GENERALIZED COMPLEX STRUCTURES ON NILMANIFOLDS

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ABSTRACT. We show that all 6-dimensional nilmanifolds admit generalized complex structures. This includes the five classes of nilmanifold which admit no known complex or symplectic structure. Furthermore, we classify all 6-dimensional nilmanifolds according to which of the four types of left-invariant generalized complex structure they admit. We also show that the two components of the left-invariant complex moduli space for the Iwasawa manifold are no longer disjoint when they are viewed in the generalized complex moduli space. Finally, we provide an 8-dimensional nilmanifold which admits no left-invariant generalized complex structure.

INTRODUCTION

Ever since Thurston [10] presented a nilmanifold as the first instance of a symplectic but non-Kähler manifold in 1976, the study of invariant geometries on nilmanifolds has been an interesting source of examples in differential geometry.

A nilmanifold is a homogeneous space $M = \Gamma \backslash G$, where G is a simply connected nilpotent real Lie group and Γ is a lattice of maximal rank in G. Such groups G of dimension ≤ 7 have been classified, and 6 is the highest dimension where there are finitely many. According to [5, 7] there are 34 isomorphism classes of connected, simply-connected 6-dimensional nilpotent Lie groups. This means that, with respect to invariant geometry, there are essentially 34 separate cases to investigate.

The question of which 6-dimensional nilmanifolds admit symplectic structure was settled by Goze and Khakimdjanov [2]: exactly 26 of the 34 classes admit symplectic forms. Subsequently, the question of left-invariant complex geometry was solved by Salamon [9]: he proved that exactly 18 of the 34 classes admit invariant complex structure. While the torus is the only nilmanifold admitting Kähler structure, 15 of the 34 nilmanifolds admit both complex and symplectic structures. This leaves us with 5 classes of 6-dimensional nilmanifolds admitting neither complex nor symplectic left-invariant geometry. See Figure 1 for illustration.

It was this result of Salamon which inspired us to ask whether the 5 outlying classes might admit *generalized complex structure*, a geometry recently introduced by Hitchin [3] and developed by the second author [1]. Generalized complex geometry contains complex and symplectic geometry as extremal special cases and shares important properties with them, such as an elliptic deformation theory as well as a local normal form (in regular neighbourhoods). The main result of this paper is to answer this question in the affirmative: all 6-dimensional nilmanifolds admit generalized complex structures.
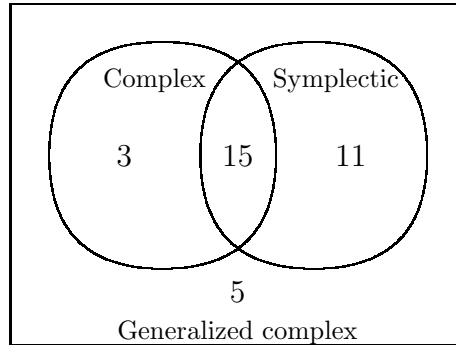
Figure 1: Left-invariant structures on the 34 six-dimensional nilpotent Lie groups.

We begin in Section 1 with a review of generalized complex geometry. A brief introduction to nilmanifolds follows in Section 2. Some results about generalized complex structures on nilmanifolds in arbitrary dimension appear in Section 3. Section 4 contains our main result: the classification of left-invariant generalized complex structures on 6-dimensional nilmanifolds. In Section 5 we show that while the moduli space of left-invariant complex structures on the Iwasawa nilmanifold is disconnected (as shown in [4]), its components can be joined using generalized complex structures. In the final section, we provide an 8-dimensional nilmanifold which does not admit a left-invariant generalized complex structure, thus precluding the possibility that all nilmanifolds admit left-invariant generalized complex geometry.

1. Generalized complex structures

We briefly review the theory of generalized complex structures; see [1] for details. A generalized complex structure on a smooth manifold M is defined to be a complex structure $\mathcal{J}$, not on the tangent bundle T, but on the sum $T \oplus T^*$ of the tangent and cotangent bundles. This complex structure is required to be orthogonal with respect to the natural inner product on sections $X + \xi, Y + \eta \in C^\infty(T \oplus T^*)$ defined by

$$\langle X + \xi, Y + \eta \rangle = \tfrac{1}{2}(\xi(Y) + \eta(X)).$$

This is only possible if the manifold has even dimension, so we suppose $\dim_{\mathbb{R}} M = 2n$. In addition, the $+i$-eigenbundle

$$L < (T \oplus T^*) \otimes \mathbb{C}$$

of $\mathcal{J}$ is required to be involutive with respect to the Courant bracket, a skew bracket operation on smooth sections of $T \oplus T^*$ defined by

$$[X + \xi, Y + \eta] = [X, Y] + \mathcal{L}_X \eta - \mathcal{L}_Y \xi - \tfrac{1}{2}d(i_X \eta - i_Y \xi),$$

where $\mathcal{L}_X$ and i_X denote the Lie derivative and interior product operations on forms.

Since $\mathcal{J}$ is orthogonal with respect to $\langle \cdot, \cdot \rangle$, the $+i$-eigenbundle L is a maximal isotropic sub-bundle of $(T \oplus T^*) \otimes \mathbb{C}$, and as such can be expressed as the Clifford annihilator of a unique line sub-bundle U_L of the complex spinors for the metric bundle $T \oplus T^*$. Since its annihilator is maximal isotropic, U_L is by definition a *pure* spinor line, and we call it the *canonical* line bundle of $\mathcal{J}$.

The bundle $\wedge^\bullet T^*$ of differential forms can in fact be viewed as a spinor bundle for $T \oplus T^*$, where the Clifford action of an element $X + \xi \in T \oplus T^*$ on a differential form ρ is given by

$$(X + \xi) \cdot \rho = i_X \rho + \xi \wedge \rho.$$

Note that $(X + \xi)^2 \cdot \rho = \langle X + \xi, X + \xi \rangle \rho$, as required. Therefore, the canonical bundle U_L may be viewed as a smooth line sub-bundle of the complex differential forms according to the relation

$$(1.1) \qquad L = \left\{ X + \xi \in (T \oplus T^*) \otimes \mathbb{C} \ : \ (X + \xi) \cdot U_L = 0 \right\}.$$

At every point, the line U_L is generated by a complex differential form of special algebraic type: purity is equivalent to the fact that it has the form

$$(1.2) \qquad \rho = e^{B + i\omega} \Omega,$$

where B, ω are real 2-forms and $\Omega = \theta_1 \wedge \cdots \wedge \theta_k$ is a complex decomposable k-form. The number k is called the *type* of the generalized complex structure, and it is not required to be constant along the manifold. Points where the type is locally constant are called *regular*. Since L is the $+i$-eigenbundle of a complex structure, we see that $L \cap \overline{L} = \{0\}$. This is equivalent to an additional constraint on ρ:

$$(1.3) \qquad \omega^{2n-2k} \wedge \Omega \wedge \overline{\Omega} \neq 0.$$

Hence we see that on a $2n$-manifold the type may take values from $k = 0$ to $k = n$. Finally, as is proven in [1], the involutivity of L under the Courant bracket is equivalent to the condition, on any local trivialization ρ of U_L, that there exist a section $X + \xi \in C^\infty(T \oplus T^*)$ such that

$$(1.4) \qquad d\rho = (X + \xi) \cdot \rho.$$

Near any regular point, this condition implies that the distribution determined by $\Omega \wedge \overline{\Omega}$ integrates to a foliation, and with (1.3), also implies that ω is a leafwise symplectic form.

