Analysis on Manifolds
lecture notes for the 2012/2013 mastermath course

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LECTURE 1
Differential operators

1.1. Differential operators I: trivial coefficients

In this section we discuss differential operators acting on spaces of functions on a manifold, while in the next section we will move to those acting on spaces of sections of vector bundles. We first discuss differential operators on an open subset $U \subset \mathbb{R}^n$.

We use the following notation for multi-indices $\alpha \in \mathbb{N}^n$:

$$|\alpha| = \sum_{j=1}^n \alpha_j; \quad \alpha! = \prod_{j=1}^n \alpha_j!.$$

Moreover, if $\beta \in \mathbb{N}^n$ we write $\alpha \leq \beta$ if and only if $\alpha_j \leq \beta_j$ for all $1 \leq j \leq n$. If $\alpha \leq \beta$ we put

$$\begin{pmatrix} \beta \\ \alpha \end{pmatrix} := \prod_{j=1}^n \begin{pmatrix} \beta_j \\ \alpha_j \end{pmatrix}.$$

Finally, we put $\partial_j = \partial/\partial x_j$ and

$$x^\alpha = \prod_{j=1}^n x_j^{\alpha_j}, \qquad \partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}. \tag{1.1}$$

Lemma 1.1.1. (Leibniz' rule) *Let $f, g \in C^\infty(U)$ and $\alpha \in \mathbb{N}^n$. Then*

$$\partial^\alpha(fg) = \sum_{\beta \leq \alpha} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \partial^\beta f \, \partial^{\alpha-\beta} g.$$

Proof Exercise. □

In what follows, we will use the notation $C^\infty(U)$ for the space of smooth complex-valued function on U; we denote by $\mathrm{End}(C^\infty(U))$ the space of $\mathbb{C}$-linear maps $P : C^\infty(U) \to C^\infty(U)$, maps that we will also call "operators on U".

Definition 1.1.2. A differential operator of order at most $k \in \mathbb{N}$ on U is an operator $P \in \mathrm{End}(C^\infty(U))$ of the form

$$P = \sum_{|\alpha| \leq k} c_\alpha \, \partial^\alpha, \tag{1.2}$$

with $c_\alpha \in C^\infty(U)$ for all α.

Hence such an operator acts on a function $f = f(x)$ on U by

$$P(f)(x) = \sum_{|\alpha| \leq k} c_\alpha(x) \, (\partial^\alpha f)(x).$$

The linear space of differential operators on U of order at most k is denoted by $\mathcal{D}_k(U)$. The union of these, for $k \in \mathbb{N}$, is denoted by $\mathcal{D}(U)$. Via Leibniz' rule one easily verifies that the composition of two differential operators from

$\mathcal{D}_k(U)$ and $\mathcal{D}_l(U)$ is again a differential operator, in $\mathcal{D}_{k+l}(U)$. Accordingly, the set $\mathcal{D}(U)$ of differential operators is a (filtered) algebra with unit.

To pass top general manifold, the most natural way to proceed is to first prove that the spaces $\mathcal{D}_k(U)$ are invariant under coordinate changes (i.e. Exercise 1.1.7), then to "glue" these spaces together. However, we will follow a shorter path which takes advantage of the fact that differential operators are local.

Definition 1.1.3. Let M be a smooth manifold. A linear operator $P \in \mathrm{End}(C^\infty(M))$ is called *local* if

$$\mathrm{supp}(P(f)) \subset \mathrm{supp}(f) \quad \forall\, f \in C^\infty(M).$$

Exercise 1.1.4. Show that P is local if and only if for any $f \in C^\infty(M)$ and any open $U \subset M$, one has the implication:

$$f|_U = 0 \Longrightarrow P(f)|_U = 0.$$

Lemma 1.1.5. *There is a unique way to associate to any local operator $P \in \mathrm{End}(C^\infty(M))$ on a manifold M and any open $U \subset M$, a "restricted operator"*

$$P_U = P|_U \in \mathrm{End}(C^\infty(U))$$

such that, for all $f \in C^\infty(M)$,

$$P_U(f|_U) = P(f)|_U$$

and, for $V \subset U$, $(P|_U)|_V = P|_V$.

Proof For $f \in C^\infty(U)$, let's look at what the value of $P_U(f) \in C^\infty(U)$ at an arbitrary point $x \in U$ can be. We choose a function $f_x \in C^\infty(M)$ which coincides with f in an open neighborhood $V_x \subset U$ of x. From the condition in the statement, we must have

$$P_U(f)(x) = P(f_x)(x).$$

We are left with checking that this can be taken as definition of P_U. All we have to check is the independence of the choice of f_x. But if g_x is another one, then $f_x - g_x$ vanishes on a neighborhood of x; since P is local, we deduce that $P(f_x) - P(g_x) = P(f_x - g_x)$ vanishes on that neighborhood, hence also at x. $\square$

The main property of local operators can be represented in local charts: if (U, κ) is a coordinate chart, then $P|_U$ can be moved to $\kappa(U)$ using the pull-back map

$$\kappa^* : C^\infty(\kappa(U)) \to C^\infty(U), \quad \kappa^*(f) = f \circ \kappa,$$

$$\begin{array}{ccc} C^\infty(U) & \xrightarrow{\ P\ } & C^\infty(U) \\ \uparrow{\scriptstyle \kappa^*} & & \uparrow{\scriptstyle \kappa^*} \\ C^\infty(\kappa(U)) & \xrightarrow[P_\kappa]{} & C^\infty(\kappa(U)) \end{array}$$

to obtain an operator

$$P_\kappa : C^\infty(\kappa(U)) \to C^\infty(\kappa(U)),\ P_\kappa = \kappa_*(P|_U) = (\kappa^*)^{-1} \circ P|_U \circ \kappa^*.$$

Definition 1.1.6. Let M be a smooth manifold. A **differential operator** of order at most k on M is a local linear operator $P \in \mathrm{End}(C^\infty(M))$ with the property that, for any coordinate chart (U, κ), $P_\kappa \in \mathcal{D}_k(\kappa(U))$.

The space of operators on M of order at most k is denoted by $\mathcal{D}_k(M)$.

Note that the condition on a coordinate chart $(U, \kappa = (x_1^\kappa, \ldots, x_n^\kappa))$ simply means that $P|_U$ is of type

$$P_U = \sum_{|\alpha| \leq k} c_\alpha(x)\, \partial_\kappa^\alpha,$$

with $c_\alpha \in C^\infty(U)$. Here ∂_κ^α act on $C^\infty(U)$ and are defined analogous to ∂^α but using the derivative along the vector fields $\partial/\partial x_j^\kappa$ induce by the chart. Note also that, in the previous definition, it would have been enough to require the condition only for a family of coordinate charts whose domains cover M. This follows from the invariance of the space of differential operators under coordinate changes:

Exercise 1.1.7. Let $h : U \to U'$ be a diffeomorphism between two open subsets of $\mathbb{R}^n$ and consider the induced map

$$h_* : \mathrm{End}(C^\infty(U)) \to \mathrm{End}(C^\infty(U')), \quad h_*(P) = (h^*)^{-1} \circ P \circ h^*.$$

Show that h_* maps $\mathcal{D}(U)$ bijectively onto $\mathcal{D}(U')$.

Deduce that a local operator $P \in \mathrm{End}(C^\infty(M))$ on a manifold M is a differential operator of order at most k if and only if for each $x \in M$ there exists a coordinate chart (U, κ) around x such that $P_\kappa \in \mathcal{D}_k(\kappa(U))$.

Exercise 1.1.8. Show that any differential operator $P \in \mathcal{D}_1(M)$ can be written as

$$P(\phi) = f\phi + L_V(\phi)$$

for some unique function $f \in C^\infty(M)$ and vector field V on M, where $L_V(f) = V(f) = df(V)$ is the derivative of f along V.

Exercise 1.1.9. This exercise provides another possible (inductive) definition of the spaces $\mathcal{D}_k(M)$. For each $f \in C^\infty(M)$, let $m_f \in \mathrm{End}(C^\infty(M))$ be the "multiplication by f" operator. The commutator of two operators P and Q is the new operator $[P, Q] = P \circ Q - Q \circ P$.

Starting with $\mathcal{D}_{-1}(M) = 0$, show that $\mathcal{D}_k(M)$ is the space of linear operators P with the property that

$$[P, m_f] \in \mathcal{D}_{k-1}(M) \quad \forall f \in C^\infty(M).$$

Next, we discuss the symbols of differential operators.

Definition 1.1.10. Let $U \subset \mathbb{R}^n$ and let $P \in \mathcal{D}_k(U)$ be of the form (1.2). The **full symbol** of the operator P is the function $\sigma(P) : U \times \mathbb{R}^n \to \mathbb{C}$ defined by

$$\sigma(P)(x, \xi) = \sum_{|\alpha| \leq k} c_\alpha(x)(i\xi)^\alpha.$$

The **principal symbol** of order k of P is the function $\sigma_k(P) : U \times \mathbb{R}^n \to \mathbb{C}$,

$$\sigma_k(P)(x, \xi) = \sum_{|\alpha| = k} c_\alpha(x)(i\xi)^\alpha.$$

A nice property of the principal symbol has, which fails for the total one is its multiplicativity property.

Exercise 1.1.11. Let $P \in \mathcal{D}_k(U)$ and $Q \in \mathcal{D}_l(U)$. Then the composition QP belongs to $\mathcal{D}_{k+l}$ and

$$\sigma_{k+l}(QP) = \sigma_l(Q)\sigma_k(P).$$

It is not difficult to check the following formulas for the symbols:

$$\sigma(P)(x,\xi) = e^{-i\xi}P(e^{i\xi})(x), \sigma_k(P)(x,\xi) = \lim_{t\to\infty} t^{-k}e^{-it\xi}P(e^{it\xi})(x).$$

Here we have identified ξ with the linear functional $x \mapsto \sum \xi_j x_j$. Accordingly, $e^{i\xi}$ stands for the function $x \mapsto e^{i\xi x}$. See also below.

Although the total symbol may look more natural then the principal one, the situation is the other way around: it is the principal symbol that can be globalized to manifolds (hence expressed coordinate free). Again, one natural way to proceed is to prove the invariance of the principal symbol under coordinate changes (but first one has to interpret the space $U \times \mathbb{R}^n$ of variables (x, ξ) correctly- and that is to view it as the cotangent bundle of U). However, we will follow a shorter path, based on the above formula for the symbol. First of all, we need a version of this formula which is more coordinate free.

Lemma 1.1.12. *Let $U \subset \mathbb{R}^n$, $P \in \mathcal{D}_k(U)$. For $(x,\xi) \in U \times \mathbb{R}^n$, choose $\varphi \in C^\infty(U)$ such that*

$$(d\varphi)_x = \sum_{j=1}^{n} \xi_j (dx_j)_x$$

(hence $\partial_j(\varphi)(x) = \xi_j$ for all j). Then

$$\sigma_k(P)(x,\xi) = \lim_{t\to\infty} t^{-k}e^{-it\varphi(x)}P(e^{it\varphi})(x).$$

Proof Left to the reader. The proof follows by application of Leibniz' rule. □

One advantage of the previous lemma is that it is most natural to view ξ as a variable in the dual of $\mathbb{R}^n$. Accordingly, $U \times \mathbb{R}^n$ should be viewed as the cotangent bundle T^*U. In other words, the principal symbol should be viewed as the function

$$\sigma_k(P) : T^*U \to \mathbb{C}, \quad \xi_1(dx_1)_x + \ldots + \xi_n(dx_n)_x \mapsto \sum_{|\alpha|=k} c_\alpha(x)(i\xi)^\alpha.$$

Exercise 1.1.13. Check directly that the principal symbol behaves well under coordinate changes. More precisely, let $h : U \to U'$ be a diffeomorphism between two opens $U, U' \subset \mathbb{R}^n$. It induces the map $T^*h : T^*U \to T^*U'$ given by $T^*h(x,\xi) = (h(x), \xi \circ T_x h^{-1})$. Accordingly we have the map $h_* : C^\infty(T^*U) \to C^\infty(T^*U')$ given by $h_*\sigma = \sigma \circ (T^*h)^{-1}$. Thus,

$$h_*\sigma(x,\xi) = \sigma(h^{-1}(x), \xi \circ T_x h).$$

Show that, for all $P \in \mathcal{D}_k(U)$,

$$\sigma_k(h_*(P)) = h_*(\sigma_k(P)).$$

Moreover, the previous characterization of the principal symbol can be taken as definition when we pass to manifolds. More precisely, given $P \in \mathcal{D}_k(M)$ on a manifold M, one defines the principal symbol (of order k) of P as the smooth function

$$\sigma_k(P) : T^*M \to \mathbb{C}$$

given by

$$\sigma_k(P)(\xi_x) = \lim_{t\to\infty} t^{-k} e^{-it\varphi(x)} P(e^{it\varphi})(x).$$

where, for $x \in M$, $\xi_x \in T^*_x M$, $\varphi \in C^\infty(M)$ is chosen so that $(d\varphi)_x = \xi_x$. The fact that this definition does not depend on the choice of φ follows from the local case (previous lemma). Indeed, choosing a coordinate chart $(U, \kappa = (x_1^\kappa, \ldots, x_n^\kappa))$ around x, the data over U consisting of x, ξ_x, P, φ is pushed forward by κ to similar data over $\kappa(U)$:

$$x_\kappa = \kappa(x),\ \xi^\kappa = \xi_x \circ (d\kappa^{-1})_{\kappa(x)},\ P_\kappa = (\kappa^{-1})^* \circ P \circ \kappa^*,\ \varphi_\kappa = \varphi \circ \kappa^{-1}$$

and it is clear that, already before taking the limit,

$$t^{-k} e^{-it\varphi_\kappa(x)} P(e^{it\varphi_\kappa})(x_\kappa) = t^{-k} e^{-it\varphi(x)} P(e^{it\varphi})(x).$$

In conclusion,

Corollary 1.1.14. *For $P \in \mathcal{D}_k(M)$, there is a well-defined smooth function*

$$\sigma_k(P) : T^*M \to \mathbb{C}$$

such that, for any coordinate chart $(U, \kappa = (x_1^\kappa, \ldots, x_n^\kappa))$,

$$T^*_x M \ni \xi_1 (dx_1^\kappa)_x + \ldots + \xi_n (dx_n^\kappa)_x \stackrel{\sigma_k(P)}{\longrightarrow} \sigma_k(P_\kappa)(\kappa(x), \xi_1, \ldots, \xi_n) \in \mathbb{C}.$$

Definition 1.1.15. Let $P \in \mathcal{D}_k(M)$. The function $\sigma_k(P) : T^*M \to \mathbb{C}$ is called **the principal symbol** of order k of the operator P.

Exercise 1.1.16. Let V be a vector field on M, $f \in C^\infty(M)$ and let P be the corresponding differential operator from Exercise 1.1.8. Show that the principal symbol of P is given by $\sigma_1(P)(x, \xi) = \xi(V_x)$.

Exercise 1.1.17. Let $P \in \mathcal{D}_k(M)$ and $Q \in \mathcal{D}_l(M)$. Show that $QP \in \mathcal{D}_{k+l}(M)$. Moreover, $\sigma_{k+l}(QP) = \sigma_l(Q)\sigma_k(P)$. Hint: use reduction to charts.

Exercise 1.1.18. Show that, for any $P \in \mathcal{D}_k(M)$ and any $f \in C^\infty(M)$ and all $\varphi \in C^\infty(M)$

$$f(x)\sigma_k(P)((d\varphi)_x) = \lim_{t\to\infty} t^{-k} e^{-it\varphi(x)} P(e^{it\varphi} f)(x). \tag{1.3}$$

Finally, here is another interpretation of the (principal) symbol, which takes into account the fact that $\sigma_k(P)(x, \xi)$ is not only smooth in ξ, but actually polynomial. Recall first the formalism that allows us to handle polynomials in a coordinate free manner. Let V be a finite dimensional vector space real or complex). Recall that a function $p : V \to \mathbb{C}$ is called polynomial of degree k if, for some (or, equivalently, any) basis $\{e_1, \ldots, e_n\}$ of V, p is of the type

$$p(v) = \sum_{|\alpha|=k} p_\alpha v^\alpha = \sum_{|\alpha|=k} p_\alpha v_1^{\alpha_1} \ldots v_n^{\alpha_n},$$

where the sum is over multi-indices $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $p_\alpha \in \mathbb{C}$. We denote by $\mathrm{Pol}^k(V)$ the space of such functions. The key remark is that this space is canonically isomorphic to the more intrinsic space S^kV^* consisting of all multilinear symmetric maps

$$p : \underbrace{V \times \ldots \times V}_{k \text{ timers}} \to \mathbb{C}.$$

The identification between S^kV^* and $\mathrm{Pol}^k(V)$ associates to the symmetric function p on k-variables, the function

$$p(v) := p(v_1, \ldots, v_k).$$

Exercise 1.1.19. Prove that, indeed, this defines a bijection between S^kV^* and $\mathrm{Pol}^k(V)$.

