concepts of Dirac structures, generalized complex structures etc. to general Courant algebroids.

In the definition of a Courant algebroid, properties C1)–C4) express natural compatibility conditions between the anchor π, the bracket $[\![\cdot, \cdot]\!]$ and the pairing $\langle \cdot, \cdot \rangle$ that will be further

discussed in Section 2.1. Property C5), on the other hand, prevents the bracket $[\![\cdot,\cdot]\!]$ from being skew-symmetric, and it implies that $\pi \circ \pi^* = 0$, so we have a chain complex

$$(6) \qquad 0 \longrightarrow T^*M \xrightarrow{\pi^*} E \xrightarrow{\pi} TM \longrightarrow 0.$$

In this paper we will restrict our attention to *exact Courant algebroids*, i.e., those for which the sequence (6) is exact. In this case, we always identify T^*M with a subspace of E via π^*.

Given an exact Courant algebroid $E \to M$, we can always choose a right splitting $\nabla : TM \to E$ of (6) which is *isotropic*, i.e., whose image in E is isotropic with respect to $\langle\cdot,\cdot\rangle$. Each such ∇ defines a "curvature" 3-form $H \in \Omega^3_{cl}(M)$ by

$$(7) \qquad H(X,Y,Z) := \langle[\![\nabla(X),\nabla(Y)]\!],\nabla(Z)\rangle, \quad \text{for } X,Y,Z \in \Gamma(TM).$$

Under the vector bundle isomorphism $\nabla + \pi^* : TM \oplus T^*M \to E$, the Courant algebroid structure on E is identified with the usual Courant algebroid structure on $TM \oplus T^*M$ defined by the H-twisted Courant bracket (5).

Remark: Exact Courant algebroids were first studied by P. Ševera, who classified them by noticing that the choice of a different isotropic splitting of (6) modifies H by an exact 3-form. As a result, the cohomology class $[H] \in H^3(M,\mathbb{R})$ is independent of the splitting and completely determines the exact Courant algebroid E up to isomorphism. We call $[H]$ the *Ševera class* of E.

2. Extended actions on Courant algebroids

In this section we briefly review the notion of *extended action*, introduced in [1] for the purpose of describing the reduction of Courant algebroids.

2.1. **Infinitesimal actions.** The action of a Lie group G on a manifold M may be described infinitesimally as a Lie algebra homomorphism $\mathfrak{g} \longrightarrow \Gamma(TM)$. The definition of an extended action on a Courant algebroid E is analogous, with E playing the role of TM.

Recall that an *infinitesimal automorphism* of a vector bundle E is a pair (F,X), where $X \in \Gamma(TM)$ and $F : \Gamma(E) \to \Gamma(E)$ satisfies

$$(8) \qquad F(fe) = fF(e) + (\mathcal{L}_X f)e, \quad e \in \Gamma(E), f \in C^\infty(M).$$

Infinitesimal bundle automorphisms form a Lie algebra with respect to the bracket

$$[(F_1, X_1),(F_2,X_2)] := (F_1F_2 - F_2F_1, [X_1,X_2]).$$

If E is a Courant algebroid, then its Lie algebra of symmetries, denoted by $\mathbf{sym}(E)$, consists of infinitesimal bundle automorphism (F,X) which preserve the bracket $[\![\cdot,\cdot]\!]$, the pairing $\langle\cdot,\cdot\rangle$, and the anchor $\pi : E \to TM$:

$$F([\![e_1,e_2]\!]) = [\![F(e_1),e_2]\!] + [\![e_1,F(e_2)]\!],$$
$$\mathcal{L}_X\langle e_1,e_2\rangle = \langle F(e_1),e_2\rangle + \langle e_1,F(e_2)\rangle,$$
$$\pi \circ F = \mathcal{L}_X \circ \pi,$$

where $e_1,e_2 \in \Gamma(E)$. Given a section $e \in \Gamma(E)$, we observe that axioms C1)–C4) in the definition of a Courant algebroid imply that the pair $(F = [\![e,\cdot]\!], X = \pi(e))$ is in $\mathbf{sym}(E)$. As a result, we obtain a map:

$$(9) \qquad \mathrm{ad} : \Gamma(E) \longrightarrow \mathbf{sym}(E), \quad e \mapsto ([\![e,\cdot]\!],\pi(e)).$$

The elements of $\mathbf{sym}(E)$ in the image of ad are called *inner symmetries* of E. Note that the map (9) extends the usual identification of vector fields $X \in \Gamma(TM)$ with infinitesimal symmetries of the Lie bracket on TM:

$$(10) \qquad \Gamma(TM) \longrightarrow \mathbf{sym}(TM), \quad X \mapsto ([X, \cdot], X).$$

It is important to note, however, that although (10) is an isomorphism, the map (9) is neither injective nor surjective in general.

Given a Lie algebra $\mathfrak{g}$, an equivariant structure on E preserving its Courant algebroid structure is defined infinitesimally by a Lie algebra homomorphism $\mathfrak{g} \to \mathbf{sym}(E)$. The particular situation that will concern us in this paper is that of $\mathfrak{g}$-actions by inner symmetries, i.e. compositions

$$(11) \qquad \mathfrak{g} \xrightarrow{\Psi} \Gamma(E) \xrightarrow{\mathrm{ad}} \mathbf{sym}(E).$$

Observe that, since the Courant bracket on $\Gamma(E)$ is *not* a Lie bracket, it is natural to replace the Lie algebra $\mathfrak{g}$ in (11) by a more general structure with "Courant-type" bracket. We call these *Courant algebras* [1] and describe them below.

2.2. **Courant algebras.** A *Courant algebra* over a Lie algebra $\mathfrak{g}$ is a Leibniz algebra $\mathfrak{a}$ [15] together with a bracket-preserving map $\pi : \mathfrak{a} \longrightarrow \mathfrak{g}$. In other words, $\mathfrak{a}$ is a vector space endowed with a bilinear bracket $[\![\cdot, \cdot]\!] : \mathfrak{a} \times \mathfrak{a} \longrightarrow \mathfrak{a}$ such that

$$[\![a_1, [\![a_2, a_3]\!]]\!] = [\![[\![a_1, a_2]\!], a_3]\!] + [\![a_2, [\![a_1, a_3]\!]]\!], \quad a_1, a_2, a_3 \in \mathfrak{a},$$

and $\pi([\![a_1, a_2]\!]) = [\pi(a_1), \pi(a_2)]$ for all $a_1, a_2 \in \mathfrak{a}$. A Courant algebra is *exact* if π is surjective and $[\![w_1, w_2]\!] = 0$ for all $w_i \in \ker(\pi)$. In this paper we only consider exact Courant algebras. Morphisms of Courant algebras are defined in a natural way.

It is clear that if E is a Courant algebroid, then $\Gamma(E)$ is a Courant algebra over the Lie algebra of vector fields, and is exact if and only if E is.