In the special case that U_L is a trivial bundle admitting a nowhere-vanishing *closed* section ρ, the structure is said to be a *generalized Calabi-Yau* structure, as in [3].

1.1. **Examples.** So far, we have explained how a generalized complex structure is equivalent to the specification of a pure line sub-bundle of the complex differential forms, satisfying the non-degeneracy condition (1.3) and the integrability condition (1.4). Now let us provide some examples of such structures.

Example 1 (Complex geometry (type n))**.** Let $J \in \text{End}(T)$ be a usual complex structure on a $2n$-manifold. The generalized complex structure corresponding to J

is

$$\mathcal{J}_J = \begin{pmatrix} -J & 0 \\ 0 & J^* \end{pmatrix},$$

where the matrix is written in the natural splitting $T \oplus T^*$. Clearly $\mathcal{J}_J^2 = -1$, and orthogonality is easily verified. The $+i$-eigenbundle of $\mathcal{J}_J$ is the maximal isotropic

$$L = T_{0,1} \oplus T_{1,0}^*,$$

where $T_{1,0} = \overline{T_{0,1}}$ is the $+i$-eigenbundle of J in the usual way. The bundle L is the Clifford annihilator of the line bundle

$$U_L = \wedge^n(T_{1,0}^*),$$

the canonical bundle associated to J. We see that $\mathcal{J}_J$ is of type n at all points in the manifold. The Courant involutivity of L is equivalent to the Lie involutivity of $T_{0,1}$, which is the usual integrability condition for complex structures. To be generalized Calabi-Yau, there must be a closed trivialization $\Omega \in C^\infty(U_L)$, which is the usual Calabi-Yau condition.

Example 2 (Symplectic geometry (type $k = 0$)). Let $\omega \in \Omega^2(M)$ be a usual symplectic structure, viewed as a skew-symmetric isomorphism $\omega : T \to T^*$ via the interior product $X \mapsto i_X\omega$. The generalized complex structure corresponding to ω is

$$\mathcal{J}_\omega = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix},$$

where the matrix is written in the natural splitting $T \oplus T^*$. Clearly $\mathcal{J}_\omega^2 = -1$, and orthogonality is easily verified. The $+i$-eigenbundle of $\mathcal{J}_\omega$ is the maximal isotropic

$$L = \{X - i\omega(X) \; : \; X \in T \otimes \mathbb{C}\},$$

which is the Clifford annihilator of the line bundle U_L with trivialization given by

$$\rho = e^{i\omega}.$$

We see that $\mathcal{J}_\omega$ is of type 0 at all points in the manifold. The Courant involutivity of L is equivalent to the constraint $d\rho = 0$, itself equivalent to the usual integrability condition $d\omega = 0$ for symplectic structures. Note that symplectic structures are always generalized Calabi-Yau.

The preceding examples demonstrate how complex and symplectic geometry appear as extremal cases of generalized complex geometry. We will now explain how one may deform these examples into new ones.

1.2. **B-fields and β-fields.** Unlike the Lie bracket, whose only symmetries are diffeomorphims, the Courant bracket is preserved by an additional group of symmetries of $T \oplus T^*$, consisting of closed 2-forms B acting via the orthogonal shear transformation

$$X + \xi \mapsto X + \xi - i_X B.$$

Such automorphisms are called *B-field* transformations, and their associated spinorial action on differential forms is via the exponential:

$$\rho \mapsto e^B \wedge \rho.$$

In this way, we see that B-field transformations do not have any effect on the type of a generalized complex structure. Nevertheless, using B-fields and diffeomorphisms, one may choose canonical coordinates for a generalized complex structure, near any regular point:

Theorem 1.1 ([1], Theorem 4.35). *Any regular point of type k in a generalized complex $2n$-manifold has a neighbourhood which is equivalent, via a diffeomorphism and a B-field transformation, to the product of an open set in $\mathbb{C}^k$ with an open set in the standard symplectic space $\mathbb{R}^{2n-2k}$.*

Although automorphisms of the Courant bracket do not affect the type of a generalized complex structure, there may be non-automorphisms which nevertheless transform a given generalized complex structure into another one, of modified type. For example, consider the action of a bivector $\beta \in C^\infty(\wedge^2 T)$ on $T \oplus T^*$ via the orthogonal shear transformation

$$X + \xi \mapsto X + \xi + i_\xi \beta.$$

The spinorial action of such β-*field* transformations on differential forms is also via the exponential

$$\rho \mapsto e^\beta \rho = (1 + i_\beta + \tfrac{1}{2} i_\beta^2 + \cdots)\rho.$$

The following proposition describes the conditions on β which ensure that it takes a complex structure into a generalized complex structure of different type.

Proposition 1.1. ([1], Section 5.3) *Let J be a complex structure on a compact $2n$-manifold, viewed as a generalized complex structure of type n. Let β be a smooth bivector of type $(2,0)$ with respect to J. Then by the above action, β deforms J into another generalized complex structure if and only if it is sufficiently small and*

$$\overline{\partial}\beta + \tfrac{1}{2}[\beta, \beta] = 0,$$

which holds if and only if each summand vanishes separately, i.e. β is a holomorphic Poisson structure. The resulting generalized complex structure has type $n - k$ at points where the bivector β has rank k.

For example, any holomorphic bivector β on $\mathbb{C}P^2$ is Poisson, and therefore a sufficiently small constant multiple of it will deform the standard complex structure into a generalized complex structure. Since β vanishes on a cubic and is of rank 2 elsewhere, the resulting generalized complex structure has type 2 along the cubic and is of type 0 elsewhere. Our purpose in introducing β-transforms is that we will use them to produce some examples of generalized complex structures on nilmanifolds.

2. NILMANIFOLDS

A nilmanifold is the quotient $M = \Gamma \backslash G$ of a connected, simply-connected nilpotent real Lie group G by the left action of a maximal lattice Γ, i.e. a discrete co-compact subgroup. By results of Malcev [6], a nilpotent Lie group admits such a lattice if and only if its Lie algebra has rational structure constants in some basis. Moreover, any two nilmanifolds of G can be expressed as finite covers of a third one.

