Complex Geometry – Mock Exam 1

Notes:

1. **Write your name and student number *****clearly*** **on each page of written solutions you hand in.**
2. You can give solutions in English or Dutch.
3. You are expected to explain your answers.
4. You are allowed to consult text books and class notes.
5. You are **not** allowed to consult colleagues, calculators, computers etc.
6. Advice: read all questions first, then start solving the ones you already know how to solve or have good idea on the steps to find a solution. After you have finished the ones you found easier, tackle the harder ones.

Questions

Exercise 1 (1.0 pt)**.** Let $n > 1$ and let $f\colon \mathbb{C}^n \to \mathbb{C}$ be holomorphic. Show that for every $w \in \mathrm{Im}(f)$ there is $z \in f^{-1}(w)$ with $\|z\| > 0$.

Proof. Assume by contradiction that there is $w_0 \in \mathbb{C}$ $f^{-1}(w_0) = \{0\}$ and consider the auxiliary function $\tilde{f}\colon \mathbb{C}^n \to \mathbb{C}$, $\tilde{f}(z) = \frac{1}{f(z)-w_0}$. Since $\tilde{f}$ is a quotient of holomorphic functions for which the numerator is nonzero at 0 but the denominator vanishes at 0 we conclude that $\tilde{f}$ has a non removable singularity at 0.

Then $\tilde{f}$ is a holomorphic function defined on $\mathbb{C}^n\backslash\{0\}$. By Hartog's theorem $\tilde{f}$ can be extended to 0 by continuity, which gives us a contradition. $\square$

Exercise 2 (2.0 pt)**.** Let $f\colon M^n \to N^n$ be a map between compact connected oriented manifolds of the same dimension. The degree of f is defined by

$$\deg(f) = \frac{\int_M f^*\rho}{\int_N \rho},$$

where ρ is any top degree form on N with $\int_N \rho \neq 0$.

- Show that $\deg(f)$ does not depend on the form ρ (hint observe that the computation defining the degree takes place in cohomology and use that for compact, connected, orientable manifolds $H^{top}(M;\mathbb{R}) = \mathbb{R}$).
- Let $p \in N$ be a regular value of f, so that p has a neighbourhood U for which $f^{-1}(U) = \cup_{i=1}^k U_i$ and $f\colon U_i \to U$ is a diffeomorphism for all i. Pick $\rho \in \Omega^n(N)$ a top degree form supported in U and show that
$$\deg(f) = \sum_{i=1}^k \epsilon(p_i),$$
where $\epsilon(p_i) = 1$ if $df_{p_i}: Tp_iM \to T_pN$ is orientation preserving and $\epsilon(p_i) = -1$ otherwise.

- Show that if M and N are complex manifolds and f is holomorphic, then $\deg(f)$ is non-negative and if f *has invertible derivative at a point*, then $\deg(f) > 0$ and equals the number of points in the pre-image of a regular value.

Proof. 1) For the first part we use that the top degree cohomology of a compact connected orientable manifold is isomorphic to $\mathbb{R}$. Since N is orientable, it has a volume form, say, ρ for which $\int_N \rho \neq 0$, in particular, we see that ρ is not exact because by Stokes theorem for and exact form we have

$$\int_N d\tau = \int_{\partial N} \tau = 0,$$

since N does not have boundary. In particular $[\rho]$ generates the top degree cohomology of N.

Given another top degree form, $\tilde{\rho}$, which represents a nonzero class, we have that $[\tilde{\rho}] = \lambda[\rho]$ for a nonzero λ, hence we can compare the degree of f computed with respect these two top degree forms:

$$\deg_{\tilde{\rho}}(f) = \frac{\int_M f^*\tilde{\rho}}{\int_N \tilde{\rho}} = \frac{\int_M f^*(\lambda\rho + d\tau)}{\int_N \lambda\rho + d\tau} = \frac{\int_M f^*(\lambda\rho)}{\int_N \lambda\rho} = \frac{\int_M f^*(\rho)}{\int_N \rho} = \deg_\rho(f).$$