If $\mathfrak{a} \longrightarrow \mathfrak{g}$ is an exact Courant algebra, then $\mathfrak{h} = \ker(\pi)$ automatically acquires a $\mathfrak{g}$-module structure: the action of $u \in \mathfrak{g}$ on $w \in \mathfrak{h}$ is

$$u \cdot w := [\![\tilde{u}, w]\!],$$

where $\tilde{u}$ is any element of $\mathfrak{a}$ such that $\pi(\tilde{u}) = u$. Since the bracket vanishes on $\mathfrak{h}$, it easily follows that this is a well defined map $\mathfrak{g} \times \mathfrak{h} \longrightarrow \mathfrak{h}$ defining a $\mathfrak{g}$-action.

The following example of an exact Courant algebra will be central in this paper.

Example 2.1 (Hemisemidirect product [1, 11])**.** If $\mathfrak{g}$ is a Lie algebra and $\mathfrak{h}$ is a $\mathfrak{g}$-module, then we can endow $\mathfrak{a} := \mathfrak{g} \oplus \mathfrak{h}$ with the structure of an exact Courant algebra by taking $\pi : \mathfrak{g} \oplus \mathfrak{h} \longrightarrow \mathfrak{g}$ to be the natural projection and defining

$$(12) \qquad [\![(u_1, w_1), (u_2, w_2)]\!] := ([u_1, u_2], u_1 \cdot w_2).$$

It is clear that $\pi : \mathfrak{a} \longrightarrow \mathfrak{g}$ is surjective and preserves brackets, and that $[\![\mathfrak{h}, \mathfrak{h}]\!] = 0$. Finally, c1) is a consequence of the Jacobi identity for $\mathfrak{g}$ and the fact that $\mathfrak{h}$ is a $\mathfrak{g}$-module. This Courant algebra first appeared in [11], where it was studied in the context of Leibniz algebras.

2.3. **Extended actions.** Let E be an exact Courant algebroid over M. We will now show how one can produce an infinitesimal action $\mathfrak{g} \to \mathbf{sym}(E)$ starting from an exact Courant algebra

$\mathfrak{h} \to \mathfrak{a} \xrightarrow{\pi} \mathfrak{g}$, and a Courant algebra morphism

$$
\begin{array}{ccc}
\mathfrak{a} & \longrightarrow & \mathfrak{g} \\
\downarrow{\scriptstyle \Psi} & & \downarrow{\scriptstyle \psi} \\
\Gamma(E) & \longrightarrow & \Gamma(TM).
\end{array}
$$

We will denote a Courant algebra morphism simply by $\Psi : \mathfrak{a} \to \Gamma(E)$, keeping in mind that it always projects to an action on M, denoted by $\psi : \mathfrak{g} \to \Gamma(TM)$. It also follows from the definitions that $\Psi(\mathfrak{h}) \subseteq \Omega^1(M)$.

Composing the map (9) with Ψ, we get a map

$$
(13) \qquad \mathrm{ad} \circ \Psi : \mathfrak{a} \longrightarrow \mathbf{sym}(E).
$$

It is important to note that, unlike the map (10), the map $\mathrm{ad} : \Gamma(E) \to \mathbf{sym}(E)$ has a nontrivial kernel: by identifying E with $TM \oplus T^*M$ through the choice of an isotropic splitting, one can directly check that $\ker(\mathrm{ad}) = \Omega^1_{cl}(M)$. Hence if Ψ maps $\mathfrak{h}$ into closed 1-forms, $\Psi(\mathfrak{h}) \subseteq \Omega^1_{cl}(M)$, the map (13) factors through $\mathfrak{a} \xrightarrow{\pi} \mathfrak{g}$. As a result, there is an induced infinitesimal $\mathfrak{g}$-action

$$
(14) \qquad \mathfrak{g} \longrightarrow \mathbf{sym}(E),
$$

as desired.

We are then led to the following definitions. Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. An *extended* $\mathfrak{g}$-*action* on an exact Courant algebroid E over M is a Courant algebra morphism $\Psi : \mathfrak{a} \to \Gamma(E)$ from an exact Courant algebra $\mathfrak{h} \to \mathfrak{a} \xrightarrow{\pi} \mathfrak{g}$ into $\Gamma(E)$ so that $\Psi(\mathfrak{h}) \subseteq \Omega^1_{cl}(M)$. We call it an *extended G-action* if the induced $\mathfrak{g}$-action (14) on E integrates to a G-action. In particular, an extended G-action on E makes it into an equivariant G-bundle, with G acting by Courant algebroid automorphisms.

Remark: Suppose that $\Psi : \mathfrak{a} \to \Gamma(E)$ is an extended $\mathfrak{g}$-action for which the projected action $\psi : \mathfrak{g} \to \Gamma(TM)$ integrates to a global G-action on M. In this case, a sufficient condition ensuring that this data defines an extended G-action on E (and not only an extended action of a cover of G) is the existence of a $\mathfrak{g}$-invariant isotropic splitting for E (these always exist, e.g., if G is compact), see [1, Sec. 2]. Indeed, such a splitting gives an identification of E with the Courant algebroid $(TM \oplus T^*M, \langle \cdot, \cdot \rangle, [\![\cdot, \cdot]\!]_H)$ satisfying

$$
(15) \qquad i_{X_a} H = d\xi_a, \qquad \text{for all } a \in \mathfrak{a},
$$

where $\Psi(a) = X_a + \xi_a$. Since M is a G-manifold, $TM \oplus T^*M$ is naturally a G-equivariant bundle, and condition (15) exactly says that this canonical G-action on $TM \oplus T^*M$ coincides infinitesimally with the one induced by Ψ under the identification $E \cong TM \oplus T^*M$.

Any Lie algebra $\mathfrak{g}$ can be thought of as a Courant algebra over itself in a trivial way, with the projection $\pi : \mathfrak{g} \to \mathfrak{g}$ given by the identity. An extended action of this Courant algebra on an exact Courant algebroid E,

$$
\begin{array}{ccc}
\mathfrak{g} & \xrightarrow{\ \mathrm{Id}\ } & \mathfrak{g} \\
\downarrow{\scriptstyle \widetilde{\psi}} & & \downarrow{\scriptstyle \psi} \\
\Gamma(E) & \xrightarrow{\ \pi\ } & \Gamma(TM),
\end{array}
$$

is called a *lifted (or trivially extended) action* on E, and denoted by $\widetilde{\psi} : \mathfrak{g} \to \Gamma(E)$. A trivial example is of course when $E = TM \oplus T^*M$, with $H = 0$, and $\widetilde{\psi} = \psi$ is an ordinary action.

In this paper, we will be only concerned with lifted actions for which the image of $\widetilde{\psi} : \mathfrak{g} \to E$ is *isotropic*, i.e., the pairing $\langle \cdot, \cdot \rangle$ vanishes on elements $\widetilde{\psi}(u)$.