A connected, simply-connected nilpotent Lie group is diffeomorphic to its Lie algebra via the exponential map and so is contractible. For this reason, the homotopy groups π_k of nilmanifolds vanish for $k > 1$, i.e. nilmanifolds are Eilenberg-MacLane spaces $K(\Gamma, 1)$. In fact, their diffeomorphism type is determined by their fundamental group. Malcev showed that this fundamental group is a finitely generated nilpotent group with no element of finite order. Such groups can be expressed as central $\mathbb{Z}$ extensions of groups of the same type, which implies that any nilmanifold can be expressed as a circle bundle over a nilmanifold of lower dimension. Because of this, one may easily use Gysin sequences to compute the cohomology ring of any nilmanifold. Nomizu used this fact to show that the rational cohomology of a nilmanifold is captured by the subcomplex of the de Rham complex $\Omega^\bullet(M)$ consisting of forms descending from left-invariant forms on G:

Theorem 2.1. (Nomizu [8]) *The de Rham complex $\Omega^\bullet(M)$ of a nilmanifold $M = \Gamma\backslash G$ is quasi-isomorphic to the complex $\wedge^\bullet \mathfrak{g}^*$ of left-invariant forms on G, and hence the de Rham cohomology of M is isomorphic to the Lie algebra cohomology of $\mathfrak{g}$.*

In this paper we will search for generalized complex structures on $\Gamma\backslash G$ which descend from left-invariant ones on G, which we will call *left-invariant* generalized complex structures. This will require detailed knowledge of the structure of the Lie algebra $\mathfrak{g}$, and so we outline its main properties in the remainder of this section.

Nilpotency implies that the central descending series of ideals defined by $\mathfrak{g}^0 = \mathfrak{g}$, $\mathfrak{g}^k = [\mathfrak{g}^{k-1}, \mathfrak{g}]$ reaches $\mathfrak{g}^s = 0$ in a finite number s of steps, called the *nilpotency index*, nil($\mathfrak{g}$) (also called the nilpotency index of any nilmanifold associated to $\mathfrak{g}$). Dualizing, we obtain a filtration of $\mathfrak{g}^*$ by the annihilators V_i of $\mathfrak{g}^i$, which can also be expressed as

$$V_i = \{v \in \mathfrak{g}^* \ : \ dv \in V_{i-1}\},$$

where $V_0 = \{0\}$. Choosing a basis for V_1 and extending successively to a basis for each V_k, we obtain a *Malcev basis* $\{e_1, \ldots, e_n\}$ for $\mathfrak{g}^*$. This basis satisfies the property

(2.1) $$de_i \in \wedge^2 \langle e_1, \ldots, e_{i-1}\rangle \quad \forall i.$$

The filtration of $\mathfrak{g}^*$ induces a filtration of its exterior algebra, and leads to the following useful definition:

Definition 2.1. With V_i as above, the *nilpotent degree* of a p-form α, which we denote by nil(α), is the smallest i such that $\alpha \in \wedge^p V_i$.

Remark. If α is a 1-form of nilpotent degree i then nil($d\alpha$) $= i - 1$.

In this paper, we specify the structure of a particular nilpotent Lie algebra by listing the exterior derivatives of the elements of a Malcev basis as an n-tuple of 2-forms $(de_k = \sum c_k^{ij} e_i \wedge e_j)_{k=1}^m$. In low dimensions we use the shortened notation ij for the 2-form $e_i \wedge e_j$, as in the following 6-dimensional example: the 6-tuple $(0, 0, 0, 12, 13, 14 + 35)$ describes a nilpotent Lie algebra with dual $\mathfrak{g}^*$ generated by 1-forms $e_1, \ldots, e_6$ and such that $de_1 = de_2 = de_3 = 0$, while $de_4 = e_1 \wedge e_2$, $de_5 = e_1 \wedge e_3$, and $de_6 = e_1 \wedge e_4 + e_3 \wedge e_5$. We see clearly that $V_1 = \langle e_1, e_2, e_3\rangle$, $V_2 = \langle e_1, e_2, e_3, e_4, e_5\rangle$, and $V_3 = \mathfrak{g}^*$, showing that the nilpotency index of $\mathfrak{g}$ is 3.

3. GENERALIZED COMPLEX STRUCTURES ON NILMANIFOLDS

In this section, we present two results concerning generalized complex structures on nilmanifolds of arbitrary dimension. In Theorem 3.1, we prove that any left-invariant generalized complex structure on a nilmanifold must be generalized Calabi-Yau, i.e. the canonical bundle U_L has a closed trivialization. In Theorem 3.2 we prove an upper bound for the type of a left-invariant generalized complex structure, depending only on crude information concerning the nilpotent structure.

We begin by observing that a left-invariant generalized complex structure must have constant type k throughout the nilmanifold M^{2n}, and its canonical bundle U_L must be trivial. Hence, by (1.2),(1.3), and (1.4) we may choose a global trivialization of the form

$$\rho = e^{B+i\omega}\Omega,$$

where B, ω are real left-invariant 2-forms and Ω is a globally decomposable complex k-form, i.e.

$$\Omega = \theta_1 \wedge \cdots \wedge \theta_k,$$

with each θ_i left-invariant. These data satisfy the nondegeneracy condition $\omega^{2n-2k} \wedge \Omega \wedge \overline{\Omega} \neq 0$ as well as the integrability condition $d\rho = (X + \xi) \cdot \rho$ for some section $X + \xi \in C^\infty(T \oplus T^*)$. Since ρ and $d\rho$ are left-invariant, we can choose $X + \xi$ to be left-invariant as well.

It is useful to order $\{\theta_1, \ldots, \theta_k\}$ according to nilpotent degree, and also to choose them in such a way that $\{\theta_j \ : \ \mathrm{nil}(\theta_j) > i\}$ is linearly independent modulo V_i; this is possible according to the following lemma.

Lemma 3.1. It is possible to choose a left-invariant decomposition $\Omega = \theta_1 \wedge \cdots \wedge \theta_k$ such that $\mathrm{nil}(\theta_i) \leq \mathrm{nil}(\theta_j)$ for $i < j$, and such that $\{\theta_j \ : \ \mathrm{nil}(\theta_j) > i\}$ is linearly independent modulo V_i for each i.