2) Picking U as in the statement of the exercise (also connected), let ρ be a top degree form with support in U and for which $\int_U \rho = 1$. We can construct one such form using a non-vanishing function on U, a bump function and normalization (so the integral is 1). Since for each U_i, $f\colon U_i \to U$ is a diffeomorphism,

$$\int_{U_i} f^*\rho = \pm 1,$$

where the sign depends on whether $f\colon U_i \to U$ is orientation preserving or orientation reversing. Since to determine if a diffeomorphism is orietation preserving or reversing can be done by checking its derivative at a point, we conclude that the sign above is $+1$ if $df_{p_i}\colon T_{p_i}M \to T_pN$ is orientation preserving and -1 otherwise.

Finally, since $\sup f^*\rho \subset \cup U_i$, we have that

$$\deg(f) = \frac{\int_M f^*\rho}{\int_N \rho} = \sum_{i=1}^{k} \int_{U_i} f^*\rho = \sum_{i=1}^{k} \epsilon(p_i).$$

3) If M and N are complex and f is holomorphic, the derivative of f is orientation preserving hence for every regular point p_i, $\epsilon(p_i) = 1$. Therefore

$$\deg(f) = |f^{-1}(p)|$$

for any regular value p.

Finally, if f has invertible derivative at a point, say p_0 and $f(p_0) = p$, then by the inverse function theorem f is invertible in a neighbouhood of p_0, say $f\colon U \subset M \to V \subset N$.By Sard's theorem there is a regular value in $q \in V$ and hence $\deg(f) = |f^{-1}(q)| > 0$, since q has at least one inverse image in U.

Remark 1: This was not part of the exercise, but it is worth pointing out. If $p \in N$ is not in the image of f, then p is a regular value and hence $\deg(f) = |f^{-1}(p)| = 0$. In particular we conclude that if the derivative of f is an isomorphism at some point, then f is surjective.

Remark 2: If M and N are complex curves, for each $p \in M$ either $df_p = 0$ or df_p is an isomorphism. That is either f has positive degree (and hence is surjective) or it has degree zero and is constant. □

Exercise 3 (2.0 pt)**.** Let Σ be a compact Riemann surface and let $p \in \Sigma$. Consider the complex line bundle $\pi\colon L_p \to \Sigma$ which has two trivializations, one over $\Sigma\backslash\{p\}$ and one over a disc, D, centered at p,

$$\Phi_0\colon L_p|_{\Sigma\backslash\{p\}} \to \Sigma\backslash\{p\} \times \mathbb{C}, \qquad \Phi_1\colon L_p|_D \to D \times \mathbb{C},$$

with transition function

$$\Phi_1 \circ \Phi_0^{-1} \colon D\backslash\{0\} \times \mathbb{C} \to D\backslash\{0\} \times \mathbb{C}, \qquad \Phi_1 \circ \Phi_0^{-1}(z, v) = (z, zv),$$

where z denotes the coordinate on the disc.

- Show that L_p has a section which vanishes at p.
- Show that L_p is not isomorphic to the trivial bundle.
- Let $p_1 \neq p_2$. Show that $L_{p_1} \otimes L_{p_2}^{-1}$ has a section with a zero at p_1 and a simple pole at p_2.
- Let $p_1 \neq p_2$. Show that if $L_{p_1} \otimes L_{p_2}^{-1}$ is isomorphic to the trivial bundle then $\Sigma = \mathbb{C}P^1$.