Passing to duals, one obtains an identification between $\mathrm{Pol}^k(V^*)$ and S^kV. As usual, the operation $V \mapsto S^kV$ extends to vector bundles so that, for any vector bundle E over a manifold M, one forms a new vector bundle S^kE over M whose fiber above $x \in M$ is S^kE_x. By the discussion above, any section of S^kE can be interpreted as a smooth function on the manifold E^*. With these at hand, it should be clear now that the principal symbol of an operator $P \in \mathcal{D}_k(M)$ becomes a section

$$\sigma_k(P) \in \Gamma(S^kTM).$$

1.2. Differential operators II: arbitrary coefficients

We shall now introduce the notion of a differential operator between smooth vector bundles E and F on a smooth manifold M, acting at the level of sections

$$P : \Gamma(E) \to \Gamma(F). \tag{1.4}$$

It is useful to have in mind that degree zero differential operators correspond to sections $C \in \Gamma(\underline{\mathrm{Hom}}\,(E, F))$ i.e. smooth maps

$$M \ni x \mapsto C_x \in \mathrm{Hom}(E_x, F_x).$$

More precisely, any such C defines an operator $C : \Gamma(E) \to \Gamma(F)$ acting on sections by

$$C(s)(x) = C_x(s(x)).$$

Exercise 1.2.1. Show that this construction defines a 1-1 correspondence between sections of $\underline{\mathrm{Hom}}\,(E, F)$ and maps from $\Gamma(E)$ to $\Gamma(F)$ which are $C^\infty(M)$-linear.

As before, differential operators $P : \Gamma(E) \to \Gamma(F)$ will have the important property of locality: for any $s \in \Gamma(E)$,

$$\mathrm{supp}(P(s)) \subset \mathrm{supp}(s).$$

And, as in the previous section (and by a similar argument), any such local operator can be restricted to opens $U \subset M$ to induce operators

$$P_U = P|_U : \Gamma(E|_U) \to \Gamma(F|_U)$$

so that $P(s)|_U = P_U(s|_U)$ and, for all $V \subset U$, $(P_U)_V = P_V$.

For the precise definition of differential operators between sections of vector bundles there are many different but equivalent ways to proceed. The most natural one is probably to assume locality, discuss the local case first (over domains of charts and trivializations of the bundles), prove that the outcome is independent of the choices, and then "glue" the local pieces together. Here is a less natural but shorter way to proceed. The idea is to pass right away to the case of of the trivial line bundles (i.e. to the previous section), in a coordinate free manner. Given a linear operator (1.4), the key remark is that any sections

$$e \in \Gamma(E), \quad \lambda \in \Gamma(F^*)$$

induce a linear operator

$$P_{e,\lambda} : C^\infty(M) \to C^\infty(M), \quad P_{e,\lambda}(f) := \lambda(P(fs))$$

(where, as above, we interpret λ as a linear map $\Gamma(F) \to C^\infty(M)$). Note that, intuitively, the choice of (arbitrary) e and λ allows us to avoid working with coordinates (with respect to (local) frames of E and F); however, $P_{e,\lambda}$ can be thought of as "the (e, λ) global coordinate of P".

Definition 1.2.2. We say that $P : \Gamma(E) \to \Gamma(F)$ is a **differential operator of order at most k from E to F** if

$$P_{e,\lambda} \in \mathcal{D}_k(M) \quad \forall \;\; e \in \Gamma(E), \; \lambda \in \Gamma(F^*).$$

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The space of such operators is denoted by $\mathcal{D}_k(M; E, F)$, or simply $\mathcal{D}_k(E, F)$.

With this definition, locality is a consequence.

Proposition 1.2.3. *Given two vector bundles E and F over M,*

(i) any differential operator $P \in \mathcal{D}_k(E, F)$ is local and, for any $U \subset M$ open, the restriction

$$P_U = P|_U : \Gamma(E|_U) \to \Gamma(F|_U)$$

belongs to $\mathcal{D}_k(E|_U, F|_U)$.

(ii) conversely, if $P \in \Gamma(E) \to \Gamma(F)$ is a local operator with the property that each point $x \in M$ has an open neighborhood U such that $P|_U \in \mathcal{D}_k(E|_U, F|_U)$, then $P \in \mathcal{D}_k(M; E, F)$.

Proof Note first that, for any vector bundle E, there exists an integer l and sections

$$e_1, \ldots, e_l \in \Gamma(E), \quad \lambda^1, \ldots, \lambda^l \in \Gamma(E^*)$$

such that, for any $s \in \Gamma(E)$,

$$s = \lambda^1(s)e_1 + \ldots + \lambda^l(s)e_l.$$

Indeed, a basic property of vector bundles is that one can always find another vector bundle E' such that $E \oplus E'$ is isomorphic to the trivial bundle $M \times \mathbb{C}^l$ for some l. This isomorphism is encoded in a global frame $\tilde{e}_1, \ldots, \tilde{e}_l$ of $E \oplus E'$. Define then e_j to be the first component of $\tilde{e}_j$ and $\lambda_j(s)$ to be the j-th coordinate of $(s, 0) \in E \oplus E'$ with respect to this frame.

Returning to our proposition, assume that $P \in \mathcal{D}_k(E, F)$, $s \in \Gamma(E)$ vanishes on $U \subset M$ and we show that also $P(s)$ vanishes on U (i.e. we use a version of

Exercise 1.1.4). But the previous discussion shows that, if $s|_U = 0$, then s can be written as a sum

$$s = f_1 e_1 + \ldots + f_l e_l$$

with $e_j \in \Gamma(E)$, $f_j \in C^\infty(M)$ with $f_i|_U = 0$ (take $f_j = \lambda^j(s)$). But then, for any $\lambda \in \Gamma(F^*)$,

$$\lambda(P(s)) = P_{e_1,\lambda}(f_1) + \ldots + P_{e_l,\lambda}(f_l)$$

hence, from the locality of the operators $P_{e,\lambda}$, we deduce that $P(s)|_U$ is killed by all $\lambda \in \Gamma(F^*)$, hence $P(s)|_U = 0$.

Next, we postpone for the moment the proof of the fact that $P|_U \in \mathcal{D}_k(E|_U, F|_U)$ and we concentrate on the last part of the proposition. Hence assume that $P \in \Gamma(E) \to \Gamma(F)$ is local and satisfies the properties from the statement. We have to show that, for any $e \in \Gamma(E)$, $\lambda \in \Gamma(F^*)$, $P_{e,\lambda} \in \mathcal{D}_k(M)$. But note that, for any U,

$$P_{e,\lambda}|_U = (P|_U)_{e|_U, \lambda|_U}$$

hence it suffices to combine the hypothesis with the last part of Exercise 1.1.7.

Finally, assume that $P \in \mathcal{D}_k(E,F)$ and we prove that the restrictions $P|_U : \Gamma(E|_U) \to \Gamma(F|_U)$ are in $\mathcal{D}_k(E|_U, F|_U)$. Let $e \in \Gamma(E|_U)$, $\lambda \in \Gamma(F^*|_U)$; we show that $Q := (P|_U)_{e,\lambda}$ belongs to $\mathcal{D}_k(U)$. As before, we have to make sure that any $x \in U$ has an open neighborhood $V_x \subset U$ such that $Q|_V \in \mathcal{D}_k(U)$. Just use $V_x = V$ so that $e|_V$ and $\lambda|_V$ admit smooth extensions $\tilde{e}$, $\tilde{\lambda}$ to M, so that $Q|_V = P_{\tilde{e},\tilde{f}}|_V$ will belong to $\mathcal{D}_k(U)$. □

Exercise 1.2.4. Show that, for any two vector bundles E and F over a manifold M, the assignment

$$U \mapsto \mathcal{D}_k(E|_U, F|_U)$$

($U \subset M$ open) is a sheaf on M.

Of course, with (ii) of the proposition in mind, we can go on and give slightly different definitions, based on the local data. For instance, if U is an open on which E and F are trivializable, with (fixed) trivialization frames

$$\mathbf{e} = \{e_1, \ldots, e_p\}, \quad \mathbf{f} = \{f_1, \ldots, f_q\},$$

then the restriction to U of a local operator $P : \Gamma(E) \to \Gamma(F)$ is of type

$$P|_U(e_j) = \sum_k P(\mathbf{e}, \mathbf{f})_j^k f_k,$$

i.e. it is determined by a matrix of operators acting on $C^\infty(U)$:

$$(P(\mathbf{e}, \mathbf{f})_j^k)_{1 \le j \le p, 1 \le k \le q}.$$

Exercise 1.2.5. Show that a linear operator $P : \Gamma(E) \to \Gamma(F)$ is in $\mathcal{D}_k(E,F)$ if and only if it is local and, for each $x \in M$, there is an open U containing x and local frames $\mathbf{e}$ and $\mathbf{f}$ of E and F over U, such that all the components $P(\mathbf{e}, \mathbf{f})_j^k$ are in $\mathcal{D}_k(U)$.

To express differential operators in perms of partial derivatives, we have to restrict to opens $U \subset M$ with

1. U is the domain of a (fixed) coordinate chart $(U, \kappa = (x_1^\kappa, \ldots, x_n^\kappa))$.

2. E is trivializable over U, with a (fixed) frame $\{s_1, \ldots\}$ over U.

Note that, in this case, we have "higher order derivatives operators"

$$\partial_\kappa^\alpha : \Gamma(E) \to \Gamma(E), \quad f^1 s_1 + \ldots \mapsto \partial_\kappa^\alpha(f^1)s_1 + \ldots.$$

Exercise 1.2.6. With the previous notations, show that a linear map $P : \Gamma(E|_U) \to \Gamma(F|_U)$ is a differential operator of order at most k if and only if it is of the form

$$P = \sum_{|\alpha| \leq k} C_\alpha \circ \partial^\alpha,$$

with $C_\alpha \in \Gamma(\underline{\mathrm{Hom}}\,(E|_U, F|_U))$.

We extend the definition of principal symbol as follows. We denote by $\pi : T^*M \to M$ the canonical projection. For a vector bundle E over M, let π^*E be the pull-back of E to T^*M (whose fiber above $\xi_x \in T_x^*M$ is E_x). For two vector bundles E and F over M, we consider the vector bundle $\underline{\mathrm{Hom}}\,(E, F)$ over M (whose fiber above $x \in M$ is $\mathrm{Hom}(E_x, F_x)$) and its pull-back to T^*M,

$$\pi^*\underline{\mathrm{Hom}}\,(E, F) \cong \underline{\mathrm{Hom}}\,(\pi^*E, \pi^*F)$$

whose fiber above $\xi_x \in T_x^*M$ is $\mathrm{Hom}(E_x, F_x)$.

Lemma 1.2.7. *Let E, F be smooth vector bundles on M and let $P \in \mathcal{D}_k(E, F)$. There exists a unique section $\sigma_k(P)$ of $\pi^*\underline{\mathrm{Hom}}\,(E, F)$ (called again* **the principal symbol** *of P), i.e. a smooth function*

$$T_x^* \ni \xi_x \mapsto \sigma_k(P)(\xi_x) \in \mathrm{Hom}(E_x, F_x),$$

with the following property: for each $x_0 \in M$ and all $s \in \Gamma(E)$ and $\varphi \in C^\infty(M)$,

$$\sigma_k(P)((d\varphi)_{x_0})(s(x_0)) = \lim_{t \to \infty} t^{-k} e^{-it\varphi(x_0)} P(e^{it\varphi}s)(x_0). \tag{1.5}$$

Moreover, for each $x \in M$ the function $\xi \mapsto \sigma_k(P)(x, \xi)$ is a degree k homogeneous polynomial function $T_x^*M \to \mathrm{Hom}(E_x, F_x)$.

Proof Uniqueness follows from the fact that for every $(x, \xi) \in T^*M$ and $v \in E_x$ there exists a $s \in \Gamma^\infty(E)$ such that $s(x) = v$ and a function $\varphi \in C^\infty(M)$ such that $d\varphi(x) = \xi$. We have to check that, fixing $x_0 \in M$, the right hand side of the formula only depends on $\xi := (d\varphi)_{x_0}$ and on $s(x_0)$, and the dependence is linear on $s(x_0)$ and polynomial in ξ. We denote this formula by $\sigma(\xi, s)$. Applying an arbitrary $\lambda \in \Gamma(F^*)$ we obtain

$$\lambda(\sigma(\xi, s)) = \sigma_k(P_{s,\lambda})(\xi),$$

hence $\lambda(\sigma(\xi, s))$ only depends on ξ (and in a polynomial fashion) and s and not on the choice of ϕ (for any λ!), from which it follows that the same is true for $\sigma(\xi, s)$. Finally, for any $f \in C^\infty(M)$, multiplying s by fs gives:

$$\lambda(\sigma(\xi, fs)) = \lim_{t \to \infty} t^{-k} e^{-it\varphi(x_0)} \lambda(P(e^{it\varphi} fs)(x_0)),$$

where $\lambda(P(e^{it\varphi} fs)(x_0)) = P_{s,\lambda}(e^{it\varphi} f)$ hence (using Exercise 1.1.18 applied to $P_{s,\lambda}$) we find

$$\lambda(\sigma(\xi, fs)) = \lambda(f(x_0)\sigma(\xi, s)).$$

Since this holds for all λ, it follows that $\sigma(\xi, fs) = f(x_0)\sigma(\xi, s)$; since $\sigma(\xi, s)$ is $\mathbb{C}$-linear in s, we deduce that it only depends $s(x_0)$ and not on $s(x_0)$. $\square$

Example 1.2.8. We consider the complexified version of the DeRham complex. I.e., we define $\Omega^k(M)_{\mathbb{C}} = \Omega^k(M) \otimes_{\mathbb{R}} \mathbb{C}$, which should be interpreted as the space of sections of the complex vector bundle $\Lambda_{\mathbb{C}} T^*M$ whose fiber at $x \in M$ consists of antisymmetric, k-multilinear maps from T_xM to $\mathbb{C}$. The exterior differentiation clearly extends to a $\mathbb{C}$-linear map $d = d_k : \Omega^k(M)_{\mathbb{C}} \to \Omega^{k+1}(M)_{\mathbb{C}}$. Let U be a coordinate patch of M with local coordinates $x_1, \ldots, x_n$. Then for each $a \in U$, the one forms $dx_1(a), \ldots, dx_n(a)$ span the cotangent space T^*_aM. Thus, $\wedge^k T^*_a M$ has the basis

$$dx_{j_1}(a) \wedge \cdots \wedge dx_{j_k}(a), \quad \text{with} j_1 < \cdots < j_k.$$

With respect to this basis, the restriction of a section $s \in \Omega^k(M)$ to U may be expressed as

$$s|_U = \sum_{j_1 < \cdots < j_k} s_{j_1,\ldots,j_k} dx_{j_1} \wedge \cdots \wedge dx_{j_k}.$$

Exterior differentiation is given by

$$ds|_U = \sum_{j_1 \cdots < j_k} d(s_{j_1,\ldots,j_k}) \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_k},$$

where $ds_{j_1,\ldots,j_k} = \sum_i \partial_i s_{j_1,\ldots,j_k}$. From this we see that d is a differential operator of order one from $\wedge^k T^*M$ to $\wedge^{k+1} T^*M$.

Exercise 1.2.9. Show that the principal symbol of exterior differentiation $d : \Gamma(\Lambda^k T^*M) \to \Gamma(\Lambda^{k+1} T^*M)$ is given by

$$\sigma_1(d)(x, \xi) : \wedge^k T^*_x M \to \wedge^{k+1} T^*_x M, \;\; \omega \mapsto i\xi \wedge \omega.$$

For E_1, E_2 smooth vector bundles on M and $P \in \mathcal{D}_k(E_1, E_2)$, the principal symbol $\sigma_k(P)$ is a section of the bundle $\underline{\mathrm{Hom}}\,(\pi^*E_1, \pi^*E_2)$. Equivalently, the symbol may be viewed as a homomorphism from the bundle π^*E_1 to π^*E_2. Thus, if E_3 is a third vector bundle and $Q \in \mathcal{D}_l(E_2, E_3)$ then the composition $\sigma_l(Q) \circ \sigma_k(P)$ is a vector bundle homomorphism from E_1 to E_3.

Lemma 1.2.10. *Let E_1, E_2, E_3 be smooth vector bundles on M. Let $P \in \mathcal{D}_k(E_1, E_2)$ and $Q \in \mathcal{D}_l(E_2, E_3)$. Then the composition $Q \circ P$ belongs to $\mathcal{D}_{k+l}(E_1, E_3)$ and*

$$\sigma_{k+l}(Q \circ P) = \sigma_l(Q) \circ \sigma_k(P).$$

Finally, we discuss the notion of formal adjoint. For this, and for later use, we need the notion of density.