Proof. Choose any left-invariant decomposition $\Omega = \theta_1 \wedge \cdots \wedge \theta_k$ ordered according to nilpotent degree, i.e. $\mathrm{nil}(\theta_i) \leq \mathrm{nil}(\theta_j)$ for $i < j$. Certainly $\{\theta_1, \ldots, \theta_k\}$ is linearly independent modulo $V_0 = \{0\}$. Now let $\pi_i : \mathfrak{g}^* \to \mathfrak{g}^*/V_i$ be the natural projection, and suppose $\{\pi_i(\theta_j) \ : \ \mathrm{nil}(\theta_j) > i\}$ is linearly independent for all $i < m$. Consider $X = \{\pi_m(\theta_j) \ : \ \mathrm{nil}(\theta_j) > m\}$. If there is a linear relation $\pi_m(\theta_p) = \sum_{l \neq p} \alpha_l \pi_m(\theta_l)$ among these elements, then we may replace θ_p with $\tilde{\theta}_p = \theta_p - \sum_{l \neq p} \alpha_l \theta_l$, which does not change Ω or affect linear independence modulo V_i, $i < m$. However, note that $\mathrm{nil}(\tilde{\theta}_p) \leq m$, i.e. we have removed an element from X. Reordering by degree and repeating the argument, we may remove any linear relation modulo V_m in this way, proving the induction step. $\square$

We require a simple linear algebra fact before moving on to the theorem.

Lemma 3.2. Let V be a subspace of a vector space W, let $\alpha \in \wedge^p V$, and suppose $\{\theta_1, \ldots, \theta_m\} \subset W$ is linearly independent modulo V. Then $\alpha \wedge \theta_1 \wedge \cdots \wedge \theta_m = 0$ if and only if $\alpha = 0$.

Proof. Let $\pi : W \to W/V$ be the projection and choose a splitting $W \cong V \oplus W/V$; $\alpha \wedge \theta_1 \wedge \cdots \wedge \theta_m$ has a component in $\wedge^p V \otimes \wedge^m(W/V)$ equal to $\alpha \otimes \pi(\theta_1) \wedge \cdots \wedge \pi(\theta_m)$, which vanishes if and only if $\alpha = 0$. $\square$

Theorem 3.1. *Any left-invariant generalized complex structure on a nilmanifold must be generalized Calabi-Yau. That is, any left-invariant global trivialization ρ of the canonical bundle must be a closed differential form. In particular, any left-invariant complex structure has holomorphically trivial canonical bundle.*

Proof. Let $\rho = e^{B+i\omega}\Omega$ be a left-invariant trivialization of the canonical bundle such that $\Omega = \theta_1 \wedge \cdots \wedge \theta_k$, with $\{\theta_1, \ldots, \theta_k\}$ ordered according to Lemma 3.1. Then the integrability condition $d\rho = (X + \xi) \cdot \rho$ is equivalent to the condition

$$(3.1) \qquad d(B+i\omega) \wedge \Omega + d\Omega = (i_X(B+i\omega)) \wedge \Omega + i_X\Omega + \xi \wedge \Omega.$$

The degree $k+1$ part of (3.1) states that

$$(3.2) \qquad\qquad d\Omega = i_X(B+i\omega) \wedge \Omega + \xi \wedge \Omega.$$

Taking wedge of (3.2) with θ_i we get

$$d\theta_i \wedge \theta_1 \wedge \cdots \wedge \theta_k = 0, \quad \forall i.$$

Now, let $\{\theta_1, \ldots, \theta_j\}$ be the subset with nilpotent degree $\leq \mathrm{nil}(d\theta_i)$. Note that $j < i$ since $\mathrm{nil}(\theta_i) = \mathrm{nil}(d\theta_i) + 1$. Then since

$$(d\theta_i \wedge \theta_1 \wedge \cdots \wedge \theta_j) \wedge \theta_{j+1} \wedge \cdots \wedge \theta_k = 0,$$

we conclude from Lemma 3.2 that

$$(3.3) \qquad\qquad d\theta_i \wedge \theta_1 \wedge \cdots \wedge \theta_j = 0, \text{ with } j < i.$$

Since this argument holds for all i, we conclude that $d\Omega = 0$. The degree $k+3$ part of (3.1) states that $d(B+i\omega) \wedge \Omega = 0$, and so we obtain that $d\rho = e^{B+i\omega}d\Omega = 0$, as required. $\qquad\square$

Equation (3.3) shows that the integrability condition satisfied by ρ leads to constraints on $\{\theta_1, \ldots, \theta_k\}$. Since these will be used frequently in the search for generalized complex structures, we single them out as follows.

Corollary 1. Assume $\{\theta_1, \ldots, \theta_k\}$ are chosen according to Lemma 3.1. Then

$$(3.4) \qquad\qquad d\theta_i \in \mathcal{I}(\{\theta_j : \mathrm{nil}(\theta_j) < \mathrm{nil}(\theta_i)\}),$$

where $\mathcal{I}(\ )$ denotes the ideal generated by its arguments. Since $\mathrm{nil}(\theta_i)$ is weakly increasing, we have, in particular,

$$d\theta_i \in \mathcal{I}(\theta_1, \ldots, \theta_{i-1}).$$

Example 3. Since $d\theta_1 \in \mathcal{I}(0)$, we see that θ_1 is always closed, and therefore it lies on V_1 or, equivalently, $\mathrm{nil}(\theta_1) = 1$.

So far, we have described constraints deriving from the integrability condition on ρ. However, nondegeneracy (in particular, $\Omega \wedge \overline{\Omega} \neq 0$) also places constraints on the θ_i appearing in the decomposition of Ω. The following example illustrates this.

Example 4. If $\theta_1, \ldots, \theta_j \in V_i$, then nondegeneracy implies that $\dim V_i \geq 2j$. For a fixed nilpotent algebra, this places an upper bound on the number of θ_j which can be chosen from each V_i.

In the next lemma we prove a similar, but more subtle constraint on the 1-forms θ_i.

Lemma 3.3. Assume that $\{\theta_1, \ldots, \theta_k\}$ are chosen according to Lemma 3.1. Suppose that no θ_i satisfies $\mathrm{nil}(\theta_i) = j$, but that there exists θ_l with $\mathrm{nil}(\theta_l) = j + 1$. Then $\theta_l \wedge \overline{\theta}_l \neq 0$ modulo V_j (i.e. in $\wedge^2(V_{j+1}/V_j)$), and in particular V_{j+1}/V_j must have dimension 2 or greater.

Proof. Assume that the hypotheses hold but $\theta_l \wedge \overline{\theta}_l = 0$ modulo V_j. Because of this, it is possible to decompose $\theta_l = v + \alpha$, where $\mathrm{nil}(\alpha) < j + 1$, and such that $v \wedge \overline{v} = 0$. Therefore, up to multiplication of θ_l (and therefore Ω) by a constant, v is real.