Proof. 1) Define a section s of L_p be declaring that on the trivialization Φ_0 the section is constant and equal to one and in the trivialization Φ_1 it is the disc coordinate, that is, s is defined by

$$\Phi_0(p, s(p)) = (p, 1) =: s_0, \qquad \Phi_1(z, s(z)) = (z, z) := s_1.$$

To conclude the exercise we just need to show that these two local expressions agree when we compare trivializations.

$$\Phi_1 \circ \Phi_0^{-1}(s_0(z)) = \Phi_1 \circ \Phi_0^{-1}(z, 1) = (z, z \cdot 1) = s_1(z).$$

2) If L_p was isomorphic to the trivial bundle it would also admit a nowhere vanishing holomorphic section, σ. Hence we can compare the section from part 1 with sigma to obtain $s = f\sigma$. Since both s and σ are holomorphic, f is also holomorphic (in a compact manifold), hence f is constant. Since $s(p) = 0$ and $\sigma(p) \neq 0$ we have $f(p) = 0$. But for $q \neq p$, $s(q) \neq 0$, hence $f(q) \neq 0$ which is a contradiction.

3) Similar to the first part, define a section constant and equal to 1 on the trivialization Φ_0, equal to z on Φ_1 and $\frac{1}{z}$ on the trivialization Φ_2 in a neighbourhood of p_2 and the same computation from before yields a holomorphic 'section' with the desired properties (notice that since it has a pole at p_2 it is not really a section.

4) If $L_{p_1} \otimes L_{p_2}$ is isomorphic to the trivial bundle, it has a nowhere vanishing section σ and we can compare s from part 3 to σs to obtain $s = f\sigma$, where f is a meromorphic function with a simple zero at p_1 that is $f \colon \Sigma \to \mathbb{C} \cup \{\infty\} = \mathbb{C}P^1$ is holomorphic, has 0 as a regular value and has degree 1, hence f is a holomorphic bijection (see exercise 2). If df vanished at some point we would have, in appropriate coordinates that $f(z) = z^k$ where k is the order of the first nonvanishing derivative of f. In particular we would have that $\deg(f) \geq k$, which is a contradiction to the fact that f has degree 1, hence df never vanishes and f is a biholomorphism. □

Exercise 4 (2.0 pt)**.** Let (M^{2n}, I, g) be an almost complex manifold with compatible Riemannian metric.

- Show that the two form defined by $\omega(X, Y) = g(X, IY)$ is of type (1,1).
- Let $\Omega \in \Omega^{n,0}(M)$ be a nonvanishing local section of the canonical bundle of M. Show that if $d\omega \wedge \Omega \neq 0$ then I is not integrable.
- Show that the almost complex structure on S^6 determined by octonionic multiplication is compatible with the Euclidean metric of $\mathbb{R}^7$ restricted to the sphere.
- Show that the almost complex structure of S^6 is not integrable.

Proof. 1) We can read off the decomposition of 2-forms into $\Omega^{1,1}$ and $\Omega^{2,0} \oplus \Omega^{0,2}$ from how I^* acts on these spaces. Namely, on 2-forms

$$I^*\alpha = \alpha \text{ iff } \alpha \in \Omega^{1,1}, \qquad I^*\alpha = -\alpha \text{ iff } \alpha \in \Omega^{2,0} \oplus \Omega^{0,2},$$

therefore, to show that ω is of type $(1,1)$ we need to show that $I^*\omega = \omega$, but this follows directly from compatibility with the metric:

$$I^*\omega(X,Y) = \omega(IX,IY) = g(IX,IIY) = g(X,IY) = \omega(X,Y), \qquad \forall X,Y.$$

2) If I is integrable, then $d\omega \in \Omega^{2,1} \oplus \Omega^{1,2}$ and hence $d\omega \wedge \Omega = 0$ as the holomorphic degree of its components is greater than the complex dimension of the manifold.

3) Is a direct check using octonionic multiplication.

4) I still have to do. $\square$

Exercise 5 (2.0 pt)**.** Let $\mathbb{Z}_2 = \langle a\colon a^2 = e\rangle$ act on $\mathbb{C}^2$ by $a \cdot (z_1, z_2) = -(z_1, z_2)$.

- Resolve the singularity of $\mathbb{C}^2/\mathbb{Z}_2$,
- Let $\Omega = dz_1 \wedge dz_2 \in \Omega^2(\mathbb{C}^2)$. Show that Ω extends as a nonvanishing section of the canonical bundle of most natural resolution $\widetilde{\mathbb{C}^2}$ of $\mathbb{C}^2$.