Reminder on densities: This is about the density bundle from the intensive reminder. Given an n-dimensional real vector space V, one defines $D_r(V)$, the space of r-densities (for any real number $r > 0$), as the set of all maps $\omega : \Lambda^n V \to \mathbb{R}$ satisfying

$$\omega(\lambda\xi) = |\lambda|^r \omega(\xi), \;\; \forall\; \xi \in \Lambda^n V.$$

Equivalently (and maybe more intuitively), one can use the set $\mathrm{Fr}(V)$ of all frames of V (i.e. ordered sets $(e_1, \ldots, e_n)$ of vectors of V which form a basis of V). Then $D_r(V)$ can also be described as the set of all functions

$$\omega : Fr(V) \to \mathbb{R}$$

with the property that, for any invertible n by n matrix A, and any frame e, for the new frame $A(e)$ one has

$$\omega(A(e)) = |\det(A)|^r \omega(e).$$

Intuitively, one may think of an r-density on V as some rule of computing volumes of the hypercubes (each frame determines such a hypercube). For each r, $D_r(V)$ is one dimensional (hence isomorphic to $\mathbb{C}$), but in a non-canonical way. Choosing a frame e of V, one has an induced r-density denoted

$$\omega_e = |e^1 \wedge \ldots \wedge e^n|_r$$

uniquely determined by the condition that $\omega_e(e) = 1$ (the e^i's in the notation stand for the dual basis of V^*).

For a manifold M, we apply this construction to all the tangent spaces to obtain a line bundle $D_r(M)$ over M, whose fiber at $x \in M$ is $D_r(T_xM)$. For $r = 1$, $D_1(M)$ is simply denoted D, or D_M whenever it is necessary to remove ambiguities. The sections of D are called densities on M.

Any local chart $(U, \kappa = (x^1_\kappa, \ldots, x^n_\kappa))$ induces a frame $(\partial/\partial x^i_\kappa)_x$ for T_xM with the dual frame $(dx^i_\kappa)_x$ for T^*_xM, for all $x \in U$. Hence we obtain an induced trivialization of $D_r(M)$ over U, with trivializing section

$$|dx^1_\kappa \wedge \ldots \wedge dx^n_\kappa|_r$$

(and, as usual, the smooth structure on D is so that these sections induced by the local charts are smooth).

An r-density on M is a section ω of $D_r(M)$. Hence, locally, with respect to a coordinate chart as before, such a density can be written as

$$\omega = f_\kappa \circ \kappa \cdot |(dx^1_\kappa \wedge \ldots \wedge dx^n_\kappa|_r$$

for some smooth function defined on $\kappa(U)$. If we consider another coordinate chart κ' on the same U then, after a short (but instructive) computation, wee see that f_κ changes according to the rule:

$$f_\kappa = |\mathrm{Jac}(h)|^r f_{\kappa'} \circ h,$$

where $h = \kappa' \circ \kappa^{-1}$ is the change of coordinates, and $\mathrm{Jac}(h)$ is the Jacobian of h. The case $r = 1$ reminds us of the usual integration and the change of variable formula: the usual integration of compactly supported functions on an open $\Omega \subset \mathbb{R}^n$ defines a map

$$\int_\Omega : C^\infty_c(\Omega; \mathbb{R}) \to \mathbb{R}$$

and, if we move via a diffeomorphism $h : \Omega \to \Omega'$, one has the change of variables formula

$$\int_\Omega f = \int_\Omega |\mathrm{Jac}(h)| \cdot f \circ h.$$

Hence, for 1-densities on the domain U of a coordinate chart, one has an induced integration map

$$\int_U : \Gamma_c(U, D|_U) \to \mathbb{R}$$

(by sending ω to $\int_{\kappa(U)} f_\kappa$) which does not depend on the choice of the coordinates. For the global integration map

$$\int_M : \Gamma_c(M, D) \to \mathbb{R},$$

one decomposes an arbitrary compactly supported density Ω on M as a finite sum $\sum_i \omega_i$, where each ω_i is supported in the domain of a coordinate chart U_i (e.g. use partitions of unity) and put

$$\int_M \omega = \sum_i \int_{U_i} \omega_i.$$

Of course, one has to prove that this does not depend on the way we decompose ω as such a sum, but this basically follows from the additivity of the usual integral.

Note that one can clearly talk about positive/negative densities. Hence any metric on D (and, in particular, any Riemannian metric on M) induces a no-where vanishing section of D (the positive one, of length 1). In other words, D is trivializable (but not canonically). The choice of a no-where zero density $d\mu$ induces an integration map:

$$\int_M \cdot \, d\mu : C_c^{\infty}(M) \to \mathbb{C}, \quad f \mapsto \int_M (f d\mu),$$

where the complex numbers show up because of the fact that $C_c^{\infty}(M)$ consists of complex valued functions; of course, $\int_M (f+ig)d\mu$ is defined as $\int_M f d\mu + i \int_M g d\mu$. Actually, to be consistent with our convention of only considering complex vector bundles, we should complexify D, i.e. consider $D \otimes_{\mathbb{R}} \mathbb{C} = D \oplus i \cdot D$; equivalently, in all the previous definitions we use $\mathbb{C}$ instead of $\mathbb{R}$ (i.e. we look at complex-valued densities). We will continue to use the notation D for the resulting complex line bundle; one obtaines the complex-valued version of the integration map

$$\int_M : \Gamma_c(M, D) \to \mathbb{C}.$$

Assume now that E and F are two vector bundles over M equipped with hermitian inner products $\langle -, - \rangle^E$[1] and $\langle -, - \rangle^F$. We also choose a strictly positive density on M, call it $d\mu$. One has an induced inner-product on the space $\Gamma_c(E)$ of compactly supported sections of E given by

$$\langle s, s' \rangle^E := \int_M \langle s(x), s' \rangle_x^E \, d\mu,$$

and similarly an inner product on $\Gamma_c(F)$. Given $P \in \mathcal{D}_k(E, F)$, a **formal adjoint** of P (with respect to the hermitian metrics and the density) is an operator $P^* \in \mathcal{D}_k(F, E)$ with the property that

$$\langle P(s_1), s_2 \rangle^F = \langle s_1, P^*(s_2) \rangle^E, \quad \forall \; s_1 \in \Gamma_c(E), s_2 \in \Gamma_c(F).$$

Proposition 1.2.11. *For any $P \in \mathcal{D}_k(E, F)$, the formal adjoint $P^* \in \mathcal{D}_k(F, E)$ exists and is unique. Moreover, the principal symbol of P^* is $\sigma_k(P^*) = \sigma_k(P)^*$, where $\sigma_k(P)^*(\xi_x)$ is the adjoint of the linear map*

$$\sigma_k(P)(\xi_x) : E_x \to F_x$$

(with respect to the inner products $\langle -, - \rangle_x^E$ and $\langle -, - \rangle_x^F$).[2]

[1] hence $\langle -, - \rangle^E$ is a family $\{\langle -, - \rangle_x^E : x \in M\}$ of inner products on the vector spaces E_x, which "varies smoothly with respect to x". The last part means, e.g., that for any $s, s' \in \Gamma(E)$, the function $\langle s, s' \rangle^E$ on M, sending x to $\langle s(x), s'(x) \rangle_x^E$ is smooth; equivalently, it has the obvious meaning in local trivializations.

[2] note that P^* depends both on the hermitian metrics on E and F as well as on the density, while it principal symbol does not depend on the density.

Proof Due to the local property of differential operators it suffices to prove the statement (both the existence as well as the uniqueness) locally. So assume that $M = U \subset \mathbb{R}^n$, where we can write $P = \sum_{|\alpha|\leq k} C_\alpha \circ \partial^\alpha$. We have $d\mu = \rho|dx|$ for some smooth function ρ on U. Writing out $\langle P(s_1), s_2\rangle^F$ and integrating by parts $|\alpha|$ times (to move ∂^α from s to s'), we find the operator P^* which does the job:

$$P^*(s') = \sum_{|\alpha|\leq k} \frac{1}{\rho}\partial^\alpha(\rho C_\alpha^* s').$$

Clearly, this is a differential operator of order at most k. For the principal symbol, we see that the only terms in this sum which matter are:

$$\sum_{|\alpha|=k} (-1)^{|\alpha|}\frac{1}{\rho}\rho C_\alpha^*\partial^\alpha(s') = \sum_{|\alpha|=k} (-1)^{|\alpha|} C_\alpha^*\partial^\alpha(s'),$$

i.e. the symbol is given by

$$\sum_{|\alpha|=k} (-1)^{|\alpha|} C_\alpha^*(i\xi)^\alpha = (\sum_{|\alpha|=k} C_\alpha^*(i\xi)^\alpha)^*.$$

The uniqueness follows from the non-degeneracy property of the integral: if $\int_U fg = 0$ for all compactly supported smooth functions, then $f = 0$. □

1.3. Ellipticity; preliminary version of the Atiyah-Singer index theorem

Definition 1.3.1. Let $P \in \mathcal{D}_k(E, F)$ be a differential operator between two vector bundles E and F over a manifold M. We say that P is an **elliptic operator** of order k if, for any $\xi_x \in T_x^*M$ non-zero,

$$\sigma_k(P)(\xi_x) : E_x \to F_x$$

is an isomorphism.

The aim of these lectures is to explain and complete the following theorem (a preliminary version of the Atiyah-Singer index theorem).

Theorem 1.3.2. *Let M be a compact manifold and let $P : \Gamma(E) \to \Gamma(F)$ be an elliptic differential operator. Then P is Fredholm (i.e. Ker(P) and Coker(P) are finite dimensional),*

$$\mathit{Index}(P) := \mathit{dim}(\mathit{Ker}(P)) - \mathit{dim}(\mathit{Coker}(P))$$

depends only on the principal symbol $\sigma_k(P)$, and Index(P) can be expressed in terms of (precise) topological data associated to $\sigma_k(P)$.

Here are a few exercises about the notion of ellipticity and Fredholmness. In these exercises we fix a a density on M and, on each vector bundle over M that we will be considering, a hermitian inner product. So, for E over M, we can talk about the adjoint D^* of differential operators $D : \Gamma(E) \to \Gamma(E)$; we will say that D is self-adjoint if $D = D^*$. We will prove later on (as part of the theorem above) that, if D is elliptic, then

(Dec) $\Gamma(E) = \mathrm{Ker}(D) + \mathrm{Im}(D)$.

(Fre) Ker(D) and Coker(D) are finite dimensional (i.e. D is Fredholm).

Exercise 1.3.3. Given a self-adjoint differential operator $D : \Gamma(E) \to \Gamma(E)$, show that:

(i) $\mathrm{Ker}(D) \cap \mathrm{Im}(D) = \{0\}$
(ii) If D satisfies condition (*Dec*), then

$$(\mathrm{Ker}(D))^{\perp} = \mathrm{Im}(D), \quad (\mathrm{Im}(D))^{\perp} = \mathrm{Ker}(D).$$

(iii) If D satisfies both conditions (*Dec*) and (*Fre*), then $\mathrm{Index}(D) = 0$.

Exercise 1.3.4. Given a self-adjoint differential operator $Q : \Gamma(E) \to \Gamma(E)$ and $D = Q \circ Q$, then

(i) D is elliptic if and only if Q is.
(ii) D satisfies conditions (*Dec*) and (*Fre*) if and only if Q does.

Exercise 1.3.5. Let $Q_+ : \Gamma(E_+) \to \Gamma(E_-)$ be a differential operator between two vector bundles E_+ and E_- and we denote by Q_- its adjoint:

$$\Gamma(E_+) \underset{Q_+}{\overset{Q_-}{\rightleftarrows}} \Gamma(E_-)\ . \tag{1.6}$$

We place ourselves in the situation of the previous exercise by taking

$$E = E_+ \oplus E_-, \quad Q(s_+, s_-) = (Q_-(s_-), Q_+(s_+)),$$

so that $D = Q \circ Q$ has components $\mathbb{D}_+ = Q_- \circ Q_+$, $D_- = Q_+ \circ Q_-$ (acting on $\Gamma(E_+)$ and $\Gamma(E_-)$, respectively). If D satisfies conditions (*Dec*) and (*Fre*), show that Q_+ is Fredholm and

$$\mathrm{Index}(Q_+) = \mathrm{Ker}(D_+) - \mathrm{Ker}(D_-).$$

Due to the the way that elliptic operators arise in geometry (via "elliptic complexes"), it is worth giving a slightly different version of the Atiyah-Singer index theorem.

Definition 1.3.6. A **differential complex** over a manifold M,

$$\mathcal{E} : \Gamma(E^0) \xrightarrow{P_0} \Gamma(E^1) \xrightarrow{P_1} \Gamma(E^2) \xrightarrow{P_2} \dots$$

consists of:

1. For each $k \geq 0$, a vector bundle E_k over M, with $E_k = 0$ for k large enough.
2. For each $k \geq 0$, a differential operator P_k from E_k to E_{k+1}, of some order d independent of k

such that, for all k, $P_{k+1} \circ P_k = 0$.

Example 1.3.7. Let $d_k : \Omega^k(M) \to \Omega^{k+1}(M)$ be exterior differentiation. Then $d_{k+1} \circ d_k = 0$ for all k. Therefore, the sequence of differential operators $d_k \in \mathcal{D}_1(\wedge^k T^*M, \wedge^{k+1} T^*M)$ forms a complex; it is called the de Rham complex.

Note that, from Lemma 1.2.10 it follows that for a complex of differential operators as above, the associated sequence $\sigma_{d_k}(P_k)$ of principal symbols is a complex of homomorphisms of the vector bundles π^*E_k on M, i.e., for any $\xi_x \in T^*_xM$, the sequence

$$E^0_x \stackrel{\sigma_d(P_0)(\xi_x)}{\longrightarrow} E^1_x \stackrel{\sigma_d(P_1)(\xi_x)}{\longrightarrow} E^2_x \stackrel{\sigma_d(P_2)(\xi_x)}{\longrightarrow} \dots$$

is a complex of vector space. In turn, this means that the composition of any two consecutive maps in this sequence is zero. Equivalently,

$$\mathrm{Ker}(\sigma_d(P_{k+1})(\xi_x)) \subset \mathrm{Im}(\sigma_d(P_k)(\xi_x)).$$

Definition 1.3.8. A differential complex $\mathcal{E}$ is called an **elliptic complex** if, for any $\xi_x \in T^*_xM$ non-zero, the sequence

$$E^0_x \stackrel{\sigma_d(P_0)(\xi_x)}{\longrightarrow} E^1_x \stackrel{\sigma_d(P_1)(\xi_x)}{\longrightarrow} E^2_x \stackrel{\sigma_d(P_2)(\xi_x)}{\longrightarrow} \dots$$

is exact, i.e.

$$\mathrm{Ker}(\sigma_d(P_{k+1})(\xi_x)) = \mathrm{Im}(\sigma_d(P_k)(\xi_x)).$$

For a general differential complex $\mathcal{E}$, one can define

$$Z^k(\mathcal{E}) = \mathrm{Ker}(P_k), \quad B^k(\mathcal{E}) = \mathrm{Im}(P_{k_1}),$$

and the k-th cohomology groups

$$H^k(\mathcal{E}) = Z^k(\mathcal{E})/B^k(\mathcal{E}).$$

The space $H^k(M, P_*)$ defined as above, is called the k-th cohomology group of the elliptic complex. One says that $\mathcal{E}$ is a **Fredholm complex** if all these groups are finite dimensional. In this case, one defines **the Euler characteristic** of $\mathcal{E}$ as

$$\chi(\mathcal{E}) := \sum_k (-1)^k \mathrm{dim}(H^k(\mathcal{E})).$$

Another version of the preliminary version of the Atiyah-Singer index theorem is the following:

Theorem 1.3.9. *If $\mathcal{E}$ is an elliptic complex over a compact manifold M, then it is also Fredholm, and the Euler characteristic $\chi(\mathcal{E})$ can be expressed in terms of topological invariants of the principal symbols associated to $\mathcal{E}$.*

Example 1.3.10. For the DeRham complex $(\Omega^*(M), d)$, the resulting cohomology in a degree k is called the k-th de Rham cohomology of M, denoted $H^k_{\mathrm{dR}}(M)$. The ellipticity of this complex (see the next exercise), together with the above result, implies that the de Rham cohomology of a compact manifold is finite dimensional. For a simpler proof of this result, involving Meyer-Vietoris sequences, we refer the reader to the book by Thornehaeve-Madsen or the book by Bott and Tu.

Exercise 1.3.11. Let V be a finite dimensional complex vector space. Let $\xi \in V^* \setminus \{0\}$. Show that the complex of linear maps $T_k : \wedge^k V^* \to \wedge^{k+1} V^*$, $\omega \mapsto \xi \wedge \omega$, is exact.