Example 2.2 (Lifted actions). Take E to be the Courant algebroid $TM \oplus T^*M$, with $H = 0$, let $\psi : \mathfrak{g} \to \Gamma(TM)$ be an action on M, and $\nu : M \to \mathfrak{g}^*$ be an equivariant map. Then

$$\widetilde{\psi}(u) := \psi(u) + d\langle \nu, u \rangle, \quad u \in \mathfrak{g},$$

is a lifted action on E. This observation is a special case of the following general fact, proven in [1, Sec. 2]: if we start with an action ψ on M, and assume that E admits an invariant splitting, with associated 3-form curvature H, then the problem of finding an isotropic lifted action $\widetilde{\psi}$ extending ψ is equivalent to finding a closed equivariant extension of H in the Cartan model.

2.4. **Moment maps for extended actions.** Let $\mathfrak{h} \to \mathfrak{a} \to \mathfrak{g}$ be an exact Courant algebra, and let $\Psi : \mathfrak{a} \to \Gamma(E)$ be an extended $\mathfrak{g}$-action on an exact Courant algebroid E over M. Recall that this implies that $\Psi(\mathfrak{a}) \subset \Omega^1_{cl}(M)$, and that $\mathfrak{h}$ is a $\mathfrak{g}$-module. Let us equip $\mathfrak{h}^*$ with the dual $\mathfrak{g}$-action. A *moment map* for the extended action Ψ is a $\mathfrak{g}$-equivariant map $\mu : M \longrightarrow \mathfrak{h}^*$ such that, for each $w \in \mathfrak{h}$,

$$\Psi(w) = d\langle \mu, w \rangle.$$

We now describe how to use equivariant maps to further extend lifted actions:

Proposition 2.3. *Let $\widetilde{\psi} : \mathfrak{g} \to \Gamma(E)$ be an isotropic lifted $\mathfrak{g}$-action on an exact Courant algebroid E, $\mathfrak{h}$ be a $\mathfrak{g}$-module, and $\mu : M \to \mathfrak{h}^*$ be an equivariant map. Then the map $\Psi : \mathfrak{g} \oplus \mathfrak{h} \to \Gamma(E)$,*

(16) $$\Psi(u, w) = \widetilde{\psi}(u) + d\langle \mu, w \rangle,$$

defines an extended $\mathfrak{g}$-action of the hemisemidirect product $\mathfrak{a} = \mathfrak{g} \oplus \mathfrak{h}$ on E with moment map μ. Moreover, the image $\Psi(\mathfrak{a}) \subseteq E$ is isotropic over $\mu^{-1}(0)$.

Proof. Let Ψ be defined as in (16). Then, using that $[\![\xi, \cdot]\!] = 0$ if $\xi \in \Omega^1_{cl}(M)$, we get

$$\begin{aligned}
[\![\Psi(u_1, w_1), \Psi(u_2, w_2)]\!] &= [\![\widetilde{\psi}(u_1), \widetilde{\psi}(u_2)]\!] + [\![\widetilde{\psi}(u_1), d\langle \mu, w_2 \rangle]\!] \\
&= \widetilde{\psi}([u_1, u_2]) + \mathcal{L}_{\psi(u_1)} d\langle \mu, w_2 \rangle \\
&= \Psi([u_1, u_2], u_1 \cdot w_2),
\end{aligned}$$

where for the last equality we used the equivariance of μ. Comparing with (12), we conclude that Ψ preserves brackets. So it defines a Courant algebra morphism. It is also clear that $\Psi(\mathfrak{h}) \subseteq \Omega^1_{cl}(M)$, hence Ψ is an extended $\mathfrak{g}$-action.

Let us now consider the pairing $\langle \Psi(u_1, w_1), \Psi(u_2, w_2) \rangle$ over $\mu^{-1}(0)$. Using that $\widetilde{\psi}$ is isotropic, i.e., $\langle \widetilde{\psi}(u_1), \widetilde{\psi}(u_2) \rangle = 0$, we get

$$\begin{aligned}
\langle \Psi(u_1, w_1), \Psi(u_2, w_2) \rangle &= \langle \widetilde{\psi}(u_1), d\langle \mu, w_2 \rangle \rangle + \langle \widetilde{\psi}(u_2), d\langle \mu, w_1 \rangle \rangle \\
&= \mathcal{L}_{\psi(u_1)} \langle \mu, w_2 \rangle + \mathcal{L}_{\psi(u_2)} \langle \mu, w_1 \rangle \\
&= \langle \mu, u_1 \cdot w_2 \rangle + \langle \mu, u_2 \cdot w_1 \rangle,
\end{aligned}$$

which clearly vanishes on points $x \in M$ where $\mu(x) = 0$. $\qquad\square$

Remark: Note that if $\widetilde{\psi}$ is a lifted G-action for a Lie group G, then Ψ is an extended G-action.

3. Reduction of exact Courant algebroids

In this section, we review the reduction procedure for exact Courant algebroids introduced in [1], giving special attention to the case described in Prop. 2.3, i.e. a lifted action extended by an equivariant map.

3.1. **The general procedure.** Let G be a connected Lie group, $\mathfrak{h} \to \mathfrak{a} \to \mathfrak{g}$ be an exact Courant algebra and $\Psi : \mathfrak{a} \to \Gamma(E)$ be an extended G-action on an exact Courant algebroid E. We consider the distribution

$$K = \Psi(\mathfrak{a}) \subset E$$

given by the image of the bundle map $\mathfrak{a} \times M \to E$ associated with Ψ.

Our goal is to use the extended action Ψ to produce a "smaller" Courant algebroid. To this end, let us suppose that $P \hookrightarrow M$ is a G-invariant submanifold of M such that

(1) $K|_P$ is isotropic;
(2) $TP = \pi(K^\perp)$.

One can directly check that $\pi(K^\perp) = \mathrm{Ann}(\Psi(\mathfrak{h}))$, so the last condition can be re-written as

$$(17) \qquad\qquad TP = \mathrm{Ann}(\Psi(\mathfrak{h}))|_P.$$

Let us assume the following regularity conditions: $K|_P$ is a vector bundle over P and the G-action on P is free and proper. As observed in [1, Sec. 3.1], it follows from the fact that Ψ is a Courant algebra morphism that the vector bundles $K|_P$ and $K^\perp|_P$ are G-equivariant subbundles of $E|_P$. It is then proven in [1, Thm. 3.3] that the vector bundle

$$(18) \qquad\qquad E^{red} := \left. \frac{K^\perp|_P}{K|_P} \right/ G$$

defines an exact Courant algebroid over the manifold P/G, called the *reduced Courant algebroid*.

Remarks:

a) As shown in [1], the assumption that $K|_P$ is isotropic is not necessary, and this is relevant for some examples. Note also that $K|_P$ is a vector bundle if and only if the distribution $\Psi(\mathfrak{h}) \subset E$ has constant rank over P.
b) The reduced Courant bracket on E^{red} is obtained canonically, by noticing that the restriction of the bracket on E to the space $\Gamma(K^\perp)^G$ of G-invariant sections of $K^\perp$ is well-defined modulo $\Gamma(K)^G$, and $\Gamma(K)^G$ is an ideal in $\Gamma(K^\perp)^G$.
c) Although the reduced Courant algebroid E^{red} is exact, it may not have a canonical splitting. Nevertheless, one can still describe the Ševera class of E^{red}, see [1, Sec. 3.2].