Let $\mathbb{Z}^2$ act on $\mathbb{C}^2$ by $(m,n) \cdot (z_1, z_2) = (z_1 + m, z_2 + n)$.

- Show that $\Omega = dz_1 \wedge dz_2 \in \Omega^2(\mathbb{C}^2)$ gives rise to a nonvanishing section of the canonical bundle of the torus $T^4 = \mathbb{C}^2/\mathbb{Z}^2$,
- Show that the $\mathbb{Z}_2$ action of the first part of this exercise gives rise to a $\mathbb{Z}_2$ action on T^4 and compute its fixed points.
- Show that the canonical bundle of the natural resolution of the singularities of $T^4/\mathbb{Z}_2$ has a nowhere vanishing holomorphic section.

Proof. 1) Following the corresponding example from the notes we blow up the origin on $\mathbb{C}^2$ and see that the $\mathbb{Z}_2$ action induces an action on $\tilde{\mathbb{C}}^2$ which has the exceptional divisor as fixed point set and rotates each fiber of the tautological bundle by π. We will need the concrete formulas here.

On $\tilde{\mathbb{C}}^2$ we have

$$u_1 = z_1, v_1 = z_2/z_1, \Rightarrow z_1 = u_1, z_2 = u_1 v_1,$$
$$u_2 = z_1/z_2, v_2 = z_2, \Rightarrow z_1 = u_2 v_2, z_2 = v_2.$$

The $\mathbb{Z}_2 = \langle a\colon a^2 = e\rangle$ action in these charts is

$$a \cdot (u_1, v_1) = (-u_1, v_1) \qquad a \cdot (u_2, v_2) = (u_2, -v_2)$$

And the quotient space is covered by two charts, say (x_1, y_1) and (x_2, y_2) related to the (u,v)-coordinates via the quotient map:

$$(x_1, y_1) = (u_1^2, v_1), \qquad (x_2, y_2) = (u_2, v_2^2).$$

The transition of coordinates between (u_1, v_1) and (u_2, v_2) gives the transition of coordinates between (x_1, y_1) and (x_2, y_2):

$$(x_1, y_1) = (u_1^2, v_1) = ((u_2 v_2)^2, 1/u_2) = (y_2 x_2^2, x_2^{-1}).$$

2) The pull back of $dz_1 \wedge dz_2$ to the (u,v)-coordinates is

$$dz_1 \wedge dz_2 = du_1 \wedge d(u_1 v_1) = u_1 du_1 \wedge dv_1 = \frac{1}{2} du_1^2 \wedge dv_1,$$

which can be pushed forward to the (x_1, y_1)-coordinates as

$$\frac{1}{2} du_1^2 \wedge dv_1 = \frac{1}{2} dx_1 \wedge dy_1.$$

A similar computation holds for the (x_2, y_2)-coordinates and using the change of coordinates between (x_1, y_1) and (x_2, y_2) we can check we didn't mess anything up

$$dx_1 \wedge dy_1 = d(y_2 x_2^2) \wedge d(x_2^{-1}) = x_2^2 dy_2 \wedge \left(\frac{-1}{x_2^2} dx_2 \right) = dx_2 \wedge dy_2.$$

3) Ω is a nonvanishing $(2,0)$-form on $\mathbb{C}^2$ invariant under the $\mathbb{Z}^2$ action, hence it descends to a nonvanishing $(2,0)$-form on T^4.

4) Using a cube as the fundamental domain of the $\mathbb{Z}_2$ action, the fixed points are those of the form (x_1, x_2, x_3, x_4) where $x_i \in \{0, 1/2\}$. Therefore there are 16 fixed points.

5) At each fixed point the action is isomorphic to the one of part 1, hence according to part 2 if we resolve each of the 16 the singularities as in part 2 the 2-form $dz_1 \wedge dz_2$ from $\mathbb{C}^2$ descended to T^4 induces a nowhere vanishing closed $(2,0)$-form on the resolution. $\square$