Deduce that the DeRham complex a manifold is an elliptic complex.

Example 1.3.12. Any elliptic operator $P \in \mathcal{D}_k(E, F)$ can be seen as an elliptic complex with $E^0 = E$, $E^1 = F$ and $E^k = 0$ for other k's, $P_0 = P$. Moreover, its Euler characteristic is just the index of P. Hence the last theorem seems to be a generalization of Theorem 1.3.2. However, there is a simple trick to go the other way around. This the next exercises.

In these exercises, we fix a a density on M and, on each vector bundle over M that we will be considering, a hermitian inner product. First we relate the notion of elliptic complexes to that of elliptic operators. For a complex of differential operators

$$\mathcal{E} : \Gamma(E^0) \xrightarrow{P_0} \Gamma(E^1) \xrightarrow{P_1} \Gamma(E^2) \xrightarrow{P_2} \ldots$$

(with hermitian inner products on each E^k) we form the total vector bundle

$$E = E^0 \oplus E^1 \oplus \ldots$$

and the Laplacian

$$\Delta := P \circ P^* + P^* \circ P : \Gamma(E) \to \Gamma(E).$$

Note that *Delta* is of type

$$\Delta = (\Delta_0, \Delta_1, \ldots) \quad \text{with} \ \ \Delta_k : \Gamma(E^k) \to \Gamma(E^k).$$

Exercise 1.3.13. For any complex of differential operators $(\mathcal{E})$, show that $\mathrm{Ker}(\Delta) = \mathrm{Ker}(P) \cap \mathrm{Ker}(P^*)$ and that the sum

$$\mathrm{Ker}(\Delta) + \mathrm{Im}(P) + \mathrm{Im}(P^*) \subset \Gamma(E)$$

is a direct sum.

Exercise 1.3.14. Show that the complex (1.6) is elliptic if and only if the Laplacian $\Delta_{\mathcal{E}}$ is an elliptic operator.

(Hint: translate this into an linear problem, in which you deal with a cochain complex consisting of finite dimensional hermitian vector spaces

$$V^0 \xrightarrow{\partial} V^1 \xrightarrow{\partial} V^2 \xrightarrow{\partial} \ldots, \tag{1.7}$$

and show that this is exact if and only if the associated "algebraic Laplacian"

$$\partial \circ \partial^* + \partial^* \circ \partial : \oplus_k V^k \to \oplus_k V^k$$

is an isomorphism.)

The next exercise is similar to the previous one but, instead of addressing ellipticity, it addresses the Fredholmness condition together with the condition (*Dec*) mentioned before:

(Dec) $\Gamma(E) = \mathrm{Ker}(\Delta) + \mathrm{Im}(\Delta)$.

The two exercises together will imply, once we show that elliptic differential operators are Fredholm, that elliptic complexes are Fredholm.

Exercise 1.3.15. With the same notation as before, for an elliptic complex $\mathcal{E}$, show that the Laplacians Fredholm and satisfies (*Dec*) if and only if the complex $\mathcal{E}$ is Fredholm and satisfies the following "Hodge decomposition"

$$\Gamma(E) = \mathrm{Ker}(P) \cap \mathrm{Ker}(P^*) + \mathrm{Im}(P) + \mathrm{Im}(P^*)$$

or, equivalently (by Exercise 1.3.13),

$$\Gamma(E) = \mathrm{Ker}(\Delta) \oplus \mathrm{Im}(P) \oplus \mathrm{Im}(P^*).$$

Moreover, in this case $\mathrm{Ker}(P) = \mathrm{Ker}(\Delta) \oplus \mathrm{Im}(P)$, so that $H^k(\mathcal{E})$ is canonically isomorphic to $\mathrm{Ker}(\Delta_k)$.

Putting the previous exercises together, we consider

$$E_+ := E^0 \oplus E^2 \oplus E^4 \oplus \dots, \quad E_- := E^1 \oplus E^3 \oplus E^5,$$

$$Q_+ := (P + P^*)|_{\Gamma(E_+)} : \Gamma(E_+) \to \Gamma(E_-)$$

and you should deduce:

Exercise 1.3.16. If $\mathcal{E}$ is Fredholm and Δ satisfies (Dec), then Q_+ is an elliptic operator and

$$\chi(\mathcal{E}) = \mathrm{Index}(Q_+).$$

1.4. Fredholm operators as tools- summary of what we need

As we have already mentioned, the aim of these lectures is to understand Theorem 1.3.2. The first few lectures will be devoted to proving that the index of any elliptic operator (over compact manifolds) is finite; after that we will spend some lectures to explain the precise meaning of "topological data associated to the symbol" (and the last lectures will be devoted to some examples). The nature of these three parts is Analysis- Topology- Geometry.

For the first part- on the finiteness of the index, we will rely on the fact that indices of operators are well behaved in the framework of Banach spaces. This is some very general theory that belongs to Functional Analysis, which we recall in this section (for more details and proofs, see the next section). In the next few lectures we will show how this theory applies to our problem (on short, we have to pass from spaces of sections of vector bundles to appropriate "Banach spaces of sections" and show that our operators have the desired compactness properties).

So, for this section we fix two Banach spaces $\mathbb{E}$ and $\mathbb{F}$ and we discuss Fredholm operators between them- i.e. operators which have a well-defined index. More formally, we denote by $\mathcal{L}(\mathbb{E}, \mathbb{F})$ the space of bounded (i.e. continuous) linear operators from $\mathbb{E}$ to $\mathbb{F}$ and we take the following:

Definition 1.4.1. A bounded operator $T : \mathbb{E} \to \mathbb{F}$ is called Fredholm if $\mathrm{Ker}(A)$ and $\mathrm{Coker}(A)$ are finite dimensional. We denote by $\mathcal{F}(\mathbb{E}, \mathbb{F})$ the space of all Fredholm operators from $\mathbb{E}$ to $\mathbb{F}$.

The index of a Fredholm operator A is defined by

$$\mathrm{Index}(A) := \dim(\mathrm{Ker}(A)) - \dim(\mathrm{Coker}(A)).$$

Note that a consequence of the Fredholmness is the fact that $R(A) = \mathrm{Im}(A)$ is closed. Here are the first properties of Fredholm operators.

Theorem 1.4.2. *Let $\mathbb{E}$, $\mathbb{F}$, $\mathbb{G}$ be Banach spaces.*

(i) *If* $B : \mathbb{E} \to \mathbb{F}$ *and* $A : \mathbb{F} \to \mathbb{G}$ *are bounded, and two out of the three operators* A, B *and* AB *are Fredholm, then so is the third, and*

$$\text{Index}(A \circ B) = \text{Index}(A) + \text{Index}(B).$$

(ii) *If* $A : \mathbb{E} \to \mathbb{F}$ *is Fredholm, then so is* $A^* : \mathbb{F}^* \to \mathbb{E}^*$ *and* [3]

$$\text{Index}(A^*) = -\text{Index}(A).$$

(iii) $\mathcal{F}(\mathbb{E}, \mathbb{F})$ *is an open subset of* $\mathcal{L}(\mathbb{E}, \mathbb{F})$*, and*

$$\text{Index} : \mathcal{F}(\mathbb{E}, \mathbb{F}) \to \mathbb{Z}$$

is locally constant.

What will be important for us is an equivalent description of Fredholm operators, in terms of compact operators. First we recall the following:

Definition 1.4.3. A linear map $T : \mathbb{E} \to \mathbb{F}$ is said to be compact if for any bounded sequence $\{x_n\}$ in $\mathbb{E}$, $\{T(x_n)\}$ has a convergent subsequence.

Equivalently, compact operators are those linear maps $T : \mathbb{E} \to \mathbb{F}$ with the property that $T(B_{\mathbb{E}}) \subset \mathbb{F}$ is relatively compact, where $B_{\mathbb{E}}$ is the unit ball of $\mathbb{E}$. Here are the first properties of compact operators.

We point out the following improvement/consequence of the Fredholm alternative for compact operators (discussed in the appendix- see Theorem 1.5.9 there).

Theorem 1.4.4. *Compact perturbations do not change Fredholmness and do not change the index, and zero index is achieved only by compact perturbations of invertible operators.*

More precisely:

(i) *If* $K \in \mathcal{K}(\mathbb{E}, \mathbb{F})$ *and* $A \in \mathcal{F}(\mathbb{E}, \mathbb{F})$*, then* $A + K \in \mathcal{F}(\mathbb{E}, \mathbb{F})$ *and* $\text{Index}(A + K) = \text{Index}(A)$*.*

(ii) *If* $A \in \mathcal{F}(\mathbb{E}, \mathbb{F})$*, then* $\text{Index}(A) = 0$ *if and only if* $A = A_0 + K$ *for some invertible operator* A_0 *and some compact operator* K*.*

Finally, there is yet another relation between Fredholm and compact operators, know as the Atkinson characterization of Fredholm operators:

Theorem 1.4.5. *Fredholmness= invertible modulo compact operators.*

More precisely, given a bounded operator $A : \mathbb{E} \to \mathbb{F}$*, the following are equivalent:*

(i) A *is Fredholm.*

(ii) A *is invertible modulo compact operators, i.e. there exist and operator* $B \in \mathcal{L}(\mathbb{F}, \mathbb{E})$ *and compact operators* K_1 *and* K_2 *such that*

$$BA = 1 + K_1, AB = 1 + K_2.$$

1.5. Appendix: Fredholm operators- more details and proofs

Here, for the curious reader, we expand the previous section, providing more details and proofs.

[3]here, $A^*(\xi)(e) = \xi(A(e)))$

1.5.1. Fredholm operators: basic properties

Let $\mathbb{E}$ and $\mathbb{F}$ be two Banach spaces. We denote by $\mathcal{L}(\mathbb{E}, \mathbb{F})$ the space of bounded linear operators from $\mathbb{E}$ to $\mathbb{F}$.

Definition 1.5.1. A bounded operator $T : \mathbb{E} \to \mathbb{F}$ is called Fredholm if $\mathrm{Ker}(A)$ and $\mathrm{Coker}(A)$ are finite dimensional. We denote by $\mathcal{F}(\mathbb{E}, \mathbb{F})$ the space of all Fredholm operators from $\mathbb{E}$ to $\mathbb{F}$.

The index of a Fredholm operator A is defined by

$$\mathrm{Index}(A) := \dim(\mathrm{Ker}(A)) - \dim(\mathrm{Coker}(A)).$$

Note that a consequence of the Fredholmness is the fact that $R(A) = \mathrm{Im}(A)$ is closed. Here are the first properties of Fredholm operators (stated already as Theorem 1.4.2 in the previous section).

Theorem 1.5.2. *Let $\mathbb{E}$, $\mathbb{F}$, $\mathbb{G}$ be Banach spaces.*

(i) If $B : \mathbb{E} \to \mathbb{F}$ and $A : \mathbb{F} \to \mathbb{G}$ are bounded, and two out of the three operators A, B and AB are Fredholm, then so is the third, and

$$\mathrm{Index}(A \circ B) = \mathrm{Index}(A) + \mathrm{Index}(B).$$

(ii) If A is Fredholm, then so is A^*, and

$$\mathrm{Index}(A^*) = -\mathrm{Index}(A).$$

(iii) $\mathcal{F}(\mathbb{E}, \mathbb{F})$ is an open subset of $\mathcal{L}(\mathbb{E}, \mathbb{F})$, and

$$\mathrm{Index} : \mathcal{F}(\mathbb{E}, \mathbb{F}) \to \mathbb{Z}$$

is locally constant.

Proof Part (i) is a purely algebraic result. We prove that if A and B are Fredholm, then so is AB (the other cases following from the arguments bellow). First of all we have a short exact sequence

$$0 \to \mathrm{Ker}(B) \to \mathrm{Ker}(AB) \stackrel{B}{\to} \mathrm{Im}(B) \cap \mathrm{Ker}(A) \to 0,$$

and this proves that AB has finite dimensional kernel with

$$\dim(\mathrm{Ker}(AB)) = \dim(\mathrm{Ker}(B)) + \dim(\mathrm{Ker}(A) \cap \mathrm{Im}(B)).$$

Next, we have the exact sequence

$$0 \to \frac{\mathrm{Im}(B) + \mathrm{Ker}(A)}{\mathrm{Im}(B)} \to \frac{\mathbb{F}}{\mathrm{Im}(B)} \stackrel{A}{\to} \frac{\mathbb{G}}{\mathrm{Im}(AB)} \to \frac{G}{\mathrm{Im}(A)} \to 0,$$

where the first map is the obvious inclusion, and the last one is the obvious projection. All the spaces in this sequence, except maybe $\mathrm{Coker}(AB)$, are finite dimensional (the first one is isomorphic to $\mathrm{Ker}(A)/\mathrm{Ker}(A) \cap \mathrm{Im}(B)$, so we deduce that also $\mathrm{Coker}(AB)$ is finite dimensional and

$$\dim(\mathrm{Coker}(AB)) = \dim(\mathrm{Coker}(A)) + \dim(\mathrm{Coker}(B)) - \dim(\mathrm{Ker}(A)) + \dim(\mathrm{Ker}(A) \cap \mathrm{Im}(B)).$$

Combining the last two identities, we get the desired equation for the index.

Part (ii) is easy.

For (iii), let $A \in \mathcal{F}(\mathbb{E}, \mathbb{F})$. We choose complements $\mathbb{E}_1$ of $\mathrm{Ker}(A)$ in $\mathbb{E}$, and $\mathbb{F}_2$ of $\mathrm{Im}(A)$ in $\mathbb{F}$. This is possible because $\mathrm{Ker}(A)$ is finite dimensional, and because $\mathrm{Im}(A)$ is closed of finite codimension, respectively. Denote by

$i_1 : \mathbb{E}_1 \to \mathbb{E}$ the canonical inclusion and by $p : \mathbb{F} \to \mathrm{Im}(A)$ the projection. To any operator $S \in \mathcal{L}(\mathbb{E}, \mathbb{F})$ we associate the operator $S_0 = pSi : \mathbb{E}_1 \to \mathrm{Im}(A)$. Since A_0 is clearly an isomorphism, there exists $\epsilon > 0$ so that, for all S such that $||S - A|| < \epsilon$, S_0 is an isomorphism. For such an S we can also say that $S_0 = pSi$ is Fredholm of index zero. But p is Fredholm of index $-\dim(\mathrm{Ker}(A))$ while i is Fredholm of index $\dim(\mathrm{Coker}(A))$. Using (i), S must be Fredholm and

$$0 = \mathrm{Index}(S_0) = -\dim(\mathrm{Ker}(A)) + \mathrm{Index}(S) + \dim(\mathrm{Coker}(A)).$$

In conclusion, for $||S - A|| < \epsilon$, S is Fredholm of index equal to $\mathrm{Index}(A)$. □

1.5.2. Compact operators: basic properties

Definition 1.5.3. A linear map $T : \mathbb{E} \to \mathbb{F}$ is said to be compact if for any bounded sequence $\{x_n\}$ in $\mathbb{E}$, $\{T(x_n)\}$ has a convergent subsequence.

We denote by $\mathcal{K}(\mathbb{E}, \mathbb{F})$ the space of such compact operators.

Equivalently, compact operators are those linear maps $T : \mathbb{E} \to \mathbb{F}$ with the property that $T(B_{\mathbb{E}}) \subset \mathbb{F}$ is relatively compact, where $B_{\mathbb{E}}$ is the unit ball of $\mathbb{E}$. Here are the first properties of compact operators.

Proposition 1.5.4. *Let $\mathbb{E}$, $\mathbb{F}$ and $\mathbb{G}$ be Banach spaces.*

(i) $\mathcal{K}(\mathbb{E}, \mathbb{F})$ is a closed subspace of $\mathcal{L}(\mathbb{E}, \mathbb{F})$.

(ii) given $T \in \mathcal{L}(\mathbb{E}, \mathbb{F})$, $S \in \mathcal{L}(\mathbb{F}, \mathbb{G})$, if T or S is compact, then so is $T \circ S$.

(iii) $T \in \mathcal{L}(\mathbb{E}, \mathbb{F})$ is compact if and only if $T^* \in \mathcal{L}(\mathbb{F}^*, \mathbb{E}^*)$ is.

In particular, $\mathcal{K}(\mathbb{E})$ is a closed two-sided $*$-ideal in $\mathcal{L}(\mathbb{E})$.

Note that, if $\mathbb{E} = \mathcal{H}$ is a Hilbert space, then $\mathcal{K}(\mathcal{H})$ is the unique non-trivial (norm-)closed ideal in $\mathcal{L}(\mathcal{H})$.