By Corollary 1, $d\theta_l \in \mathcal{I}(\{\theta_i : \mathrm{nil}(\theta_i) < j + 1\})$. By hypothesis, there is no θ_i with nilpotent degree j, therefore we obtain

$$dv + d\alpha = \sum_{\mathrm{nil}(\theta_i) < j} \xi_i \wedge \theta_i,$$

where $\xi_i \in \mathfrak{g}^*$. Since $\mathrm{nil}(dv) = j$, there is an element $w \in \mathfrak{g}^{j-1}$ such that $i_w dv \neq 0$. On the other hand, the nilpotent degrees of $d\alpha$ and the $\{\theta_i\}$ in the sum above are less than j, hence interior product with w annihilates all these forms. In particular,

$$0 \neq i_w dv = \sum_{\mathrm{nil}(\theta_i) < j} (i_w \xi_i) \theta_i.$$

Therefore, $i_w \xi_i$ is nonzero for some i. But, the left hand side is real, thus

$$0 = i_w dv \wedge \overline{i_w dv} = \left(\sum_{\mathrm{nil}(\theta_i) < j} (i_w \xi_i) \theta_i \right) \wedge \left(\sum_{\mathrm{nil}(\theta_i) < j} (i_w \overline{\xi}_i) \overline{\theta}_i \right).$$

By the nondegeneracy condition, the right hand side is nonzero, which is a contradiction. Hence $\theta_l \wedge \overline{\theta}_l \neq 0$ modulo V_j. $\qquad\square$

From this lemma, we see that if $\dim V_{j+1}/V_j = 1$ occurs in a nilpotent Lie algebra, then it must be the case, either that some θ_i has nilpotent degree j, or that no θ_i has nilpotent degree $j+1$. In this way, we see that the size of the nilpotent steps $\dim V_{j+1}/V_j$ may constrain the possible types of left-invariant generalized complex structures, as we now make precise.

Theorem 3.2. *Let M^{2n} be a nilmanifold with associated Lie algebra $\mathfrak{g}$. Suppose there exists a $j > 0$ such that, for all $i \geq j$,*

$$(3.5) \qquad\qquad \dim(V_{i+1}/V_i) = 1.$$

Then M cannot admit left-invariant generalized complex structures of type k for $k \geq 2n - \mathrm{nil}(\mathfrak{g}) + j$.

In particular, if M has maximal nilpotency index (i.e. $\dim V_1 = 2$, $\dim V_i/V_{i-1} = 1 \ \forall i > 1$), then it does not admit generalized complex structures of type k for $k \geq 2$.

Proof. First observe that for any nilpotent Lie algebra $\mathrm{nil}(\mathfrak{g}) \leq 2n - 1$, so the theorem only restricts the existence of structures of type $k > 1$.

According to Lemma 3.3, if none of the θ_i have nilpotent degree j, then there can be none with nilpotent degree $j + 1, j + 2, \ldots$ by the condition (3.5). Hence we obtain upper bounds for the nilpotent degrees of $\{\theta_1, \ldots, \theta_k\}$, as follows. First, θ_1 has nilpotent degree 1 (by Example 3). Then, if $\mathrm{nil}(\theta_2) \geq j + 2$, this would imply that no θ_i had nilpotent degree $j + 1$, which is a contradiction by the previous paragraph. Hence $\mathrm{nil}(\theta_2) < j + 2$. In general, $\mathrm{nil}(\theta_i) < j + i$. Suppose that M admits a generalized complex structure of type $k > 1$. Then we see that $\mathrm{nil}(\theta_k) < j + k$. By Example 4, we see this means that $\dim V_{j+k-1} \geq 2k$.

On the other hand, $\dim V_{j+k-1} = 2n - \dim \mathfrak{g}^*/V_{j+k-1}$, and since $\mathfrak{g}^* = V_{\mathrm{nil}(\mathfrak{g})}$, we have

$$\mathfrak{g}^*/V_{j+k-1} = V_{\mathrm{nil}(\mathfrak{g})}/V_{j+k-1} \cong V_{\mathrm{nil}(\mathfrak{g})}/V_{\mathrm{nil}(\mathfrak{g})-1} \oplus \cdots \oplus V_{j+k}/V_{j+k-1},$$

and the $\mathrm{nil}(\mathfrak{g}) - j - k + 1$ summands on the right have dimension 1, by hypothesis. Hence $\dim V_{j+k-1} = 2n - \mathrm{nil}(\mathfrak{g}) + j + k - 1$, and combining with the previous inequality, we obtain

$$k < 2n - \mathrm{nil}(\mathfrak{g}) + j,$$

as required. For the last claim, observe that M^{2n} has maximal nilpotency index when $\mathrm{nil}(M) = 2n - 1$, in which case (3.5) holds for $j = 1$. $\square$

Constraints beyond those mentioned here may be obtained if one considers the fact that $\Omega \wedge \overline{\Omega}$ defines a foliation and that ω restricts to a symplectic form on each leaf. Both the leafwise nondegeneracy of ω and the requirement of being closed on the leaves lead to useful constraints on what types of generalized complex structure may exist, as we shall see in the following sections.

4. Generalized complex structures on 6-nilmanifolds

In this section, we turn to the particular case of 6-dimensional nilmanifolds. The problem of classifying those which admit left-invariant complex (type 3) and symplectic (type 0) structures has already been solved [9, 2], so we are left with the task of determining which 6-nilmanifolds admit left-invariant generalized complex structures of types 1 and 2. The result of our classification is presented in Table 1, where explicit examples of all types of left-invariant generalized complex structures are given, whenever they exist. The main results establishing this classification are Theorems 4.1 and 4.2. Throughout this section we often require the use of a Malcev basis $\{e_1, \ldots, e_6\}$ as well as its dual basis $\{\partial_1, \ldots, \partial_6\}$. We use $e_{i_1 \cdots i_p}$ as an abbreviation for $e_{i_1} \wedge \cdots \wedge e_{i_p}$.

4.1. Generalized complex structures of type 2.
By the results of the last section, a left-invariant structure of type 2 is given by a closed form $\rho = \exp(B + i\omega)\theta_1\theta_2$, where $\omega \wedge \theta_1\theta_2\overline{\theta}_1\overline{\theta}_2 \neq 0$. As a consequence of Theorem 3.2, any 6-nilmanifold with maximal nilpotence step cannot admit this kind of structure. We now rule out some additional nilmanifolds, using Lemma 3.3.

Lemma 4.1. If a 6-nilmanifold M has nilpotent Lie algebra given by $(0, 0, 0, 12, 14, -)$, and has nilpotency index 4, then M does not admit left-invariant generalized complex structures of type 2.