3.2. **A special case of moment map reduction.** Let E be an exact Courant algebroid over M. We will now specialize the reduction procedure for actions arising as in Prop. 2.3.

Definition 3.1. Let $\widetilde{\psi} : \mathfrak{g} \to \Gamma(E)$ be an isotropic lifted G-action on E and let $\mu : M \to \mathfrak{h}^*$ be a G-equivariant map, for $\mathfrak{h}$ some $\mathfrak{g}$-module. Assume that 0 is a regular value of μ, and that the G-action on $\mu^{-1}(0)$ is free and proper. We refer to the triple $(\widetilde{\psi}, \mathfrak{h}, \mu)$ as *reduction data*.

It follows from Prop. 2.3 that reduction data define an extended G-action Ψ of the hemisemidirect product $\mathfrak{g} \oplus \mathfrak{h}$ on E, with image

$$(19) \qquad\qquad K = \{\widetilde{\psi}(u) + d\langle \mu, w \rangle, \ u \in \mathfrak{g}, w \in \mathfrak{h}\} \subseteq E.$$

Proposition 3.2. *Let E be an exact Courant algebroid over M, and let $(\widetilde{\psi}, \mathfrak{h}, \mu)$ be reduction data. Then $K|_{\mu^{-1}(0)}$ is an equivariant G-bundle over $\mu^{-1}(0)$, and the quotient vector bundle*

$$E^{red} := \left. \frac{K^{\perp}|_{\mu^{-1}(0)}}{K|_{\mu^{-1}(0)}} \right/ G$$

defines an exact Courant algebroid over $\mu^{-1}(0)/G$.

Proof. Note that $\mathrm{Ann}(\Psi(\mathfrak{h}))|_{\mu^{-1}(0)} = \mathrm{Ann}(d\mu)|_{\mu^{-1}(0)} = T(\mu^{-1}(0))$, so (17) holds for $P = \mu^{-1}(0)$. The fact that $K|_{\mu^{-1}(0)}$ is isotropic follows from Prop. 2.3. Using that the G-action on $\mu^{-1}(0)$ is free and that $\Psi(\mathfrak{h})|_{\mu^{-1}(0)}$ is a bundle, one concludes that $K|_{\mu^{-1}(0)}$ is a vector bundle. The result now follows from the construction outlined in Section 3.1. $\square$

Example 3.3. Let $(TM \oplus T^*M, \langle \cdot, \cdot \rangle, [\![\cdot, \cdot]\!]_H)$ be the Courant algebroid E. If the closed 3-form H is basic with respect to a G-action on M, then we may regard this G-action as a lifted action on E, i.e., $\widetilde{\psi} = \psi : \mathfrak{g} \to \Gamma(TM) \subset \Gamma(E)$. In this case, for any choice of $\mathfrak{g}$-module $\mathfrak{h}$ and equivariant map $\mu : M \to \mathfrak{h}^*$, the reduced Courant algebroid is naturally split, as follows. Since $K = \psi(\mathfrak{g}) \oplus d\langle \mu, \mathfrak{h} \rangle$ and $K^{\perp} = T\mu^{-1}(0) \oplus \mathrm{Ann}(\psi(\mathfrak{g}))$, we have

$$E^{red} = \left. \frac{T\mu^{-1}(0)}{\psi(\mathfrak{g})} \oplus \frac{\mathrm{Ann}(\psi(\mathfrak{g}))}{d\langle \mu, \mathfrak{h} \rangle} \right/ G = TM^{red} \oplus T^*M^{red},$$

where $M^{red} = \mu^{-1}(0)/G$. The curvature 3-form of E^{red} with respect to this natural splitting is just the pushdown of the (basic) 3-form ι^*H, where $\iota : \mu^{-1}(0) \hookrightarrow M$ is the inclusion.

This example has two simple extreme cases: if we take $\mathfrak{h} = \{0\}$, then $E^{red} = T(M/G) \oplus T^*(M/G)$, whereas if we pick $\mathfrak{g} = \{0\}$, then $E^{red} = T\mu^{-1}(0) \oplus T^*\mu^{-1}(0)$.

Remarks:

    a) In general, the reduced Courant algebroid E^{red} will not be canonically split, but one can construct splittings by choosing connections for the G-bundle $\mu^{-1}(0) \longrightarrow \mu^{-1}(0)/G$, see [1, Sec. 3] for a detailed discussion.

    b) A description of the Ševera class of a reduced Courant algebroid is presented in [1, Sec. 3.2], including explicit examples of a trivial Ševera class that is reduced to a non-trivial one.

4. REDUCTION OF GEOMETRICAL STRUCTURES

In this section we explain how various geometrical structures can be transported to the reduced Courant algebroid in the presence of an extended action. For simplicity, we restrict our attention to the special case of extended actions determined by the reduction data $(\widetilde{\psi}, \mathfrak{h}, \mu)$, as in section 3.2.

4.1. Reduction of Dirac and generalized complex structures.
As in Section 3.2, E is an exact Courant algebroid over M, $(\widetilde{\psi}, \mathfrak{h}, \mu)$ defines reduction data, K is given by (19), and the reduced Courant algebroid E^{red} over $M^{red} = \mu^{-1}(0)/G$ is given in Prop. 3.2.

Suppose that $L \subset E$ is a G-invariant Dirac structure, defining a G-equivariant subbundle of E. (Infinitesimally, this means that $[\![\widetilde{\psi}(u), \Gamma(L)]\!] \subseteq \Gamma(L)$.) Consider the following distribution of E^{red}:

$$(20) \qquad\qquad L^{red} := \left. \frac{(L \cap K^{\perp} + K)|_{\mu^{-1}(0)}}{K|_{\mu^{-1}(0)}} \right/ G.$$

One can directly verify that $L^{red\perp} = L^{red}$. So, at each point, (20) defines a Lagrangian subspace of E^{red}. However, the distribution L^{red} may not form a smooth vector bundle over M^{red}. (A sufficient, but not necessary, condition is that $L \cap K|_{\mu^{-1}(0)}$ has constant rank.) If it does, then it is shown in [1, Sec. 4.1] that L^{red} is automatically integrable with respect to the reduced Courant bracket, and hence defines a Dirac structure in E^{red}. The same construction holds for complex Dirac structures if we replace K by its complexification $K \otimes \mathbb{C} \subseteq E \otimes \mathbb{C}$ in (20), yielding a reduced complex Dirac structure in $E^{red} \otimes \mathbb{C}$.