Proof We prove that, if $T_n \to T$ and T_n are all compact, then T is compact. Since $T(B_{\mathbb{E}})$ is bounded and $\mathbb{F}$ is Banach, it suffices to show that $T(B_{\mathbb{E}})$ is precompact, i.e. that it can be covered by a finite number of balls of arbitrarily small radius ϵ. So, let $\epsilon > 0$. Choose n such that $||T_n - T|| < \epsilon/2$ and cover $T_n(B_{\mathbb{E}})$ by a finite number of balls $B(f_i, \epsilon/2)$. Then the balls $B(f_i, \epsilon)$ cover $T(B_{\mathbb{E}})$.

We now prove (iii) (the remaining statements are immediate). Assume first that T is relatively compact, and let $K \subset \mathbb{F}$ be the closure of $T(B_{\mathbb{E}})$ (compact). Let v_n be a sequence in the unit ball of $\mathbb{F}^*$. We want to prove that $T^*(v_n) = v_n \circ T$ has a convergent subsequence. We consider the space $\mathcal{C}(K)$ of continuous functions on K, and the subspace $\mathcal{H}$ consisting of the restrictions $\phi_n = v_n|_K$. We claim we can apply Ascoli to $\mathcal{H}$. Equicontinuity: since $||v_n|| \leq 1$, we have

$$|\phi_n(x) - \phi_n(y)| \leq ||x - y||$$

for all x and y. Equiboundedness: since $||v_n|| \leq 1$ and any $y \in K$ has norm less then $||T||$, we have

$$|\phi_n(y)| \leq ||T||$$

for all $y \in K$ and all n. By Ascoli, we find a subsequence of ϕ_n, which we may assume is ϕ_n itself, which is convergent in norm. We use that it is Cauchy:

$$sup_{y \in K} |\phi_n(y) - \phi_m(y)| \to 0.$$

Since $T(B_{\mathbb{E}}) \subset K$, this clearly implies that $T^*(v_n)$ is Cauchy in $\mathbb{E}^*$, hence convergent. For the converse of (iii), we apply the first half to conclude that $T^{**} : \mathbb{E}^{**} \to \mathbb{F}^{**}$ is compact. Viewing $\mathbb{E} \subset \mathbb{E}^**$ as a closed subspace, and similarly for $\mathbb{F}$, we have $T(B_{\mathbb{E}}) = T^{**}(B_{\mathbb{E}})$-relatively compact. □

Next, we discuss the relationship with finite rank operators.

Definition 1.5.5. A linear map $T : \mathbb{E} \to \mathbb{F}$ is said to be of finite rank if it is continuous and its image is a finite dimensional space. We denote by $\mathcal{K}_{\text{fin}}(\mathbb{E}, \mathbb{F})$ the space of compact operators from $\mathbb{E}$ to $\mathbb{F}$.

Equivalently, $\mathcal{K}_{\text{fin}}(\mathbb{E}, \mathbb{F})$ is the image of the canonical inclusion

$$\mathbb{E}^* \otimes \mathbb{F} \to \mathcal{L}(\mathbb{E}, \mathbb{F}), \quad \sum e_i^* \otimes f_i \mapsto \sum e_i^*(-) f_i$$

i.e. the finite rank operators are those of type $T(x) = \sum e_i^*(x) f_i$ (finite sum) with $e_i^* \in \mathbb{E}^*$, $f_i \in \mathbb{F}$. It is clear that

$$\mathcal{K}_{\text{fin}}(\mathbb{E}, \mathbb{F}) \subset \mathcal{K}(\mathbb{E}, \mathbb{F}) \subset \mathcal{L}(\mathbb{E}, \mathbb{F})$$

and $\mathcal{K}_{\text{fin}}(\mathbb{E}, \mathbb{F})$ has all the properties of $\mathcal{K}(\mathbb{E}, \mathbb{F})$ from the previous proposition, except from being closed. All we can say in general is that

$$\overline{\mathcal{K}_{\text{fin}}(\mathbb{E}, \mathbb{F})} \subset \mathcal{K}(\mathbb{E}, \mathbb{F}),$$

and the next proposition[4] gives conditions on $\mathbb{F}$ so that this inclusion becomes equality. For this, we recall that a Schauder basis for $\mathbb{F}$ is a countable family $\{f_k : k \geq 1\}$ of elements of $\mathbb{F}$ with the property that each $y \in \mathbb{F}$ can be uniquely written as

$$y = \sum_{k=1}^{\infty} t_k f_k$$

with t_k-scalars. Clearly, any separable Hilbert space admits a Schauder basis, but also spaces such as L^p with $p \geq 1$ do.

Proposition 1.5.6. *If $\mathbb{F}$ admits a Schauder basis then an operator $T \in \mathcal{L}(\mathbb{E}, \mathbb{F})$ is compact if and only if it is the limit of a sequence of finite rank operators; in other words,*

$$\mathcal{K}(\mathbb{E}, \mathbb{F}) = \overline{\mathcal{K}_{fin}(\mathbb{E}, \mathbb{F})}.$$

Proof We still have to show that any compact T is a limit of finite rank ones. Let $\{f_k : k \geq 1\}$ be a Schauder basis, and let $f^k : \mathbb{F} \to \mathbb{C}$ be the coordinate functions. It is known that the Schauder basis can be chosen such that f^k are continuous. We put $T_k \in \mathcal{L}(\mathbb{E}, \mathbb{F})$,

$$T_k(x) = \sum_{i}^{k} f^i(T(x)) f_i.$$

To prove $T_k \to T$, let $\epsilon > 0$. For any x of norm less then 1, we find N such that

$$\sum_{i=k}^{\infty} f^i(T(x)) f_i|| < \epsilon$$

[4]this proposition is just for your curiosity.

for all $k \geq N$. But since $T(B_{\mathbb{E}})$ is relatively compact, we can choose N uniform with respect to $x \in B_{\mathbb{E}}$. Hence $||T - T_k|| < \epsilon$ for all $k \geq N$. □

Finally, to give an alternative description of compact operators, we recall that a linear map $T : \mathbb{E} \to \mathbb{F}$ is said to be completely continuous if it carries weakly convergent sequences into norm convergent sequences.

Proposition 1.5.7. *Any compact operator $T : \mathbb{E} \to \mathbb{F}$ is completely continuous. The converse is true if $\mathbb{E}$ is reflexive.*

1.5.3. Compact operators: the Fredholm alternative

In this section, $\mathbb{E} = \mathbb{F}$ (a Banach space). One of the versions of the Fredholm alternative says that, if K is a compact operator on $\mathbb{E}$, then the associated equation $x = Kx + y$ behaves like in the finite dimensional case: if the homogeneous equation $x = Kx$ has only the trivial solution $x = 0$, then the inhomogeneous equations

$$x = Kx + y$$

has a unique solution $x \in \mathbb{E}$, for every $y \in \mathbb{E}$. More precisely, we have the following:

Proposition 1.5.8. *For $K \in \mathcal{K}(\mathbb{E})$, the following are equivalent:*

(i) $1 - K$ *is injective.*
(ii) $1 - K$ *is surjective.*
(iii) $1 - K$ *is bijective.*

The general version of the Fredholm alternative is best expressed in terms of Fredholm operators.

Theorem 1.5.9. *For any compact operator K on $\mathbb{E}$, $1 - K$ is a Fredholm operator of index zero.*

Before turning to the proofs, let us point out that these results are naturally cast as properties of the spectrum of compact operators [5]. Recall that, for an operator $T : \mathbb{E} \to \mathbb{E}$, the spectrum $\sigma(T)$ consists of those complex numbers λ with the property that $\lambda - T$ is not invertible. A particular case of this is when the equation $Tx = \lambda x$ has a non-trivial solution $x \in \mathbb{E}$. In this case λ is called an eigenvalue of T, the space $N_\lambda = \{x \in \mathbb{E} : Tx = \lambda x\}$ is called the λ-eigenspace of T, and the set of all eigenvalues of T is denoted by $\sigma_p(T)$ (called the point-spectrum of T). With these, we have:

Theorem 1.5.10. *Assume that $\mathbb{E}$ is infinite dimensional. For any compact operator $K \in \mathcal{K}(\mathbb{E})$,*

(i) $\sigma(K) = \sigma_p(K) \cup \{0\}$, *and this is either finite or it is a countable sequence of complex number converging to zero.*
(ii) for any non-zero eigenvalue λ, the corresponding eigenspace $N_\lambda(K)$ is finite dimensional.

We now turn to the proofs of these results. We will use the Riesz lemma:

[5] again, this (i.e. the next theorem) is just for your curiosity.

Lemma 1.5.11. *If $M \subset \mathbb{E}$ is a closed subspace, $M \neq \mathbb{E}$, then for every $\epsilon > 0$, there exists $x_\epsilon \in \mathbb{E}$ such that*

$$||x_\epsilon|| = 1, d(x_\epsilon, M) > 1 - \epsilon.$$

Proof Choose $x \in \mathbb{E} - M$ and put $d = d(x, M) > 0$. Since $d(x, M) < d/(1-\epsilon)$, we find $m_\epsilon \in M$ such that $||x - m_\epsilon|| < d/(1-\epsilon)$. Put

$$x_\epsilon = \frac{x - m_\epsilon}{||x - m_\epsilon||}.$$

□

Let us also point out the following simple consequence, known as the Theorem of Riesz, which is interesting on its own, and which immediately implies (ii) of Theorem 1.5.10.

Corollary 1.5.12. *If the unit ball of a Banach space $\mathbb{E}$ is compact, then $\mathbb{E}$ is finite dimensional.*

Proof Cover $B_{\mathbb{E}}$ by a finite number of balls of radius $1/2$. Denote by M the subspace spanned by the centers of these balls; if $M \neq \mathbb{E}$, we can apply the previous lemma with $\epsilon = 1/2$ and we obtain a contradiction. In conclusion, $\mathbb{E} = M$ is finite dimensional. □

Proof [(of Proposition 1.5.8 and of Theorem 1.5.9)] We first claim that, for any compact operator K, the image of $1 - K$ is closed in $\mathbb{E}$. Denote $S = 1 - K$, $N = \mathrm{Ker}(S)$. Consider $y \in \overline{S(\mathbb{E})}$, and write

$$y = \lim_{n \to \infty} S(x_n)$$

for some sequence $\{x_n\}$ in $\mathbb{E}$. We will show that $\{x_n\}$ may be chosen to be bounded. From the compactness of K, this implies that $\{x_n\}$ may be assumed to converge to an element $x \in \mathbb{E}$, hence $y = S(x) \in S(\mathbb{E})$. To achieve the boundedness of $\{x_n\}$, it suffices to show that $d(x_n, N)$ is bounded. Indeed, in this case we find $a_n \in N$ such that $\{||x_n - a_n||\}$ is bounded and we may replace x_n by $x_n - a_n$.

So, let us assume that $\{d(x_n, N)\}$ is unbounded and we will obtain a contradiction. First of all, we may assume that this unbounded sequence converges to ∞. Put

$$z_n = \frac{1}{d(x_n, N)} x_n.$$

This has the properties:

$$d(z_n, N) = 1, \ \lim_{n \to \infty} S(z_n) = 0.$$

We may assume that z_n is bounded (otherwise, by the first property above we find $z_n^{'} \in z_n + N$ such that $||z_n^{'}|| \leq 2$ and $\{z_n^{'}\}$ has the same properties). Since K is compact, we may also assume that $K(z_n)$ converges to an element $a \in \mathbb{E}$. From the properties of z_n, we find that $d(a, N) = 1$ and that $z_n = S(z_n) + K(z_n)$ converges to a. The last statement and the definition of a imply that $K(a) = a$, i.e. $a \in N$, which contradicts $d(a, N) = 1$. This finishes the proof of the fact that $\mathrm{Im}(1 - K)$ is closed.

With this property proven, to finish the proof of Proposition 1.5.8, one can go on with a "direct" argument that does not use any of the properties of Fredholm operators. Alternatively, one can now prove Theorem 1.5.9, which clearly implies the proposition.

Proof [(end of proof of Proposition 1.5.8)] We now prove that (i) implies (ii). Hence, let us assume that S is injective and $S(\mathbb{E}) \neq \mathbb{E}$. We consider the decreasing sequence of subspaces of $\mathbb{E}$:

$$\ldots \subset \mathbb{E}_3 \subset \mathbb{E}_2 \subset \mathbb{E}_1 = \mathbb{E}$$

where $\mathbb{E}_n = S^n(\mathbb{E})$. Note that $K(\mathbb{E}_n) \subset \mathbb{E}_n$. Since the restriction of K to each $\mathbb{E}_n$ is compact, the first part of the proof implies that each $\mathbb{E}_n$ is a closed subspace of $\mathbb{E}_{n-1}$, while the injectivity of S implies that these inclusions are proper. From the Riesz Lemma we find $x_n \in \mathbb{E}_n$ with

$$||x_n|| = 1, d(x_n, \mathbb{E}_{n+1}) \geq \frac{1}{2}.$$

However, for each $n > m$ one has

$$Kx_n - Kx_m = Kx_n - x_m + Sx_m \in E_n - x_m + E_{m1} \subset E_{m1} - x_m,$$

hence

$$||Kx_n - Kx_m|| > \frac{1}{2},$$

and then $\{Kx_n\}$ cannot have a convergent subsequence, which contradicts the compactness of K.

Finally, the inverse implication (ii) $\Longrightarrow$ (i) is a consequence of the direct implication and the general fact that $\mathrm{Ker}(T^*) = \mathrm{Im}(T)^{\perp}$: if $S = 1 - K$ is surjective, it follows that $Ker(S^*) = \mathrm{Im}(S)^{\perp} = \{0\}$, i.e. S^* must be injective. Applying (i) $\Longrightarrow$ (ii) to K^* (we do know that K^* is compact!), S^* is surjective, hence $\mathrm{Ker}(S) = \mathrm{Im}(S^*)^{\perp} = \{0\}$, i.e. S is injective. □

Proof [(end of the proof of Theorem 1.5.9), hence also of proof 2 of Proposition 1.5.8)] The Riesz Lemma immediately implies that $\mathrm{Ker}(1 - K)$ is finite dimensional. Applying this to K^*, we deduce that also $\mathrm{Im}(1 - K)^{\perp} = \mathrm{Ker}(1 - K^*)$ is finite dimensional. Since $\mathrm{Im}(1 - K)$ is closed (see the first part of the previous proof), we deduce $\mathrm{Im}(1 - K)$ is of finite codimension. Hence $1 - K$ is Fredholm. We then have a continuous family $\{1 - tK : t \in \mathbb{R}\}$ of Fredholm operators. By the properties of the index, the index at $t = 1$ coincides with the index at $t = 0$, which is zero. □

□

Proof [(of Theorem 1.5.10)] The only thing still to be proven is that $\sigma_p(K)$ is either finite, or a countable sequence converging to zero. It suffices to show that for any sequence $\{\lambda_n\}$ of distinct eigenvalues of K which converge to λ (finite or infinite), $\lambda = 0$. Assume that $\{\lambda_n\}$ is such a sequence. Choose eigenvectors x_n corresponding to λ_n, $x_n \neq 0$ and put

$$\mathbb{E}_n = \mathrm{span}\{x_1, \ldots, x_n\}.$$

Since the λ_i are distinct, it follows that

$$\mathbb{E}_1 \subset \mathbb{E}_2 \subset \ldots$$

is a strictly increasing sequence of subspaces of $\mathbb{E}$. From the Riesz Lemma with $\epsilon = 1/2$ we find

$$u_n \in \mathbb{E}_n, \quad ||u_n|| = 1, \quad d(u_n, \mathbb{E}_{n-1}) > \frac{1}{2}.$$

Note also that

$$T(\mathbb{E}_n) \subset \mathbb{E}_n, \quad (T - \lambda_m \mathrm{Id})(\mathbb{E}_m) \subset \mathbb{E}_{m-1}.$$

We deduce that for $m > n$,

$$\frac{Tu_n}{\lambda_n} - \frac{Tu_m}{\lambda_m} \in E_n + E_{m-1} - u_m = E_{m-1} - u_m,$$

hence

$$||\frac{Tu_n}{\lambda_n} - \frac{Tu_m}{\lambda_m}|| \geq \frac{1}{2},$$

and $\{Tu_n/\lambda_n\}$ cannot have a convergent subsequence. But, since T is compact, $\{Tu_n\}$ does posses a convergent subsequence, so λ must equal 0. □

1.5.4. The relation between Fredholm and compact operators

We have already seen from the Fredholm alternative that, for any compact operator $K \in \mathcal{K}(\mathbb{E})$, $1 - K$ is a Fredholm operator of index zero. Much more precisely, we have the following (stated as Theorem 1.4.4 in the previous section).