Proof. Suppose that M admits a structure ρ of type 2. Since M has nilpotency index 4, V_{i+1}/V_i is 1-dimensional for $i \geq 1$. From $d\theta_1 = 0$ and Lemma 3.3, we see that $\theta_1 = z_1 e_1 + z_2 e_2 + z_3 e_3$ and $\mathrm{nil}(\theta_2) \leq 2$, thus $\theta_2 = w_1 e_1 + w_2 e_2 + w_3 e_3 + w_4 e_4$. The conditions $d(\theta_1 \theta_2) = 0$ and $\theta_1 \theta_2 \overline{\theta_1 \theta_2} \neq 0$ together imply $z_3 = 0$. Further, the annihilator of $\theta_1 \theta_2 \overline{\theta_1 \theta_2}$ is generated by $\{\partial_5, \partial_6\}$. Hence, the nondegeneracy condition $\omega^2 \wedge \Omega \wedge \overline{\Omega} \neq 0$ implies that

$$B + i\omega = (k_1 e_1 + \ldots k_4 e_4 + k_5 e_5)e_6 + \alpha,$$

where $k_5 \neq 0$ and $\alpha \in \wedge^2 \langle e_1, \cdots, e_5 \rangle$. But, using the structure constants, we see that $d\rho$ must contain a nonzero multiple of e_6, and so cannot be closed. $\qquad\square$

Lemma 4.2. Nilmanifolds associated to the algebras defined by

$$(0, 0, 0, 12, 14, 13 - 24),$$
$$(0, 0, 0, 12, 14, 23 + 24)$$

do not admit left-invariant generalized complex structures of type 2.

Proof. Each of these nilmanifolds has $\mathrm{nil}(\mathfrak{g}) = 3$, with $\dim V_1 = 3$ and $\dim V_2/V_1 = 1$. Suppose either nilmanifold admitted a structure ρ of type 2. If $\mathrm{nil}(\theta_2) = 2$, we could use the argument of the previous lemma to obtain a contradiction. Hence, suppose $\mathrm{nil}(\theta_2) = 3$. Lemma 3.3 then implies that $\theta_2 \wedge \overline{\theta_2} \neq 0 \pmod{V_2}$, which means it must have nonzero e_5 and e_6 components.

Now, if θ_1 had a nonzero e_3 component, then $d\theta_2 \wedge \theta_1$ would have nonzero e_{234} component, contradicting (3.4). Hence

$$(4.1) \qquad \theta_1 = z_1 e_1 + z_2 e_2 \qquad \theta_2 = \sum_{i=1}^{6} w_i e_i.$$

But for these, the coefficient of e_{123} in $d\theta_2 \wedge \theta_1$ would be $-z_2 w_6$ (for the first nilmanifold) or $z_1 w_6$ (for the second), in each case implying $\theta_1 \wedge \overline{\theta_1} = 0$, a contradiction. $\qquad\square$

Lemma 4.3. Nilmanifolds associated to the algebras defined by

$$(0, 0, 12, 13, 23, 14),$$
$$(0, 0, 12, 13, 23, 14 - 25)$$

do not admit left-invariant generalized complex structures of type 2.

Proof. Each of these nilmanifolds have $\mathrm{nil}(\mathfrak{g}) = 4$, with $\dim V_1 = 2$, $\dim V_2 = 3$, and $\dim V_3 = 5$. Suppose either nilmanifold admitted a structure ρ of type 2. V_4/V_3 is 1-dimensional and so Lemma 3.3 implies that $\mathrm{nil}(\theta_2) \neq 4$. Since θ_1 is closed and satisfies $\theta_1 \overline{\theta}_1 \neq 0$, we may rescale it to obtain $\theta_1 = e_1 + z_2 e_2$. The condition $d\theta_2 \in \mathcal{I}(\theta_1)$ implies we can write $\theta_2 = w_2 e_2 + w_3 e_3 + w_4(e_4 + z_2 e_5)$. Now let

$$B + i\omega = \sum_{i<j} k_{ij} e_{ij}$$

and differentiate ρ using the structure constants. In both cases, $d\rho = 0$ implies $B + i\omega = \xi_1 \theta_1 + \xi_2 \theta_2$ for 1-forms ξ_i. Therefore ω is degenerate on the leaves defined by $\mathrm{Ann}(\theta_1 \theta_2 \overline{\theta_1 \theta_2})$, contradicting the requirement $\omega^2 \wedge \Omega \wedge \overline{\Omega} \neq 0$. $\qquad\square$

Theorem 4.1. *The only 6-dimensional nilmanifolds not admitting left-invariant generalized complex structures of type 2 are those with maximal nilpotency index and those excluded by Lemmas 4.1, 4.2, and 4.3.*

Proof. In Table 1, we provide explicit left-invariant generalized complex structures of type 2 for all those not excluded by the preceding lemmas and Theorem 3.2. □

4.2. **Generalized complex structures of type 1.** A left-invariant generalized complex structure of type 1 is given by $\rho = \exp(B + i\omega)\theta_1$, where $\omega^2 \wedge \theta_1\overline{\theta_1} \neq 0$. Note that this implies that ω is a symplectic form on the 4-dimensional leaves of the foliation determined by $\theta_1 \wedge \overline{\theta_1}$.

Theorem 4.2. *The only 6-nilmanifolds which do not admit left-invariant generalized complex structures of type 1 are those associated to the algebras defined by* $(0, 0, 12, 13, 23, 14)$ *and* $(0, 0, 12, 13, 23, 14 - 25)$.

Proof. In Table 1, we provide explicit forms defining type 1 structures for all 6-nilmanifolds except the two mentioned above.

Suppose the nilmanifold is associated to the Lie algebra $(0, 0, 12, 13, 23, 14)$. Then up to a choice of Malcev basis, a generalized complex structure of type 1 can be written
$$\rho = \exp(B + i\omega)(e_1 + z_2 e_2), \qquad B + i\omega = \sum_{i<j} k_{ij} e_{ij}.$$

The condition $d\rho = 0$ implies that
$$(-k_{36}e_{314} + k_{45}e_{135} - k_{45}e_{423} + k_{46}e_{136} + k_{56}e_{236} - k_{56}e_{514})(e_1 + z_2 e_2) = 0.$$

The vanishing of the e_{1245}, e_{1236}, e_{1235}, and e_{1234} components imply successively that k_{56}, k_{46}, k_{45}, and k_{36} all vanish.