If now $\mathcal{J}$ is a G-invariant generalized complex structure on E, then its $+i$-eigenbundle L is a G-invariant complex Dirac structure in $E \otimes \mathbb{C}$. So one may attempt to transport $\mathcal{J}$ to the reduced Courant algebroid E^{red} by reducing this Dirac structure as in (20). Assuming that L^{red} is a smooth bundle over M^{red}, then it defines a reduced generalized complex structure $\mathcal{J}^{red}$ in E^{red} if and only if it satisfies the extra condition

$$(21) \qquad\qquad L^{red} \cap \overline{L^{red}} = \{0\}.$$

As observed in [1, Sec. 5.1], this last condition can be equivalently expressed in terms of the operator $\mathcal{J}$ as follows: over each point in $\mu^{-1}(0)$, we must have

$$(22) \qquad\qquad \mathcal{J}K \cap K^{\perp} \subset K.$$

A particularly simple condition implying both the smoothness of L^{red} and condition (22) is

$$(23) \qquad\qquad \mathcal{J}K = K \quad \text{over } \mu^{-1}(0),$$

as discussed in [1, Sec. 5]. We summarize the discussion by citing the following result:

Theorem 4.1 ([1], Thm. 5.2). *Let $\mathcal{J}$ be a generalized complex structure on the exact Courant algebroid E and let $(\widetilde{\psi}, \mathfrak{h}, \mu)$ be reduction data. If $\mathcal{J}$ is G-invariant and satisfies (23), then L^{red} defines a reduced generalized complex structure $\mathcal{J}^{red}$ on E^{red}.*

We now illustrate this reduction procedure with two simple examples, in which the Courant algebroid E is taken to be $TM \oplus T^*M$, with $H = 0$.

Example 4.2 (Hamiltonian reduction). Consider a hamiltonian G-manifold (M, ω), with action $\psi : \mathfrak{g} \to \Gamma(TM)$ and moment map $\mu : M \to \mathfrak{g}^*$. We can describe Hamiltonian reduction as generalized reduction as follows: the triple $(\psi, \mathfrak{g}, \mu)$ defines reduction data, and, according to Example 3.3, the reduced Courant algebroid is $E^{red} = TM^{red} \oplus T^*M^{red}$, where $M^{red} = \mu^{-1}(0)/G$. Viewing ω as a generalized complex structure $\mathcal{J}_\omega$, it follows from the *moment map condition*

$$(24) \qquad\qquad i_{\psi(u)}\omega = d\langle \mu, u \rangle, \quad u \in \mathfrak{g}$$

that the reduction data and the generalized complex structure are related by

$$(25) \qquad\qquad \mathcal{J}_\omega d\langle \mu, u \rangle = \psi(u),$$

and this immediately implies condition (23). So we can carry out generalized reduction to obtain $\mathcal{J}_\omega^{red} = \mathcal{J}_{\omega_{red}}$, where ω_{red} is the reduced symplectic form on M^{red} obtained by Marsden-Weinstein reduction.

Remark: The compatibility (25) of the previous example can be generalized as follows: instead of ordinary actions ψ, one can consider more general lifted actions, for example those of the form $\widetilde{\psi}(u) = \psi(u) + d\langle \nu, u \rangle$, where $\nu : M \to \mathfrak{g}^*$ is equivariant (see Example 2.2); one can also consider more general $\mathfrak{g}$-modules $\mathfrak{h}$, and then impose the condition

$$\mathcal{J}d\langle \mu, u \rangle = \widetilde{\psi}(u) = \psi(u) + d\langle \nu, u \rangle,$$

which directly implies (23). If $\mathfrak{h} = \mathfrak{g}$, these are the actions studied in [10, 13] (where the map $\mu + i\nu$ is called the moment map). For the examples of interest in this paper, we will need $\mathfrak{g} \neq \mathfrak{h}$ and a weaker version of (23) to hold (c.f. Example 4.8).

Example 4.3 (Holomorphic quotients). Suppose that a complex group G acts holomorphically on a complex manifold (M, I). We now consider the reduction data $(\psi, \mathfrak{h} = \{0\}, \mu = 0)$. In this case, the reduced Courant algebroid is $T(M/G) \oplus T^*(M/G)$. On the other hand, condition (23) is a direct consequence of the fact that the action is holomorphic: $\psi(u + iv) = \psi(u) + I\psi(v)$, $u + iv \in \mathfrak{g}$. Carrying out generalized reduction for $\mathcal{J}_I$, we obtain $\mathcal{J}_I^{red} = \mathcal{J}_{I^{red}}$, where I^{red} is the quotient complex structure on M/G.

The framework of reduction described in this section is general enough to include more exotic examples (see e.g. [1, Sec. 5.2]), such as the case of a lifted action preserving a symplectic structure whose reduction is a complex structure.

4.2. **Generalized Hermitian reduction.** We now describe a situation in which a generalized complex structure may have a natural reduction even when condition (23) fails to hold. While we state our results for extended actions defined by reduction data $(\widetilde{\psi}, \mathfrak{h}, \mu)$, this is not essential; more general extended actions may be treated in the same way.

A *generalized (Riemannian) metric* on a Courant algebroid E is an orthogonal, self-adjoint bundle automorphism $\mathcal{G} : E \longrightarrow E$ satisfying

$$\langle \mathcal{G}e, e \rangle > 0, \quad \forall e \neq 0.$$

A generalized metric is compatible with a generalized complex structure $\mathcal{J}$ if they commute. The pair $(\mathcal{J}, \mathcal{G})$ is then called a *generalized Hermitian structure*.

If $K \subset E$ and $\mathcal{G}$ is a generalized metric on E, then we define

$$(26) \qquad K^{\mathcal{G}} := \mathcal{G}K^{\perp} \cap K^{\perp},$$

which is the $\mathcal{G}$-orthogonal complement of K in $K^{\perp}$.

Theorem 4.4 (Generalized Hermitian reduction). *Let E be an exact Courant algebroid over M, with reduction data $(\widetilde{\psi}, \mathfrak{h}, \mu)$. Suppose that E is equipped with a G-invariant generalized Hermitian structure $(\mathcal{J}, \mathcal{G})$. If*

$$(27) \qquad \mathcal{J}K^{\mathcal{G}} = K^{\mathcal{G}} \quad over \ \mu^{-1}(0),$$

then $\mathcal{J}$ can be reduced to E^{red}, and $\mathcal{G}$ induces a compatible generalized metric on E^{red}.

Proof. The proof follows closely the ideas in [1, Sec. 6.1]. Let us first notice that condition (27) implies (22): Since $\mathcal{J}$ is orthogonal, we see that

$$\langle \mathcal{J}K, K^{\mathcal{G}} \rangle = \langle K, K^{\mathcal{G}} \rangle = 0,$$

hence $\mathcal{J}K \subset K^{\mathcal{G}\perp}$ and $\mathcal{J}K \cap K^{\perp} \subset K^{\mathcal{G}\perp} \cap K^{\perp} = K$. This shows that the Dirac reduction L^{red} of the i-eigenbundle of $\mathcal{J}$ satisfies (21), hence it defines a reduced generalized complex structure as long as it is a smooth bundle, a fact which we now verify.