Theorem 1.5.13. *Compact perturbations do not change Fredholmness and do not change the index, and zero index is achieved only by compact perturbations of invertible operators.*

More precisely:

(i) If $K \in \mathcal{K}(\mathbb{E}, \mathbb{F})$ and $A \in \mathcal{F}(\mathbb{E}, \mathbb{F})$, then $A + K \in \mathcal{F}(\mathbb{E}, \mathbb{F})$ and $\mathrm{Index}(A + K) = \mathrm{Index}(A)$.

(ii) If $A \in \mathcal{F}(\mathbb{E}, \mathbb{F})$, then $\mathrm{Index}(A) = 0$ *if and only if $A = A_0 + K$ for some invertible operator A_0 and some compact operator K.*

There is yet another relation between Fredholm and compact operators, know as the Atkinson characterization of Fredholm operators (Theorem 1.4.5 in the previous section).

Theorem 1.5.14. *Fredholmness= invertible modulo compact operators.*

More precisely, given a bounded operator $A : \mathbb{E} \to \mathbb{F}$, the following are equivalent:

(i) A is Fredholm.

(ii) A is invertible modulo compact operators, i.e. there exist and operator $B \in \mathcal{L}(\mathbb{F}, \mathbb{E})$ and compact operators K_1 and K_2[6] such that

$$BA = 1 + K_1, AB = 1 + K_2.$$

[6]from the proof we will see that one can actually choose K_1 and K_2 to be finite rank operators, and B so that $ABA = A$, $BAB = B$.

We now turn to the proofs of these results.

Proof [(of Theorem 1.5.14)] Assume first the existence of B, K_1 and K_2. Since identity plus compact is Fredholm, we deduce that the kernel of A is finite dimensional (since is included in the kernel of $1 + K_1$) and, similarly, the cokernel of A is finite dimensional. Hence A is Fredholm.

Assume now that A is Fredholm. Choose a complement $\mathbb{E}_1$ of $\mathrm{Ker}(A)$ in $\mathbb{E}$ and a complement $\mathbb{F}_1$ of $\mathrm{Im}(A)$ in $\mathbb{F}$. Then $A_1 = A|_{\mathbb{E}_1}$ is an isomorphism from $\mathbb{E}_1$ into $\mathrm{Im}(A)$ and we define B such that $B = (A_1)^{-1}$ on $\mathrm{Im}(A)$ and $B = 0$ on $\mathbb{F}_2$. Then the resulting K_1 will be a projection onto $\mathrm{Ker}(A)$ and $1 + K_2$ will be a projection onto $\mathrm{Im}(A)$; hence K_1 and K_2 will have the desired properties. □

Proof [(of Theorem 1.5.13)] Part (i) follows easily from Atkinson's characterization and the Fredholm alternative: choose B, K_1 and K_2 as above. We deduce that B is itself Fredholm of index $-\mathrm{index}(A)$ (here we used the additivity of the index and the Fredholm alternative). We remark that $(A+K)B = 1+(K_1+KB)$ and $BA = 1 + (K_2 + BK)$, where $K_1 + KB$ and $K_2 + BK$ are compact. We then deduce that $A + K$ is Fredholm of index equal to $-\mathrm{index}(B) = \mathrm{index}(A)$.

We still have to prove that $\mathrm{Index}(A) = 0$ can only happen for compact perturbations of invertible operators. As above, we choose a complement $\mathbb{E}_1$ of the kernel of A and a complement $\mathbb{F}_1$ of the image of A. With respect to these decompositions, A is just $(x, y) \mapsto (A_1(x), 0)$, where $A_1 : \mathbb{E}_1 \to \mathrm{Im}(A)$ is an isomorphism (the restriction of A to $\mathbb{E}_1$). That A has zero index means that the dimension of $\mathrm{Ker}(A)$ equals to the dimension of $\mathbb{F}_1$ (bot finite!). Choosing an isomorphism $\phi : \mathrm{Ker}(A) \to \mathbb{F}_1$, the map $K : (x, y) \mapsto (0, \phi(y))$ is compact and $A + K = (A_1, \phi)$ is an isomorphism. □

LECTURE 2
Distributions on manifolds

As explained in the previous lecture, to show that an elliptic operator between sections of two vector bundles E and F,

$$P : \Gamma(M, E) \to \Gamma(M, F)$$

has finite index, we plan to use the general theory of Fredholm operators between Banach spaces. In doing so, we first have to interpret our P's as operators between certain "Banach spaces of sections". The problem is that the usual spaces of smooth sections $\Gamma(M, E)$ have no satisfactory Banach space structure. Given a vector bundle E over M, by a "Banach space of sections of E", $\mathbb{B}(M, E)$, one should understand (some) Banach space which contains the space $\Gamma(M, E)$ of all the smooth sections of E as a (dense) subspace. One way to introduce such Banach spaces is to consider the completion of $\Gamma(M, E)$ with respect to various norms of interest. This can be carried out in detail, but the price to pay is the fact that the resulting "Banach spaces of sections" have a rather abstract meaning (being defined as completions). We will follow a different path, which is based on the following remark: there is a very general (and natural!) notion of "generalized sections of a vector bundle E over M", hence a space $\Gamma_{\text{gen}}(M; E)$ of such generalized sections (namely the space $\mathcal{D}'(M, E)$ of distributions, discussed in this lecture), so general that all the other "Banach spaces of sections" are subspaces of $\Gamma_{\text{gen}}(M; E)$. The space $\Gamma_{\text{gen}}(M; E)$ itself will not be a Banach space, but all the Banach spaces of sections which will be of interest for us can be described as subspaces of $\Gamma_{\text{gen}}(M; E)$ satisfying certain conditions (and that is how we will define them).

Implicit in our discussion is the fact that all the spaces we will be looking at will be vector spaces endowed with a topology (t.v.s.'s= topological vector spaces). Although our final aim is to deal with Banach spaces, the general t.v.s.'s will be needed along the way (however, all the spaces we will be looking at will be l.c.v.s.'s= locally convex vector spaces, i.e., similarly to Banach spaces, they can be defined using certain seminorms).

In this lecture, after recalling the notion of t.v.s. (topological vector space) and the special case of l.c.v.s. (locally convex vector space), we will discuss the space of generalized functions (distributions) on opens in $\mathbb{R}^n$ and then their generalizations to functions on manifolds or, more generally, to sections of vector bundles over manifolds. Since t.v.s.'s, l.c.v.s.'s and the local theory of generalized functions (distributions) on opens in $\mathbb{R}^n$ have already been discussed in the intensive reminder, our job will be to pass from local (functions on opens in $\mathbb{R}^n$) to global (sections of vector bundles over arbitrary manifolds). However, these lecture notes also contain some of the local theory that has been discussed in the "intensive reminder".

2.1. Reminder: Locally convex vector spaces

We start by recalling some of the standard notions from functional analysis (which have been discussed in the intensive reminder).

Topological vector spaces

First of all, a **t.v.s.** (**topological vector space**) is a vector space V (over $\mathbb{C}$) together with a topology $\mathcal{T}$, such that the two structures are compatible, i.e. the vector space operations

$$V \times V, (v, w) \mapsto v + w, \ \mathbb{C} \times V \to V, (\lambda, v) \mapsto \lambda v$$

are continuous. Recall that associated to the topology $\mathcal{T}$ and to the origin $0 \in V$, one has the family of all open neighborhoods of 0:

$$\mathcal{T}(0) = \{D \in \mathcal{T} : 0 \in D\}.$$

Since the translations $\tau_x : V \to V$, $y \mapsto y + x$ are continuous, the topology $\mathcal{T}$ is uniquely determined by $\mathcal{T}(0)$: for $D \subset V$, we have

$$D \in \mathcal{T} \iff \forall\, x \in D \quad \exists\, B \in \mathcal{T}(0) \text{ such that } \ x + B \subset D. \tag{2.1}$$

In this characterization of the opens inside V, one can replace $\mathcal{T}(0)$ by any basis of neighborhoods of 0, i.e. by any family $\mathcal{B}(0) \subset \mathcal{T}(0)$ with the property that

$$D \in \mathcal{T}(0) \Longrightarrow \ \exists\, B \in \mathcal{B} \text{ such that } \ B \subset D.$$

In other words, if we knows a basis of neighborhoods $\mathcal{B}(0)$ of $0 \in V$, then we know the topology $\mathcal{T}$.

Exercise 2.1.1. Given a family $\mathcal{B}(0)$ of subsets of a vector space V containing the origin, what axioms should it satisfy to ensure that the resulting topology (defined by (2.1)) is indeed a topology which makes $(V, \mathcal{T})$ into a t.v.s.?

Note that, in a t.v.s. $(V, \mathcal{T})$, also the convergence can be spelled out in terms of a (any) basis of neighborhoods $\mathcal{B}(0)$ of 0: a sequence $(v_n)_{n \geq 1}$ of elements of V **converges to** $v \in V$, written $v_n \to v$, if and only if:

$$\forall\, B \in \mathcal{B}(0), \ \ \exists\, n_B \in \mathbb{N} \ \text{ such that } \ v_n - v \in B \ \ \forall\, n \geq n_B.$$

Of course, this criterion can be used for $\mathcal{B}(0) = \mathcal{T}(0)$, but often there are smaller bases of neighborhoods $\mathcal{B}(0)$ at hand (after all, "b" is just the first letter of the word "ball"). For instance, if $(V, || - ||)$ is a normed space, then the resulting t.v.s. has as basis of neighborhoods

$$\mathcal{B}(0) = \{B(0, r) : r \geq 0\},$$

where

$$B(0, r) = \{v \in V : ||v|| < r\}.$$

In a t.v.s. $(V, \mathcal{T})$, one can also talk about the notion of Cauchy sequence: a sequence $(v_n)_{n\geq 1}$ in V is called a **Cauchy sequence** if:

$$\forall\ D \in \mathcal{T}(0)\ \ \exists\ \ n_D \in \mathbb{N}\ \text{ such that }\ v_n - v_m \in D\ \ \forall\ n, m \geq n_D.$$

Again, if we have a basis of neighborhoods $\mathcal{B}(0)$ at our disposal, it suffices to require this condition for $D = B \in \mathcal{B}(0)$.

In particular, one can talk about completeness of a t.v.s: one says that $(V, \mathcal{T})$ is (sequentially) **complete** if any Cauchy sequence in V converges to some $v \in V$.

Locally convex vector spaces

Recall also that a **l.c.v.s.** (**locally convex vector space**) is a t.v.s. $(V, \mathcal{T})$ with the property that "there are enough convex neighborhoods of the origin". That means that

$$\mathcal{T}_{\text{convex}}(0) := \{C \in \mathcal{T}(0) : C \text{ is convex}\}$$

is a basis of neighborhoods of $0 \in V$ or, equivalently:

$$\forall\ D \in \mathcal{T}(0)\ \ \exists\ \ C \in \mathcal{T}(0) \text{ convex, such that }\ C \subset D.$$

In general, l.c.v.s.'s are associated to families of seminorms (and sometimes this is taken as "working definition" for locally convex vector spaces). First recall that a **seminorm** on a vector space V is a map $p : V \to [0, \infty)$ satisfying

$$p(v + w) \leq p(v) + p(w),\ \ p(\lambda v) = |\lambda| p(v),$$

for all $v, w \in V$, $\lambda \in \mathbb{C}$ (and it is called a norm if $p(v) = 0$ happens only for $v = 0$).

Associated to any family

$$P = \{p_i\}_{i\in I}$$

of seminorms (on a vector space V), one has a notion of balls:

$$B^r_{i_1,\ldots,i_n} := \{v \in V : p_{i_k}(v) < r,\ \forall 1 \leq k \leq n\},$$

defined for all $r > 0$, $i_1, \ldots, i_n \in I$. The collection of all such balls form a family $\mathcal{B}(0)$, which will induce a locally convex topology $\mathcal{T}_P$ on V (convex because each ball is convex). Note that, the convergence in the resulting topology is the expected one:

$$v_n \to v\ \text{ in } (V, \mathcal{T}_P) \Longleftrightarrow\ p_i(v_n - v) \to 0\ \ \forall\ i \in I.$$

(and there is a similar characterization for Cauchy sequences). The fact that, when it comes to l.c.v.s.'s it suffices to work with families of seminorms, follows from the following:

Theorem 2.1.2. *A t.v.s. $(V, \mathcal{T})$ is a l.c.v.s. if and only if there exists a family of seminorms P such that $\mathcal{T} = \mathcal{T}_P$.*

Proof Idea of the proof: to produce seminorms, one associates to any $C \subset V$ convex the functional

$$p_C(v) = \inf\{r > 0 : x \in rC\}.$$

Choosing C "nice enough", this will be a seminorm. One then shows that one can find a basis of neighborhoods of the origin consisting of "nice enough" convex neighborhoods. □

By abuse of terminology, we also say that (V, P) is a l.c.v.s. (but one should keep in mind that all that matters is not the family of seminorms P but just the induced topology $\mathcal{T}_P$).

Remark 2.1.3. In most of the examples of l.c.v.s.'s, the seminorms come first (quite naturally), and the topology is the associated one. However, there are some examples in which the topology comes first and one may even not care about what the seminorms actually are (see the general construction of inductive limit topologies at the end of this section).

On the other hand, one should be aware that different sets of seminorms may induce the same l.c.v.s. (i.e. the same topology). For instance, if $P_0 \subset P$ is a smaller family of seminorms which has the property that for any $p \in P$, there exists $p_0 \in P_0$ such that $p_0 \leq p$ (i.e. $p_0(v) \leq p(v)$ for all $v \in V$), then P and P_0 define the same topology. This trick will be repeatedly used in the examples.

Exercise 2.1.4. Prove the last statement.

Next, it will be useful to have a criteria for continuity of linear maps between l.c.v.s.'s in terms of the seminorms. The following is a very good exercise.

Proposition 2.1.5. *Let (V, P) and (W, Q) be two l.c.v.s.'s and let*

$$A : V \to W$$

be a linear map. Then T is continuous if and only if, for any $q \in Q$, there exist $p_1, \ldots, p_n \in P$ and a constant $C > 0$ such that

$$q(A(v)) \leq C \cdot max\{p_1(v), \ldots, p_n(v)\} \ \ \forall \ v \in V.$$

Note that we will deal only with l.c.v.s.'s which are separated (Hausdorff).

Exercise 2.1.6. Let (V, P) be a l.c.v.s., where $P = \{p_i\}_{i \in I}$ is a family of seminorms on V. Show that it is Hausdorff if and only if, for $v \in V$, one has the implication:

$$p_i(v) = 0 \ \ \forall \ i \in I \Longrightarrow \ v = 0.$$

Finally, recall that a **Frechet space** is a t.v.s. V with the following properties:

1. it is complete.
2. its topology is induced by a countable family of semi-norms $\{p_1, p_2, \ldots\}$.

In this case, it follows that V is metrizable, i.e. the topology of V can also be induced by a (complete) metric:

$$d(v, w) := \sum_{n \geq 1} \frac{1}{2^n} \frac{p_n(v - w)}{1 + p_n(v - w)}.$$

Example 2.1.7. Of course, any Hilbert or Banach space is a l.c.v.s. This applies in particular to all the familiar Banach spaces such as the L^p-spaces on an open $\Omega \subset \mathbb{R}^n$

$$L^p(\Omega) = \{f : \Omega \to \mathbb{C} : f \text{ is measurable }, \int_\Omega |f|^p < \infty\},$$

with the norm

$$||f||_{L^p} = (\int_\Omega |f|^p)^{1/p}.$$

Recall that, for $p = 2$, this is a Hilbert space with inner product

$$\langle f, g\rangle_{L^2} = \int_\Omega f\overline{g}.$$

Example 2.1.8. Another class of examples come from functions of a certain order, eventually with restrictions on their support. For instance, for an open $\Omega \subset \mathbb{R}^n$, $r \in \mathbb{N}$ and $K \subset \Omega$ compact, we consider the space

$$\mathcal{C}^r_K(\Omega) = \{\phi : \Omega \to \mathbb{C} : \phi \text{ is of class } C^r \text{ and } \mathrm{supp}(\phi) \subset K\}.$$

The norm which is naturally associated to this space is $|| \cdot ||_{K,r}$ defined by

$$||\phi||_{r,K} = \sup\{|\partial^\alpha \phi(x)| : x \in K, |\alpha| \leq r\}.$$

With this norm, $\mathcal{C}^r_K(\Omega)$ becomes a Banach space. Note that convergence in this space is uniform convergence on K of all derivatives up to order r.

However, if we consider $r = \infty$, then $C^\infty_K(\Omega)$ should be considered with the family of seminorms $\{|| \cdot ||_{K,r} : r \in \mathbb{N}\}$. The result is a Frechet space. Note that convergence in this space is uniform convergence on K of all derivatives.