The leaves of the distribution $\mathrm{Ann}(\theta_1\overline{\theta_1})$ are the tori generated by $\partial_3, \partial_4, \partial_5, \partial_6$, and the previous conditions on $B + i\omega$ imply that on these leaves, ω restricts to $\mathrm{Im}(k_{34})e_{34} + \mathrm{Im}(k_{35})e_{35}$ which is degenerate, contradicting $\omega^2 \wedge \theta_1\overline{\theta_1} \neq 0$. An identical argument holds for the nilpotent algebra $(0, 0, 12, 13, 23, 14 - 25)$. □

5. β-transforms of generalized complex structures

In this section, we will use Proposition 1.1 to show that any left-invariant complex structure on a nilmanifold may be deformed into a left-invariant generalized complex structure of type 1. By connecting the type 3 and type 1 structures in this way, we go on to show that the two disconnected components of the left-invariant complex moduli space on the Iwasawa manifold may be joined by paths of generalized complex structures.

Theorem 5.1. *Every left-invariant complex structure 2n-dimensional nilmanifold can be deformed, via a β-field, into a left-invariant generalized complex structure of type $n - 2$.*

Proof. According to Proposition 1.1, such a deformation can be obtained if we find a holomorphic Poisson structure. Let us construct such a bivector β.

By Theorem 3.1, a left-invariant complex structure on a nilmanifold has a holomorphically trivial canonical bundle. Let $\Omega = \theta_1 \wedge \cdots \wedge \theta_n$ be a holomorphic volume form, and choose the θ_i according to Lemma 3.1, so that, by Corollary 1, the differential forms $\theta_1, \theta_1 \wedge \theta_2, \ldots, \theta_1 \wedge \cdots \wedge \theta_n$ are all holomorphic. Now let $\{x_1, \ldots, x_n\}$ be a dual basis for the holomorphic tangent bundle. By interior product with Ω, we see that the multivectors $x_n, x_{n-1} \wedge x_n, \ldots, x_1 \wedge \cdots \wedge x_n$ are all holomorphic as well. In particular, we have a holomorphic bivector $\beta = x_{n-1} \wedge x_n$. Calculating the Schouten bracket of this bivector with itself, we obtain

$$[\beta, \beta] = [x_{n-1} \wedge x_n, x_{n-1} \wedge x_n] = 2[x_{n-1}, x_n] \wedge x_{n-1} \wedge x_n = 0,$$

where the last equality follows from the fact that $[x_{n-1}, x_n] \in \langle x_{n-1}, x_n \rangle$, since $\theta_i([x_{n-1}, x_n]) = -d\theta_i(x_{n-1}, x_n) = 0$ for $i < n - 1$, by Corollary 1.

Hence β gives rise to a deformation of the generalized complex structure. The resulting structure $\tilde{\rho}$ is given by the following differential form:

$$\tilde{\rho} = e^{i_\beta}\rho = \rho + i_\beta \rho = e^{\theta_{n-1} \wedge \theta_n} \theta_1 \wedge \cdots \wedge \theta_{n-2},$$

and we see immediately that it is a left-invariant generalized complex structure of type $n - 2$. $\qquad\square$

In [4], Ketsetzis and Salamon study the space of left-invariant complex structures on the Iwasawa nilmanifold. This manifold is the quotient of the complex 3-dimensional Heisenberg group of unipotent matrices by the lattice of unipotent matrices with Gaussian integer entries. As a nilmanifold, it has structure $(0, 0, 0, 0, 13 - 24, 14 + 23)$. Ketsetzis and Salamon observe that the space of left-invariant complex structures with fixed orientation has two connected components which are distinguished by the orientation they induce on the complex subspace $\langle \partial_1, \partial_2, \partial_3, \partial_4 \rangle$.

Connecting the two components. Consider the left-invariant complex structures defined by the closed differential forms $\rho_1 = (e_1 + ie_2)(e_3 + ie_4)(e_5 + ie_6)$ and $\rho_2 = (e_1 + ie_2)(e_3 - ie_4)(e_5 - ie_6)$. These complex structures clearly induce opposite orientations on the space $\langle \partial_1, \ldots, \partial_4 \rangle$, so lie in different connected components of the moduli space of left-invariant complex structures.

By Theorem 5.1, the first complex structure can be deformed, by the β-field $\beta_1 = \frac{-1}{4}(x_3 - ix_4)(x_5 - ix_6)$ into the generalized complex structure

$$e^\beta \rho_1 = e^{-(e_{35} - e_{46}) - i(e_{45} + e_{36})}(e_1 + ie_2),$$

and then, by the action of the closed B-field $B_1 = e_{35} - e_{46}$, into

$$\rho = e^{-i(e_{45} + e_{36})}(e_1 + ie_2).$$

On the other hand, the second complex structure can be deformed, via the β-field $\beta_2 = \frac{1}{4}(x_3 + ix_4)(x_5 + ix_6)$, into the type 1 generalized complex structure

$$e^\beta \rho_2 = e^{(e_{35} - e_{46}) - i(e_{45} + e_{36})}(e_1 + ie_2),$$

and then by the B-field $B_2 = -(e_{35} - e_{46})$ into

$$\rho = e^{-i(e_{45} + e_{36})}(e_1 + ie_2),$$

which is the same generalized complex structure obtained from ρ_1.

Therefore, by deforming by β- and B-fields, the two disconnected components of the moduli space of left-invariant complex structures can be connected, through generalized complex structures.

6. An 8-dimensional nilmanifold

We have established that all 6-dimensional nilmanifolds admit generalized complex structures. One might ask whether every even-dimensional nilmanifold admits left-invariant generalized complex geometry. In this section, we answer this question in the negative, by presenting an 8-dimensional nilmanifold which does not admit any type of left-invariant generalized complex structure.

Example 5. Consider a nilmanifold M associated to the 8-dimensional nilpotent Lie algebra defined by

$$(0, 0, 12, 13, 14, 15, 16, 36 - 45 - 27).$$

Since it has maximal nilpotency index, Theorem 3.2 implies that it may only admit left-invariant generalized complex structures of types 1 and 0. We exclude each case in turn:

- *Type 1*: Suppose there is a type 1 structure, defined by the left-invariant form $\rho = e^{B+i\omega}\theta_1$. Then $d\theta_1 = 0$ and $\theta_1\overline{\theta}_1 \neq 0$ imply that $\theta_1\overline{\theta}_1$ is a multiple of e_{12} and therefore ω must be symplectic along the leaves defined by $\langle\partial_3,\ldots,\partial_8\rangle$. These leaves are actually nilmanifolds associated to the nilpotent algebra defined by $(0, 0, 0, 0, 0, 12 + 34)$, which admits no symplectic structure, and so we obtain a contradiction.
- *Type 0*: The real second cohomology of M is given by

$$H^2(M, \mathbb{R}) = \langle e_{23}, e_{34} - e_{25}, e_{17}\rangle,$$

  and since e_8 does not appear in any of its generators, it is clear that any element in $H^2(M, \mathbb{R})$ has vanishing fourth power, hence excluding the existence of a symplectic structure.

In this way, we see that the 8-dimensional nilmanifold M given above admits no left-invariant generalized complex structures at all.