Using the $\mathcal{G}$-orthogonal decomposition $K^{\perp} = K \oplus K^{\mathcal{G}}$ over $\mu^{-1}(0)$, we obtain an identification of vector bundles

$$K^{\mathcal{G}}|_{\mu^{-1}(0)}/G \cong E^{red}.$$

Since $K^{\mathcal{G}}$ is $\mathcal{J}$ invariant, this identification induces a generalized almost complex structure $\mathcal{J}^{red}$ on E^{red}, whose $+i$-eigenbundle agrees with the reduced Dirac structure L^{red}. This implies that

L^{red} is smooth, and hence integrable, and that $\mathcal{J}^{red}$ is the generalized complex structure associated to it.

Since $\mathcal{G}(K^{\mathcal{G}}) = K^{\mathcal{G}}$, we can also transport the generalized metric $\mathcal{G}$ to a generalized metric $\mathcal{G}^{red}$ on E^{red}, and since $\mathcal{G}$ and $\mathcal{J}$ commute pointwise, the same holds for their restrictions to $K^{\mathcal{G}}$. Thus the reduced metric and generalized complex structure are compatible. $\qquad\square$
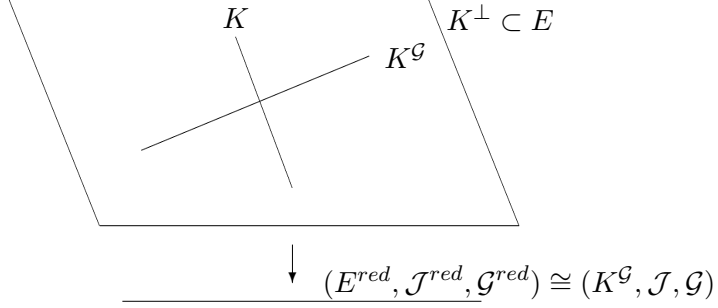

FIGURE 1. The generalized metric and complex structure on the reduced Courant algebroid are modeled on the $\mathcal{G}$-orthogonal complement of K inside $K^{\perp}$.

Intuitively, thinking of E as a generalized tangent bundle to M, the distribution K plays the role of the tangent distribution to the orbits of the extended action. The construction above can be interpreted as saying that the orthogonal complement of the generalized "G-orbit" is a model for the quotient; see Figure 1. Despite the clarity of condition (27), it is often easier to verify its orthogonal complement:

$$(28) \qquad\qquad \mathcal{J}(K + \mathcal{G}K) = K + \mathcal{G}K.$$

For further details concerning the reduction of generalized metrics, see [3].

4.3. Generalized Kähler and hyper-Kähler reductions.
A *generalized Kähler structure* is a generalized complex structure $\mathcal{J}$ together with a compatible generalized Riemannian metric $\mathcal{G}$ such that $\mathcal{J}' := \mathcal{J}\mathcal{G}$ is also a generalized complex structure.

A *generalized hyper-Kähler* structure is a triple of generalized complex structures, $\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3$, each of which forms a generalized Kähler structure with the same generalized Riemannian metric $\mathcal{G}$, and such that

$$\mathcal{J}_1\mathcal{J}_2 = -\mathcal{J}_2\mathcal{J}_1 = \mathcal{J}_3.$$

Remark: Observe that given a generalized complex structure $\mathcal{J}$ with compatible metric $\mathcal{G}$, the product $\mathcal{J}\mathcal{G}$ is always a generalized almost complex structure, i.e., $(\mathcal{J}\mathcal{G})^2 = -\text{Id}$. The nontrivial requirement is the integrability of this structure. Note also that $\mathcal{J}\mathcal{G}$ is compatible with $\mathcal{G}$.

We may now apply Theorem 4.4 to obtain generalized Kähler and generalized hyper-Kähler reductions. As before, let $(\widetilde{\psi}, \mathfrak{h}, \mu)$ be reduction data for the exact Courant algebroid E.

Theorem 4.5 (Generalized Kähler reduction)**.** *Suppose that $(\mathcal{J}, \mathcal{G})$ is a G-invariant generalized Kähler structure on E. If $\mathcal{J}K^{\mathcal{G}} = K^{\mathcal{G}}$ over $\mu^{-1}(0)$, then the generalized Kähler structure $(\mathcal{J}, \mathcal{G})$ reduces to the Courant algebroid E^{red} over $\mu^{-1}(0)/G$.*

Proof. Letting $\mathcal{J}' = \mathcal{J}\mathcal{G}$, it is clear that $\mathcal{J}'$ is also G-invariant and that both $(\mathcal{J}, \mathcal{G})$ and $(\mathcal{J}', \mathcal{G})$ satisfy the conditions of Theorem 4.4. Hence they can be reduced to E^{red}, and the reduced metric $\mathcal{G}^{red}$ is compatible with both reduced generalized complex structures. Since the reduced structures are identified with the restrictions of $\mathcal{J}, \mathcal{J}'$ and $\mathcal{G}$ to $K^{\mathcal{G}}$, the fact that $\mathcal{J}' = \mathcal{J}\mathcal{G}$ on $K^{\mathcal{G}}$ implies that the same holds in E^{red}. Hence $(\mathcal{J}^{red}, \mathcal{G}^{red})$ is a generalized Kähler structure. $\square$

We have the following analogous result for generalized hyper-Kähler structures:

Theorem 4.6 (Generalized hyper-Kähler reduction). *Suppose that $(\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3, \mathcal{G})$ is a G-invariant generalized hyper-Kähler structure on E. If $\mathcal{J}_1 K^{\mathcal{G}} = \mathcal{J}_2 K^{\mathcal{G}} = \mathcal{J}_3 K^{\mathcal{G}} = K^{\mathcal{G}}$ over $\mu^{-1}(0)$, then the generalized hyper-Kähler structure $(\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3, \mathcal{G})$ reduces to a generalized hyper-Kähler structure in E^{red} over $\mu^{-1}(0)/G$.*

4.4. **Examples.** We now describe how one recovers the classical Kähler and hyper-Kähler quotients [6, 9, 12] using our methods. As a final example we show that the classical hyper-Kähler quotient may be viewed as a non-trivial example of a generalized Kähler quotient.

For the examples in this section, the Courant algebroid in question is $E = TM \oplus T^*M$, with $H = 0$, and with reduction data $(\psi, \mathfrak{h}, \mu)$, where the lifted action ψ is an ordinary action.

Remark: In a separate paper [2], we describe examples of generalized Kähler and hyper-Kähler quotients involving non-vanishing twists H and non-trivial lifted actions, such as the construction of generalized Kähler and hyper-Kähler structures on certain moduli spaces of instantons and Lie groups.

Example 4.7 (Kähler reduction). Let (I, ω) be a Kähler structure on M, preserved by a Hamiltonian G-action $\psi : \mathfrak{g} \to \Gamma(TM)$, with moment map $\mu : M \to \mathfrak{g}^*$. Let $g = \omega I$ be its associated Kähler metric.

We then view the Kähler structure as a generalized Kähler structure $(\mathcal{J}_\omega, \mathcal{G})$, where

$$(29) \qquad \mathcal{G} = \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix},$$

and carry out its reduction using the reduction data $(\psi, \mathfrak{g}, \mu)$. The reduced Courant algebroid, as described in Example 3.3, is simply $TM^{red} \oplus T^*M^{red}$, where $M^{red} = \mu^{-1}(0)/G$.