Yet another natural space is the space of all smooth functions $C^\infty(\Omega)$. A nice topology on this space is the one induced by the family of seminorms

$$\{|| \cdot ||_{K,r} : K \subset \Omega \text{ compact}, r \in \mathbb{N}\}.$$

Using an exhaustion of Ω by compacts, i.e. a sequence $(K_n)_{n\geq 0}$ of compacts with

$$\Omega = \cup_n K_n, \quad K_n \subset \mathrm{Int}(K_{n+1}),$$

we see that the original family of seminorms can be replaced by a countable one:

$$\{|| \cdot ||_{K_n,r} : n, r \in \mathbb{N}\}$$

(using Remark 2.1.3, check that the resulting topology is the same!). Hence $C^\infty(\Omega)$ with this topology has the chance of being Frechet- which is actually the case.

Note that convergence in this space is uniform convergence on compacts of all derivatives.

Example 2.1.9. As a very general construction: for any t.v.s. (locally convex or not), there are (at least) two important l.c. topologies on the continuous dual:

$$V^* := \{u : V \to \mathbb{R} : u \text{ is linear and continuous}\}.$$

The first topology, denoted $\mathcal{T}_s$, is the one induced by the family of seminorms $\{p_v\}_{v\in V}$, where

$$p_v : V^* \to \mathbb{R},\ p_v(u) = |u(v)|.$$

This topology is called the weak* topology on V^*, or the topology of simple convergence. Note that $u_n \to u$ in this topology if and only if $u_n(v) \to u(v)$ for all $v \in V$.

The second topology, denoted $\mathcal{T}_b$, called the strong topology (or of uniform convergence on bounded sets) is defined as follows. First of all, recall that a subset $B \subset V$ is called bounded if, for any neighborhood of the origin, there exists $\lambda > 0$ such that $B \subset \lambda V$. If the topology of V is generated by a family of seminorms P, this means that for any $p \in P$ there exists $\lambda_p > 0$ such that

$$B \subset B_p(r_p) = \{v \in V : p(v) < r_p\}.$$

This implies (see also Proposition 2.1.5) that for any continuous linear functional $u \in V^*$,

$$p_B(u) := \sup\{|u(v)| : v \in B\} < \infty.$$

In this way we obtain a family $\{p_B\}_B$ of seminorms (indexed by all the bounded sets B), and $\mathcal{T}_b$ is defined as the induced topology.

A related topology on V^* is the topology $\mathcal{T}_c$ of uniform convergence on compacts, induced by the family of seminorms $\{p_C : C \subset V^* \text{ compact}\}$.

Some explanations (for your curiosity): In this course, when dealing with a particular l.c.v.s. V, what will be of interest to us is to understand the convergence in V, understand continuity of linear maps defined on V or the continuity of maps with values in V (i.e., in practical terms, one may forget the l.c. topology and just keep in mind convergence and continuity). From this point of view, in almost all the cases in which we consider the dual V^* of a l.c.v.s. V (e.g. the space of distributions), in this course we will be in the fortunate situation that it does not make a difference if we use $\mathcal{T}_s$ or $\mathcal{T}_b$ on V^* (note: this does not mean that the two topologies coincide- it just means that the specific topological aspects we are interested in are the same for the two).

What happens is that the spaces we will be dealing with in this course have some very special properties. Axiomatising these properties, one ends up with particular classes of l.c.v.s.'s which can be understood as part of the general theory of l.c.v.s.'s. Here we give a few more details of what is really going on (the references below send you to the book "Topological vector spaces, distributions and kernels" by F. Treves).

First of all, as a very general fact: for any t.v.s. V, $\mathcal{T}_s$ and $\mathcal{T}_c$ induce the same topology on any equicontinuous subset $H \subset V^*$ (Prop. 32.5, pp. 340). Recall that H is called equicontinuous if, for every $\epsilon > 0$, there exists a neighborhood B of the origin such that

$$|u(v)| \leq \epsilon, \quad \forall\ v \in B, \quad \forall\ u \in H.$$

An important class of t.v.s.'s is the one of barreled space, which we now recall. A barrel in a t.v.s. V is a non-empty closed subset $A \subset V$ with the following properties:

1. A is absolutely convex: $|\alpha|A + |\beta|A \subset A$ for all $\alpha, \beta \in \mathbb{C}$ with $|\alpha| + |\beta| = 1$.
2. A is absorbing: $\forall\ \ v \in V,\ \ \exists\ \ r > 0$ such that $v \in rA$.

A t.v.s. V is said to be barreled if any barrel in V is a neighborhood of zero. For instance, all Frechet spaces are barreled.

For a barreled space V, given $H \subset V^*$, the following are equivalent (Theorem 33.2, pp. 349):

1. H is weakly bounded (i.e. bounded in the l.c.v.s. $(V^*, \mathcal{T}_s)$).
2. H is strongly bounded (i.e. bounded in the l.c.v.s. $(V^*, \mathcal{T}_b)$).

3. H is relatively compact in the weak topology (i.e. the closure of H in $(V^*, \mathcal{T}_s)$ is compact there).
4. H is equicontinuous.

Hence, for such spaces, the notion of "bounded" is the same in $(V^*, \mathcal{T}_s)$ and $(V^*, \mathcal{T}_b)$, and we talk simply about "bounded subsets of V^*. However, the notion of convergence of sequences may still be different; of course, strong convergence implies weak convergence, but all we can say about a weakly convergent sequence is that it is bounded in the strong topology. More can be said for a more special class of t.v.s.'s.

A t.v.s. is called a Montel space if V is barreled and every closed bounded subset of E is compact. Note that this notion is much more restrictive than that of barreled space. For instance, while all Banach spaces are barreled, the only Banach spaces which are Montel are the finite dimensional ones (because the unit ball is compact only in the finite dimensinal Banach spaces). On the other hand, while all Frechet spaces are barreled, there are Frechet spaces which are Montel, but also others which are not Montel. The main examples of Montel spaces which are of interest for us are: the space of smooth functions, and the space of test functions (discussed below).

For a Montel space V, it follows that the topologies $\mathcal{T}_c$ and $\mathcal{T}_b$ are the same (Prop. 34.5, pp. 357). From the general property of equicontinuous subsets H mentioned above, we deduce that on such H's,

$$\mathcal{T}_s|_H = \mathcal{T}_b|_H.$$

(also, by the last result we mentioned, H being equicontinuous is equivalent to being bounded). Taking for H the set of elements of a weakly convergent sequence and its weak limit (clearly weakly bounded!), it follows that the sequence is also strongly convergent; hence convergence w.r.t. $\mathcal{T}_s$ and w.r.t. $\mathcal{T}_b$ is the same. Note that this does not imply that the two topologies are the same: we know from point-set topology that the notion of convergence w.r.t. a topology $\mathcal{T}$ does not determine the topology uniquely unless the topology satisfies the first axiom of countability (e.g. if it is metrizable).

As a summary, for Montel spaces V,

1. the notion of boundedness in $(V^*, \mathcal{T}_s)$ and in $(V^*, \mathcal{T}_b)$ is the same (and coincides with equicontinuity).
2. $\mathcal{T}_s$ and $\mathcal{T}_b$ induce the same topology on any bounded $H \subset V^*$.
3. a sequence in V^* is weakly convergent if and only it is strongly convergent.

Inductive limits

As we saw in all examples (and we will see in almost all the other examples), l.c.v.s.'s usually come with naturally associated seminorms and the topology is just the induced one. However, there is an important example in which the topology comes first (and one usually doesn't even bother to find seminorms inducing it): the space of test functions (see next section). This example fits into a general construction of l.c. topologies, known as "the inductive limit". The general framework is the following. Start with

$$X = \text{vector space}, \; X_\alpha \subset X \text{ vector subspaces such that } \; X = \cup_\alpha X_\alpha,$$

where α runs in an indexing set I. We also assume that, for each α, we have given:

$$\mathcal{T}_\alpha - \text{locally convex topology on } X_\alpha.$$

One wants to associate to this data a topology $\mathcal{T}$ on X, so that

1. $(X, \mathcal{T})$ is a l.c.v.s.

2. all inclusions $i_\alpha : X_\alpha \to X$ become continuous.

There are many such topologies (usually the "very small" ones, e.g. the one containing just $\emptyset$ and X itself) and, in general, if $\mathcal{T}$ works, then any $\mathcal{T}' \subset \mathcal{T}$ works as well. The question is: is there "the best one" (i.e. the smallest one)? The answer is yes, and that is what the inductive limit topology on X (associated to the initial data) is. In short, this is induced by the following basis of neighborhoods:

$$\mathcal{B}(0) := \{B \subset X : B - \text{convex such that } B \cap X_\alpha \in \mathcal{T}_\alpha(0) \text{ for all } \ \alpha \in I\},$$

(show that one gets a l.c. topology and it is the largest one!). One should keep in mind that what is important about $(X, \mathcal{T})$ is to recognize when a function on X is continuous, and when a sequence in X converges. The first part is a rather easy exercise with the following conclusion:

Proposition 2.1.10. *Let X be endowed with the inductive limit topology $\mathcal{T}$, let Y be another l.c.v.s. and let*

$$A : X \to Y$$

be a linear map. Then A is continuous if and only if each

$$A_\alpha := A|_{X_\alpha} : X_\alpha \to Y$$

is.

The recognition of convergent subsequences is a bit more subtle and, in order to have a more elegant statement, we place ourselves in the following situation: the indexing set I is the set $\mathbb{N}$ of positive integers,

$$X_1 \subset X_2 \subset X_3 \subset \ldots, \quad , X_n - \text{closed in } X_{n+1}, \quad \mathcal{T}_n = \mathcal{T}_{n+1}|_{X_n}$$

(i.e. each $(X_n, \mathcal{T}_n)$ is embedded in $(X_{n+1}, \mathcal{T}_{n+1})$ as a closed subspace). We assume that all the inclusions are strict. The following is a quite difficult exercise.

Theorem 2.1.11. *In the case above, a sequence $(x_n)_{n\geq 1}$ of elements in X converges to $x \in X$ (in the inductive limit topology) if and only if the following two conditions hold:*

1. *$\exists$ n_0 such that $x, x_m \in X_{n_0}$ for all m.*
2. *$x_m \to x$ in X_{n_0}.*

(note: one can also show that $(X, \mathcal{T})$ cannot be metrizable).

2.2. Distributions: the local theory

In this section we recall the main functional spaces on $\mathbb{R}^n$ or, more generally, on any open $\Omega \subset \mathbb{R}^n$. Recall that, for $K \subset \mathbb{R}^n$ and $r \in \mathbb{N}$, one has the seminorm $|| \cdot ||_{K,r}$ on $C^\infty(\Omega)$ given by:

$$||f||_{r,K} = \sup\{|\partial^\alpha f(x)| : x \in K, |\alpha| \leq r\}.$$

$\mathcal{E}(\Omega)$: smooth functions:

One defines

$$\mathcal{E}(\Omega) := C^{\infty}(\Omega),$$

endowed with the locally convex topology induced by the family of seminorms $\{||\cdot||_{K,r}\}_{K\subset\Omega \text{ compact}, r\in\mathbb{N}}$ (see also Example 2.1.8). Hence, in this space, convergence means: $f_n \to f$ if and only if for each multi- index α and each compact $K \subset \Omega$, $\partial^{\alpha} f_n \to \partial^{\alpha} f$ uniformly on K.

As a l.c.v.s, it is a Frechet space (and is also a Montel space).

Algebraically, $\mathcal{E}(\Omega)$ is also a ring (or even an algebra over $\mathbb{C}$), with respect to the usual multiplication of functions. Note that this algebraic operation is continuous.

$\mathcal{D}(\Omega)$: compactly supported smooth functions (test functions):

One defines

$$\mathcal{D}(\Omega) := C_c^{\infty}(\Omega),$$

the space of smooth functions with compact support, with the following topology. First of all, for each $K \subset \Omega$, we consider

$$\mathcal{E}_K(\Omega) := C_K^{\infty}(\Omega),$$

the space of smooth functions with support inside K, endowed with the topology induced from the topology of $\mathcal{E}(\Omega)$ (which is the same as the topology discussed in Example 2.1.8, i.e. induced by the family of seminorms $\{||\cdot||_{K,r}\}_{r\in\mathbb{N}}$. While, set theoretically (or as vector spaces),

$$\mathcal{D}(\Omega) = \cup_K \mathcal{E}_K(\Omega)$$

(union over all compacts $K \subset \Omega$), we consider the inductive limit topology on $\mathcal{D}(\Omega)$ (see the end of the previous section).

Convergent sequences are easy to recognize here: $f_n \to f$ in $\mathcal{D}(\Omega)$ if and only if there exist a compact K such that $f_n \in \mathcal{E}_K$ for all n, and $f_n \to f$ in $\mathcal{E}_K$ (indeed, using an exhaustion of Ω by compacts (see again Example 2.1.8), we see that we can place ourselves under the conditions which allow us to apply Theorem 2.1.11).

As a l.c.v.s., $\mathcal{D}(\Omega)$ is complete but it is not Frechet (see the end of Theorem 2.1.11). (However, it is a Montel space).

Algebraically, $\mathcal{D}(\Omega)$ is also an algebra over $\mathbb{C}$ (with respect to pointwise multiplication), which is actually an ideal in $\mathcal{E}(\Omega)$ (the product between a compactly supported smooth function and an arbitrary smooth function is again compactly supported).

$\mathcal{D}'(\Omega)$: distributions:

The space of distributions on $\mathbb{R}^n$ is defined as the (topological) dual of the space of test functions:

$$\mathcal{D}'(\Omega) := (\mathcal{D}(\Omega))^*$$

(see also Example 2.1.9). An element of this space is called a distribution on Ω. Unraveling the inductive limit topology on $\mathcal{D}(\Omega)$, one gets a more explicit description of these space. More precisely, using Proposition 2.1.10 to recognize the continuous linear maps by restricting to compacts, and using Proposition 2.1.5 to rewrite the resulting continuity conditions in terms of seminorms, one finds the following:

Corollary 2.2.1. *A distribution on Ω is a linear map*

$$u : C_c^{\infty}(\Omega) \to \mathbb{C}$$

with the following property: for any compact $K \subset \Omega$, there exists $C = C_K > 0$, $r = r_K \in \mathbb{N}$ such that

$$|u(\phi)| \leq C||\phi||_{K,r} \quad \forall\, \phi \in C_K^{\infty}(\Omega).$$

As a l.c.v.s., $\mathcal{D}'(U)$ will be endowed with the strong topology (the topology of uniform convergence on bounded subsets- see Example 2.1.9). Note however, when it comes to convergence of sequences (u_n) of distributions, the strong convergence is equivalent to simple (pointwise) convergence. [1]

In general, any smooth function f induces a distribution u_f

$$\phi \mapsto \int_{\mathbb{R}^n} f\phi,$$

and this correspondence defines a continuous inclusion of

$$i : \mathcal{E}(\Omega) \hookrightarrow \mathcal{D}'(\Omega).$$

For this reason, distributions are often called "generalized functions", and one often identifies f with the induced distribution u_f.

Algebraically, the multiplication on $\mathcal{E}(\Omega)$ extends to a $\mathcal{E}(\Omega)$-module structure on $\mathcal{D}'(\Omega)$

$$\mathcal{E}(\Omega) \times \mathcal{D}'(\Omega) \to \mathcal{D}'(\Omega),\ (f, u) \mapsto fu,$$

where

$$(fu)(\phi) = u(f\phi).$$

---

[1] Explanation (for your curiosity): When it comes to the following notions:

1. bounded subsets of $\mathcal{D}'(\Omega)$,
2. convergence of sequences in $\mathcal{D}'(\Omega)$,
3. continuity of a linear map $A : V \to \mathcal{D}'(\Omega)$ defined on a Frechet space V (e.g. $V = \mathcal{E}(\Omega')$),
4. continuity of a linear map $A : V \to \mathcal{D}'(\Omega)$ defined on a l.c.v.s. V which is the inductive limit of Frechet spaces (e.g. $V = \mathcal{D}(\Omega')$).

(notions which depend on what topology we use on $\mathcal{D}'(\Omega)$), it does not matter whether we use the strong topology $\mathcal{T}_b$ or the weak topology $\mathcal{T}_s$ on $\mathcal{D}'(\Omega)$: the a priori different resulting notions will actually coincide.

For boundedness and convergence this follows from the fact that $\mathcal{D}(\Omega)$ is a Montel space (Theorem 34.4, pp. 357 in the book by Treves). For continuity of linear maps defined on a Frechet space, one just uses that, because V is metrizable, continuity is equivalent to sequential continuity (i.e. the property of sending convergent sequences to convergent sequences) and the previous part. If V is an inductive limit of Frechet spaces one uses the characterization of continuity of linear maps defined on inductive limits (Proposition 2.1.10).