Nilmanifold class	b_1	b_2	Complex (type 3)	Type 2	Type 1	Symplectic (type 0)
$(0,0,12,13,14,15)$	2	3	–	–	$(1+i2)\exp i(36-45)$	$16+34-25$
$(0,0,12,13,14,34+52)$	2	2	–	–	$(1+i2)\exp(-45+36+i(36+45))$	–
$(0,0,12,13,14,23+15)$	2	3	–	–	$(1+i2)\exp i(36-45)$	$16+24+34-25$
$(0,0,12,13,23,14)$	2	4	–	–	–	$15+24+34-26$
$(0,0,12,13,23,14-25)$	2	4	–	–	–	$15+24-35+16$
$(0,0,12,13,23,14+25)$	2	4	$(1+i2)(4+i5)(3+i6)$	$(1+i2)(4+i5)\exp i(36)$	$(1+i2)\exp(43-56+i(46-35))$	$15+24+35+16$
$(0,0,12,13,14+23,34+52)$	2	2	–	–	$(1+i2)\exp(45-35+36+i(-36+45-16))$	–
$(0,0,12,13,14+23,24+15)$	2	3	–	–	$(1+i2)\exp(2\times 35+i(36-45))$	$16+2\times 34-25$
$(0,0,0,12,13,14+35)$	3	5	–	$(1+2+i3)(4+i5)\exp i(26)$	$(1+i2)\exp i(36+45)$	–
$(0,0,0,12,13,14+23)$	3	6	$(1+i2)(3-2\times i4)(5+2\times i6)$	$(2+i3)(4+i5)\exp i(16-34)$	$(1+i2)\exp i(36+45)$	$16-2\times 34-25$
$(0,0,0,12,13,24)$	3	6	$(1+i2)(3+4+i4)(5+6-i6)$	$(1+2+i3)(4+i5)\exp i(26)$	$(1+i2)\exp i(35+46)$	$26+14+35$
$(0,0,0,12,13,14)$	3	6	$(1+i2)(3+i4)(5+i6)$	$(2+i3)(4+i5)\exp i(16)$	$(1+i2)\exp(35-46+i(36+45))$	$16+24+35$
$(0,0,0,12,13,23)$	3	8	$(1+i2)(3+i4)(5+i6)$	$(1+i2)(5+i6)\exp i(16-34)$	$(1+i2)\exp(35-46+i(36+45))$	$15+24+36$
$(0,0,0,12,14,15+23)$	3	5	–	–	$(1+i3)\exp i(26-45)$	$13+26-45$
$(0,0,0,12,14,15+23+24)$	3	5	–	–	$(1+i3)\exp i(26-45)$	$13+26-45$
$(0,0,0,12,14,15+24)$	3	5	–	–	$(1+i3)\exp i(26-45)$	$13+26-45$
$(0,0,0,12,14,15)$	3	5	–	–	$(1+i3)\exp i(26-45)$	$13+26-45$
$(0,0,0,12,14,24)$	3	5	$(1+i2)(3+i4)(5+i6)$	$(1+i2)(5+i6)\exp i(34)$	$(1+i2)\exp(35-46+i(45+36))$	–
$(0,0,0,12,14,13+42)$	3	5	$(1+i2)(3+i4)(2\times 5-i6)$	–	$(1+i2)\exp i(35+46)$	$15+26+34$
$(0,0,0,12,14,23+24)$	3	5	$(1+i2)(3+4+i3)(5+6+i6)$	–	$(1+i2)\exp(35+46+i(35-46))$	$16-34+25$
$(0,0,0,12,23,14+35)$	3	5	–	$(1+2+i3)(5+i4)\exp(3+i1)6$	$(1+i2)\exp(36+45+i(36-45))$	–
$(0,0,0,12,23,14-35)$	3	5	$(1+i3)(4-i5)(2+i6)$	$(1+i3)(4-i5)\exp i(26)$	$(1+i3)\exp(24+56+i(25+46))$	–
$(0,0,0,12,14-23,15+34)$	3	4	–	$(1+i2)(3+i4)\exp i(56)$	$(2+i3)\exp i(16+35+45-26)$	$16+35+24$
$(0,0,0,12,14+23,13+42)$	3	5	$(1+i2)(3-i4)(5+i6)$	$(1+i2)(3-i4)\exp i(56)$	$(1+i2)\exp(35+46+i(36-45))$	$15+2\times 26+34$
$(0,0,0,0,12,15+34)$	4	6	–	$(1+i2)(3+i4)\exp i(56)$	$(3+i4)\exp i(25+16)$	–
$(0,0,0,0,12,15)$	4	7	–	$(1+i2)(3+i4)\exp i(56)$	$(1+i2)\exp i(34+56)$	$16+25+34$
$(0,0,0,0,12,14+25)$	4	7	$(1+i2)(4+i5)(3+i6)$	$(1+i2)(4+i5)\exp i(36)$	$(1+i2)\exp(34+56+i(35-46))$	$13+26+45$
$(0,0,0,0,12,14+23)$	4	8	$(1+i2)(3-i4)(5+i6)$	$(1+i2)(3-i4)\exp i(56)$	$(1+i2)\exp(35+46+i(36-45))$	$13+26+45$
$(0,0,0,0,12,34)$	4	8	$(1+i2)(3+i4)(5+i6)$	$(1+i2)(3+i4)\exp i(56)$	$(1+i2)\exp(35-46+i(45+36))$	$15+36+24$
$(0,0,0,0,12,13)$	4	9	$(2+i3)(1+i4)(5+i6)$	$(2+i3)(5+i6)\exp i(14)$	$(2+i3)\exp(15-46+i(16+45))$	$16+25+34$
$(0,0,0,0,13+42,14+23)$	4	8	$(1+i2)(3-i4)(5+i6)$	$(1+i2)(3+i4)\exp i(56)$	$(1+i2)\exp(35+46+i(36-45))$	$16+25+34$
$(0,0,0,0,0,12+34)$	5	9	$(1+i2)(3+i4)(5+i6)$	$(1+i2)(3+i4)\exp i(56)$	$(1+i2)\exp i(36+45)$	–
$(0,0,0,0,0,12)$	5	11	$(1+i2)(3+i4)(5+i6)$	$(1+i2)(3+i4)\exp i(56)$	$(1+i2)\exp i(36+45)$	$16+23+45$
$(0,0,0,0,0,0)$	6	15	$(1+i2)(3+i4)(5+i6)$	$(1+i2)(3+i4)\exp i(56)$	$(1+i2)\exp i(36+45)$	$12+34+56$

Table 1: Differential forms defining left-invariant Generalized Calabi-Yau structures. The symbol '-' denotes nonexistence.

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