As in Example 4.2, the moment map condition (24) for ω immediately implies that $\mathcal{J}_\omega K = K$. Hence $K^{\mathcal{G}} = \mathcal{J}_I \mathcal{J}_\omega K^\perp \cap K^\perp = \mathcal{J}_I K^\perp \cap K^\perp$ and therefore

$$\mathcal{J}_\omega K^{\mathcal{G}} = \mathcal{J}_\omega(\mathcal{J}_I K^\perp \cap K^\perp) = \mathcal{J}_I K^\perp \cap K^\perp = K^{\mathcal{G}}.$$

So we can apply Theorem 4.5 and reduce the Kähler structure as a generalized Kähler structure. We now check that the reduced structure $\mathcal{J}_I^{red}$ and $\mathcal{J}_\omega^{red}$ agree with the generalized structures associated with the usual reduced Kähler structure on M^{red}. First, as observed in Example 4.2, the generalized reduction of the symplectic structure is just the usual symplectic reduction. Let us now discuss the reduction of $\mathcal{J}_I$. We know that, at each point of M^{red}, $\mathcal{J}_I^{red}$ is described by its restriction to

$$K^{\mathcal{G}} = \{X + \xi \in TP \oplus T^*P : X \perp \psi(\mathfrak{g}) \text{ and } \xi(\psi(\mathfrak{g})) = 0\},$$

where $P = \mu^{-1}(0)$. We have the identifications

$$\{X \in TP : X \perp \psi(\mathfrak{g})\} = TM^{red} \quad \text{and} \quad \{\xi \in T^*P : \xi(\psi(\mathfrak{g})) = 0\} = T^*M^{red},$$

and the space on the left is invariant under $\mathcal{J}_I$. It follows that $\mathcal{J}_I^{red}$ preserves TM^{red}, and hence it is of complex type. One sees from this description that $\mathcal{J}_I^{red}$ agrees with the generalized

complex structure associated with the complex structure obtained by the usual Kähler reduction procedure [6, 12].

Example 4.8 (hyper-Kähler reduction). Let (I_1, I_2, I_3, g) be a hyper-Kähler structure on M preserved by a Hamiltonian G-action $\psi : \mathfrak{g} \to \Gamma(TM)$, in the sense that there exist moment maps $\mu_1, \mu_2, \mu_3 \in C^\infty(M, \mathfrak{g}^*)^G$ satisfying

$$(30) \qquad i_{\psi(u)}\omega_j = d\langle \mu_j, u \rangle, \;\; \forall u \in \mathfrak{g}, \text{ and } j = 1, 2, 3,$$

where $\omega_j = gI_j$ are the Kähler forms.

We consider the generalized complex structures $\mathcal{J}_j$ associated with the complex structures I_j, $j = 1, 2, 3$, and the generalized metric $\mathcal{G}$ as in (29). Then $(\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3, \mathcal{G})$ defines a generalized hyper-Kähler structure, and we consider the reduction data $(\psi, \mathfrak{h}, \mu)$, where now

$$\mathfrak{h} = \mathfrak{g} \oplus \mathfrak{g} \oplus \mathfrak{g}, \quad \text{and} \quad \mu = (\mu_1, \mu_2, \mu_3) : M \to \mathfrak{h}^*.$$

It follows that

$$K = \{\psi(u) + d\langle \mu_1, w_1 \rangle + d\langle \mu_2, w_2 \rangle + d\langle \mu_3, w_3 \rangle, \; u \in \mathfrak{g}, \;\; w_1, w_2, w_3 \in \mathfrak{h}\}.$$

In order to apply Theorem 4.6, we must check that

$$(31) \qquad \mathcal{J}_j(K + \mathcal{G}K) = K + \mathcal{G}K$$

over $\mu^{-1}(0)$, $j = 1, 2, 3$. Using (30), we see that

$$K + \mathcal{G}K = \psi(\mathfrak{g}) + \omega_1(\psi(\mathfrak{g})) + \omega_2(\psi(\mathfrak{g})) + \omega_3(\psi(\mathfrak{g})) + g(\psi(\mathfrak{g})) + I_1\psi(\mathfrak{g}) + I_2\psi(\mathfrak{g}) + I_3\psi(\mathfrak{g}).$$

It now follows from the relations $I_1 I_2 = I_3$, $I_1\omega_2 = -\omega_3$ and $I_1\omega_1 = -g$ that (31) holds. Hence we can reduce the hyper Kähler-structure to $M^{red} = \mu^{-1}(0)/G$ as a generalized hyper-Kähler structure.

As in Example 4.7, we have the identification

$$(32) \qquad K^{\mathcal{G}} = \{X \in TP : X \perp \psi(\mathfrak{g})\} \oplus \{\xi \in T^*P : \xi(\psi(\mathfrak{g})) = 0\} = TM^{red} \oplus T^*M^{red},$$

for $P = \mu^{-1}(0)$. Since $K^{\mathcal{G}}$ is invariant under $\mathcal{J}_1$, $\mathcal{J}_2$ and $\mathcal{J}_3$ and these structures are of complex type, it follows that the space

$$\{X \in TP : X \perp \psi(\mathfrak{g})\} \cong TM^{red},$$

is also invariant by these structures. Hence $\mathcal{J}_1^{red}, \mathcal{J}_2^{red}$ and $\mathcal{J}_3^{red}$ are complex structures and the generalized hyper-Kähler structure obtained in the reduced manifold is precisely the usual hyper-Kähler reduction of M from [9].

In [5], it was shown that a generalized Kähler structure $(\mathcal{J}, \mathcal{G})$ on $E = TM \oplus T^*M$ with $H = 0$ determines and is uniquely determined by a quadruple (I_+, I_-, g, b), where g is a Riemannian metric on M, I_+ and I_- are Hermitian complex structures (hence defining a *bihermitian* structure), and b is a 2-form such that

$$(33) \qquad d_-^c \omega_- = -d_+^c \omega_+ = db,$$

where $\omega_\pm = gI_\pm$ and $d^c = i(\overline{\partial} - \partial)$ is defined by the appropriate complex structure.

The bihermitian structure is obtained as follows (see [5] for details). Since $\mathcal{G}^2 = \text{Id}$, we can write $E = C_+ \oplus C_-$, where $C_\pm$ is the ± 1-eigenbundle of $\mathcal{G}$. The spaces C_+ and C_- intersect T^*M trivially, so the projection $\pi : E \to TM$ induces identifications $C_\pm \cong TM$. The metric g on M

is induced by the restriction of $\langle \cdot, \cdot \rangle$ to C_+, whereas $J_\pm$ come from the restrictions of $\mathcal{J}$ to $C_\pm$. Conversely, the generalized Kähler structure may be written in terms of (I_+, I_-, g, b) as follows.

$$\mathcal{J} = \frac{1}{2} \begin{pmatrix} 1 & \\ b & 1 \end{pmatrix} \begin{pmatrix} I_+ + I_- & -(\omega_+^{-1} - \omega_-^{-1}) \\ \omega_+ - \omega_- & -(I_+^* + I_-^*) \end{pmatrix} \begin{pmatrix} 1 & \\ -b & 1 \end{pmatrix},$$

$$G = \begin{pmatrix} 1 & \\ b & 1 \end{pmatrix} \begin{pmatrix} & g^{-1} \\ g & \end{pmatrix} \begin{pmatrix} 1 & \\ -b & 1 \end{pmatrix}.$$

It follows from this bihermitian interpretation of generalized Kähler geometry that any hyper-Kähler structure (g, I_1, I_2, I_3) determines a generalized Kähler structure, by simply choosing $I_+ = I_1$ and $I_- = I_2$, for example. We now show that this "forgetful functor" commutes with reduction, i.e. intertwines the notion of generalized Kähler reduction from Theorem 4.5 with the classical hyper-Kähler quotient procedure.

Example 4.9 (Hyper-Kähler reduction versus generalized Kähler reduction). A hyper-Kähler structure (g, I_1, I_2, I_3) defines a bihermitian structure (g, I_1, I_2) satisfying (33) for $b = 0$, hence it defines a generalized Kähler structure $(\mathcal{G}, \mathcal{J})$ as above. The generalized complex structure $\mathcal{J}$ may be described as a bundle automorphism of $E = C_+ \oplus C_-$ as follows: on C_+,

$$\mathcal{J}(X + g(X)) = I_1 X + g(I_1 X),$$

and on C_-,

$$\mathcal{J}(X - g(X)) = I_2 X - g(I_2 X).$$

Given a Hamiltonian G-action $\psi : \mathfrak{g} \longrightarrow \Gamma(TM)$ preserving the hyper-Kähler structure, and with moment maps $\mu_j : M \longrightarrow \mathfrak{g}^*$, $j = 1, 2, 3$, we consider the same reduction data as in Example 4.8,

$$(\psi, \mathfrak{h} = \mathfrak{g} \oplus \mathfrak{g} \oplus \mathfrak{g}, \mu = (\mu_1, \mu_2, \mu_3)),$$

but we will now use it to reduce the generalized Kähler structure $(\mathcal{G}, \mathcal{J})$, rather than the original hyper-Kähler structure.

Following Example 4.8, we know that $K^\mathcal{G}$ is given by (32), and each summand in that expression is invariant under I_j, $j = 1, 2, 3$. Note that we can write

$$K^\mathcal{G} = (K^\mathcal{G} \cap C_+) \oplus (K^\mathcal{G} \cap C_-).$$

If $X + g(X) \in K^\mathcal{G} \cap C_+$, then $I_1 X - I_1^* g(X) = I_1 X + g(I_1 X) \in K^\mathcal{G}$, and, similarly, $X - g(X) \in K^\mathcal{G} \cap C_-$ implies that $I_2 X + I_2^* g(X) = I_2 X - g(I_2 X) \in K^\mathcal{G}$. Hence $\mathcal{J} K^\mathcal{G} = K^\mathcal{G}$ and, by Theorem 4.5, we may reduce the generalized complex structure $(\mathcal{G}, \mathcal{J})$ to $M^{red} = \mu^{-1}(0)/G$.

The bihermitian structure on M_{red} associated with $(\mathcal{G}^{red}, \mathcal{J}^{red})$ can be described as follows: $\mathcal{G}$ restricted to $K^\mathcal{G}$ is just the reduced metric g^{red} obtained by hyper-Kähler reduction as in Example 4.8, whereas the restriction of $\mathcal{J}$ to $C_\pm \cap K^\mathcal{G}$ defines complex structures via the projection to TM^{red}. It is easily verified that the restriction of $\mathcal{J}$ to $C_+ \cap K^\mathcal{G}$ defines I_1^{red}, and the restriction to $C_- \cap K^\mathcal{G}$ gives I_2^{red}, where $(g^{red}, I_1^{red}, I_2^{red}, I_3^{red})$ is the reduced hyper-Kähler structure, as required.

REFERENCES

[1] H. Bursztyn, G. R. Cavalcanti, and M. Gualtieri. Reduction of Courant algebroids and generalized complex structures. Math.DG/0509640. To appear in Adv. Math..

[2] H. Bursztyn, G. R. Cavalcanti, and M. Gualtieri. In preparation.

[3] G. R. Cavalcanti. Reduction of metric structures on Courant algebroids. In preparation.

[4] T. Courant. Dirac manifolds. *Trans. Amer. Math. Soc.*, 319:631–661, 1990.

[5] M. Gualtieri. *Generalized Complex Geometry*. D.Phil. thesis, Oxford University, 2003. math.DG/0401221.

[6] V. Guillemin and S. Sternberg. Geometric quantization and multiplicities of group representations. *Invent. Math.*, 67(3):515–538, 1982.

[7] N. J. Hitchin. Generalized Calabi-Yau manifolds. *Q. J. Math.*, 54:281–308, 2003.

[8] N. J. Hitchin. Instantons, Poisson structures and generalized Kaehler geometry. *Comm. Math. Phys.*, 265: 131-164, 2006. math.DG/0503432.

[9] N. J. Hitchin, A. Karlhede, U. Lindström, and M. Roček. Hyper-Kähler metrics and supersymmetry. *Comm. Math. Phys.*, 108(4):535–589, 1987.

[10] S. Hu. Hamiltonian symmetries and reduction in generalized geometry. math.DG/0509060.

[11] M. K. Kinyon and A. Weinstein. Leibniz algebras, Courant algebroids, and multiplications on reductive homogeneous spaces. *Amer. J. Math.*, 123(3):525–550, 2001.

[12] F. C. Kirwan. *Cohomology of quotients in symplectic and algebraic geometry*, volume 31 of *Mathematical Notes*. Princeton University Press, Princeton, NJ, 1984.

[13] Y. Lin, S. Tolman. Symmetries in generalized Kähler geometry. math.DG/0509069.

[14] Z.-J. Liu, A. Weinstein, P. Xu: Manin triples for Lie algebroids. *J. Differential Geom.*, 45: 547–574, 1997.

[15] J.-L. Loday. Une version non commutative des algèbres de Lie: les algèbres de Leibniz. In *R.C.P. 25, Vol. 44 (French) (Strasbourg, 1992)*, volume 1993/41 of *Prépubl. Inst. Rech. Math. Av.*, pages 127–151. Univ. Louis Pasteur, Strasbourg, 1993.

[16] R. Moraru and M. Verbitsky. Stable bundles on hypercomplex surfaces. math.DG/0611714.

[17] P. Ševera and A. Weinstein. Poisson geometry with a 3-form background. *Progr. Theoret. Phys. Suppl.*, 144:145–154, 2001. Noncommutative geometry and string theory (Yokohama, 2001).

[18] M. Stienon, P. Xu. Reduction of generalized complex structures. math.DG/0509393.

[19] I. Vaisman. Reduction and submanifolds of generalized complex manifolds. math.DG/0511013.

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