$\mathcal{E}'(\Omega)$: compactly supported distributions:

The space of compactly supported distributions on Ω is defined as the (topological) dual of the space of all smooth functions

$$\mathcal{E}'(\Omega) := (\mathcal{E}(\Omega))^*.$$

Using Proposition 2.1.5 to rewrite the continuity condition, we find:

Corollary 2.2.2. *A compactly supported distribution on Ω is a linear map*

$$u : C^\infty(\Omega) \to \mathbb{C}$$

with the following property: there exists a compact $K \subset \Omega$, $C > 0$ and $r \in \mathbb{N}$ such that

$$|u(\phi)| \leq C||\phi||_{K,r} \quad \forall\ \phi \in C^\infty(\Omega).$$

Again, as in the case of $\mathcal{D}'(\Omega)$, we endow $\mathcal{E}'(\Omega)$ with the strong topology.[2]

Note that the dual of the inclusion $\mathcal{D}(\Omega) \hookrightarrow \mathcal{E}$ induces a continuous inclusion

$$\mathcal{E}'(\Omega) \hookrightarrow \mathcal{D}'(\Omega).$$

Explicitly, any linear functional on $C^\infty(\Omega)$ can be restricted to a linear functional on $C_c^\infty(\Omega)$, and the estimates for the compactly supported distributions imply the ones for distributions.

Hence the four distributional spaces fit into a diagram

$$\begin{array}{ccc} \mathcal{D} & \longrightarrow & \mathcal{E} \\ \downarrow & & \downarrow \\ \mathcal{E}' & \longrightarrow & \mathcal{D}' \end{array} ,$$

in which all the arrows are (algebraic) inclusions which are continuous, and the spaces on the left are (topologically) the compactly supported version of the spaces on the right.

Change of coordinates

In general, a change of coordinates (i.e. a diffeomorphism) $\chi : \Omega_1 \to \Omega_2$ induces maps χ_* from the four distributional spaces of Ω_1 to the ones of Ω_2, in a way which is compatible with the diagrams themselvs. At the level of sections it is simply

$$\chi_* : \mathcal{D}(\Omega_1) \to \mathcal{D}(\Omega_2),\ \chi_*(\phi) = \phi \circ \chi^{-1}$$

(and similarly for $\mathcal{E}$). At the level of distributions, since we want χ_* to be compatible with the inclusion $f \mapsto u_f$ of smooth functions into distributions, we would like to have $\chi_*(u_f) = u_{\chi_*(f)} = u_{f\circ\chi^{-1}}$, i.e.

$$\chi_*(u_f)(\phi) = \int_{\Omega_2} f \circ \chi^{-1} \cdot \phi = \int_{\Omega_1} |Jac(\chi)| f \cdot \phi \circ \chi = u_f(|Jac(\chi) \cdot \phi \circ \chi).$$

[2]Explanation (for your curiosity): The same discussion as in the case of $\mathcal{D}'(\Omega)$ applies also to $\mathcal{E}'(\Omega)$. This is due to the fact that also $\mathcal{E}(\Omega)$ is a Montel space (with the same reference as for $\mathcal{D}(\Omega)$). Hence, when it comes to bounded subsets, convergent sequences, continuity of linear maps from a (inductive limit of) Frechet space(s) to $\mathcal{E}'(\Omega)$, it does not matter whether we use the strong topology $\mathcal{T}_b$ or the simple topology $\mathcal{T}_s$ on $\mathcal{E}'(\Omega)$.

This brings us to the definition of χ_* on all distributions:

$$\chi_* : \mathcal{D}'(\Omega_1) \to \mathcal{D}'(\Omega_2),\ \chi_*(u)(\phi) = u(|Jac(\chi)| \cdot \phi \circ \chi). \tag{2.2}$$

Supports of distributions

Next, we recall why $\mathcal{E}'(\Omega)$ is called the space of *compactly supported* distributions. The main remark is that the assignment

$$\Omega \mapsto \mathcal{D}'(\Omega)$$

defines a sheaf and, as for any sheaf, one can talk about sections with compact support. What happens is that the elements in $\mathcal{D}'(\Omega)$ which have compact support in this sense, are precisely the ones in the image of the inclusion $\mathcal{E}'(\Omega) \hookrightarrow \mathcal{D}'(\Omega)$.

Here are some details. First of all, for any two opens $\Omega \subset \Omega'$, one has an inclusion ("extension by zero")

$$\mathcal{D}(\Omega) \hookrightarrow \mathcal{D}(\Omega'),\ f \mapsto \tilde{f},$$

where $\tilde{f}$ is f on Ω and zero outside. Dualizing, we get a "restriction map",

$$\mathcal{D}'(\Omega') \to \mathcal{D}'(\Omega),\ u \mapsto u|_{\Omega}.$$

The sheaf property of the distributions is the following property which follows immediately from a partition of unity argument:

Lemma 2.2.3. *Assume that $\Omega = \cup_i \Omega_i$, with $\Omega_i \subset \mathbb{R}^n$ opens, and that u_i are distributions on Ω_i such that, for all i and j,*

$$u_i|_{\Omega_i \cap \Omega_j} = u_j|_{\Omega_i \cap \Omega_j}.$$

Then there exists a unique distribution u on Ω such that

$$u|_{\Omega_i} = u_i$$

for all i.

Proof Use partitions of unity. □

From this it follows that, for any $u \in \mathcal{D}'(\Omega)$, there is a largest open $\Omega_u \subset \Omega$ on which u vanishes (i.e. $u|_{\Omega_u} = 0$).

Definition 2.2.4. For $u \in \mathcal{D}'(\Omega)$, define its support

$$\text{supp}(u) = \Omega - \Omega_u = \{x \in \Omega : u|_{V_x} = 0 \text{ for any neighborhood } V_x \subset \Omega \text{ of } x\}.$$

We say that u is compactly supported if $\text{supp}(f)$ is compact.

Example 2.2.5. For any $x \in \Omega$, one has the distribution δ_x defined by

$$\delta_x(\phi) = \phi(x).$$

It is not difficult to check that its support is precisely $\{x\}$.

Exercise 2.2.6. Show that $u \in \mathcal{D}'(\Omega)$ has compact support if and only if it is in the image of the inclusion

$$\mathcal{E}'(\Omega) \hookrightarrow \mathcal{D}'(\Omega).$$

Derivatives of distributions and Sobolev spaces

Finally, we discuss one last property of distributions which is of capital importance: one can talk about the partial derivatives of any distribution! The key (motivating) remark is the following, which follows easily from integration by parts.

Lemma 2.2.7. *Let $f \in C^\infty(\Omega)$ and let u_f be the associated distribution.*

Let $\partial^\alpha f \in C^\infty(\Omega)$ be the higher derivative of f associated to a multi-index α, and let $u_{\partial^\alpha f}$ be the associated distribution.

Then $u_{\partial^\alpha f}$ can be expressed in terms of u_f by:

$$u_{\partial^\alpha f}(\phi) = (-1)^{|\alpha|} u_f(\partial^\alpha f).$$

This shows that the action of the operator ∂^α on smooth functions can be extended to distributions.

Definition 2.2.8. For a distribution u on Ω and a multi-index α, one defines the new distribution $\partial^\alpha u$ on Ω, by

$$(\partial^\alpha u)(\phi) = (-1)^{|\alpha|} u(\partial^\alpha \phi), \quad \forall\, \phi \in C_c^\infty(\Omega).$$

Example 2.2.9. The distribution u_f makes sense not only for smooth functions on Ω, but also for functions $f : \Omega \to \mathbb{C}$ with the property that $\phi f \in L^1(\Omega)$ for all $\phi \in C_c^\infty(\Omega)$ (so that the integral defining u_f is absolutely convergent). In particular it makes sense for any $f \in L^2(\Omega)$ and, as before, this defines an inclusion

$$L^2(\Omega) \hookrightarrow \mathcal{D}'(\Omega).$$

We now see one of the advantages of the distributions: any $f \in L^2$, although it may even not be continuous, has derivatives $\partial^\alpha f$ of any order! Of course, they may fail to be functions, but they are distributions. In particular, it is interesting to consider the following spaces.

Definition 2.2.10. For any $r \in \mathbb{N}$, $\Omega \subset \mathbb{R}^n$ open, we define the Sobolev space on Ω of order r as:

$$H_r(\Omega) := \{u \in \mathcal{D}'(\Omega) : \partial^\alpha(u) \in L^2(\Omega) \text{ whenever } |\alpha| \leq r\},$$

endowed with the inner product

$$\langle u, u' \rangle_{H_r} = \sum_{|\alpha| \leq r} \langle \partial^\alpha u, \partial^\alpha u' \rangle_{L^2}.$$

In this way, $H_r(\Omega)$ becomes a Hilbert space.

2.3. Distributions: the global theory

The l.c.v.s.'s $\mathcal{E}(\Omega)$, $\mathcal{D}(\Omega)$, $\mathcal{D}'(\Omega)$ and $\mathcal{E}'(\Omega)$ can be extended from opens $\Omega \subset \mathbb{R}^n$ to arbitrary manifold M (allowing us to talk about distributions on M, or generalized functions on M) and, more generally, to arbitrary vector bundles E over a manifold M (allowing us to talk about distributional sections of E, or generalized sections of E). To explain this extension, we fix M to be an n-dimensional manifold, and let E be a complex vector bundle over M of rank p.

$\underline{\mathcal{E}(M,E)}$ **(smooth sections):**

One defines

$$\mathcal{E}(M,E) := \Gamma(E),$$

the space of all smooth sections of E endowed with the following locally convex topology. To define it, we choose a cover $\mathcal{U} = \{U_i\}_{i\in I}$ of M by opens which are domains of "total trivializations" of E, i.e. both of charts (U_i, κ_i) for M as well as of trivializations $\tau_i : E|_{U_i} \to U_i \times \mathbb{C}^p$ for E. This data clearly induces an isomorphism of vector spaces

$$\phi_i : \Gamma(E|_{U_i}) \to C^\infty(\kappa_i(U_i))^p$$

(see also subsection 2.3 below). Altogether, and after restricting sections of E to the various $U_i's$, these define an injection

$$\phi : \Gamma(E) \to \Pi_i C^\infty(\kappa_i(U_i))^p = \Pi_i \mathcal{E}(\kappa_i(U_i))^p.$$

Endowing the right hand side with the product topology, the topology on $\Gamma(E)$ is the induced topology (via this inclusion). Equivalently, considering as indices $\gamma = (i, l, K, r)$ consisting of $i \in I$ (to index the open U_i), $1 \le l \le p$ (to index the l-th component of $\phi(s|_{U_i})$), $K \subset \kappa(U_i)$ compact and r- non-negative integer, one has seminorms $|| \cdot ||_\gamma$ on $\Gamma(E)$ as follows: for $s \in \Gamma(E)$, restrict it to U_i, move it to $\mathcal{E}(\kappa_i(U_i))^p$ via ϕ_i, take its l-th component, and apply the seminorm $|| \cdot ||_{K,r}$ of $\mathcal{E}(\kappa_i(U_i))$:

$$||s||_\gamma = ||\phi(s|_{U_i})^l||_{r,K}.$$

Putting together all these seminorms will define the desired l.c. topology on $\Gamma(E)$.

Exercise 2.3.1. Show that this topology does not depend on the choices involved.

Note that, since the cover $\mathcal{U}$ can be chosen to be countable (our manifolds are always assumed to satisfy the second countability axiom!), it follows that our topology can be defined by a countable family of seminorms. Using the similar local result, you can now do the following:

Exercise 2.3.2. Show that $\mathcal{E}(M,E)$ is a Frechet space.

Finally, note that a sequence $(s_m)_{m\ge 1}$ converges to s in this topology if and only if, for any open U which is the domain of a local chart κ for M and of a local frame $\{s_1, \ldots, s_p\}$ for E, and for any compact $K \subset U$, writing $s_m = (f_m^1, \ldots, f_m^p)$, $s = (f^1, \ldots, f^p)$ with respect to the frame, all the derivatives $\partial_\kappa^\alpha(f_m^i)$ converge uniformly on K to $\partial_\kappa^\alpha(f^i)$ (when $m \to \infty$).

When $E = \mathbb{C}_M$ is the trivial line bundle over M, we simplify the notation to $\mathcal{E}(M)$. As in the local theory, this is an algebra (with continuous multiplication). Also, the multiplication of sections by functions makes $\mathcal{E}(M,E)$ into a module over $\mathcal{E}(M)$.

$\underline{\mathcal{D}(M,E)}$ **(compactly supported smooth sections):**

One defines

$$\mathcal{D}(M,E) := \Gamma_c(E),$$

the space of all compactly supported smooth sections endowed with the following l.c. topology defined exactly as in the local case: one writes

$$\mathcal{D}(M,E) = \cup_K \mathcal{E}_K(M,E),$$

where the union is over all compacts $K \subset M$, and $\mathcal{E}_K(M,E) \subset \mathcal{E}(M,E)$ is the space of smooth sections supported in K, endowed with the topology induced from $\mathcal{E}(M,E)$; on $\mathcal{D}(M,E)$ we consider the inductive limit topology.

Exercise 2.3.3. Describe more explicitly the convergence in $\mathcal{D}(M,E)$.

Again, when $E = \mathbb{C}_M$ is the trivial line bundle over M, we simplify the notation to $\mathcal{D}(M)$.

$\underline{\mathcal{D}'(M,E)}$ **(generalized sections):**

This is the space of distributional sections of E, or the space of generalized sections of E. To define it, we do not just take the dual of $\mathcal{D}(M,E)$ as in the local case, but we first:

1. Consider the complexification of the density line bundle, still denoted by $D = D_M$ on M (see the previous chapter). All we need to know about D is that its compactly supported sections can be integrated over M without any further choice, i.e. there is an integral

   $$\int_M : \Gamma_c(D) \to \mathbb{C}.$$

   If you are more familiar with integration of (top-degree) forms, you may assume that M has an orientation, $D = \Lambda^n T^*M \otimes \mathbb{C}$- the space of $\mathbb{C}$-valued n-forms (an identification induced by the orientation), and $\int_M$ is the integral that you already know. Or, if you are more familiar with integration of functions on Riemannian manifolds, you may assume that M is endowed with a metric, D is the trivial line bundle (an identification induced by the metric) and that $\int_M$ is the integral that you already know.
2. Consider the "functional dual" of E:

   $$E^\vee := E^* \otimes D = \mathrm{Hom}(E,D),$$

   the bundle whose fiber at $x \in M$ is the complex vector space consisting of all ($\mathbb{C}$-)linear maps $E_x \to D_x$.

   The main point about $E^\vee$ is that it comes with a "pairing" (pointwise the evaluation map)

   $$< -,- >: \Gamma(E^\vee) \times \Gamma(E) \to \Gamma(D)$$

   (and its versions with supports) and then, using the integration of sections of D, we get *canonical* pairings

   $$[-,-] : \Gamma_c(E^\vee) \times \Gamma(E) \to \mathbb{C}, \ \ (s_1,s_2) \mapsto \int_M < s_1, s_2 > .$$

We now define

$$\mathcal{D}'(M,E) := (\mathcal{D}(M,E^{\vee}))^*$$

(endowed with the strong topology). Note that, it is precisely because of the way that $E^{\vee}$ was constructed, that we have canonical (i.e. independent of any choices, and completely functorial) inclusions

$$\mathcal{E}(M,E) \hookrightarrow \mathcal{D}'(M,E),$$

sending a section s to the functional $u_s := < \cdot, s >$. And, as before, we identify s with the induced distribution u_s.

When $E = \mathbb{C}_M$, we simplify the notation to $\mathcal{D}'(M)$.

As for the algebraic structure, as in the local case, $\mathcal{D}'(M,E)$ is a module over $\mathcal{E}(M)$, with continuous multiplication

$$\mathcal{E}(M) \times \mathcal{D}'(M,E) \to \mathcal{D}'(M,E)$$

defined by

$$(fu)(s) = u(fs).$$

Example 2.3.4. Special care has to be taken when $M = \Omega$ is an open in $\mathbb{R}^n$ and E is a trivial vector bundle. Strictly speaking, we have now two spaces represented by the same notation $\mathcal{D}'(\Omega)$:

1. the space from the local theory, which is the dual of $\mathcal{D}(\Omega) = C_c^{\infty}(\Omega)$ - call it $\mathcal{D}'(\Omega)_{old}$.
2. the space from the global theory, which is the dual of $\mathcal{D}(\Omega, D) = \Gamma_c(\Omega, D)$- call it $\mathcal{D}'(\Omega)_{new}$ (where D is the density line bundle of Ω, complexified).

The two are identified by the canonical identification of D with the trivial line bundle. At the level of distributions, the identification is

$$\mathcal{D}'(\Omega)_{old} \overset{\sim}{\longleftrightarrow} \mathcal{D}'(\Omega)_{new}, \quad \widehat{\xi} \longleftrightarrow \xi,$$

where

$$\widehat{\xi}(\phi) := \xi(\phi|dx_1 \dots dx_n|), \quad \text{for } \phi \in C_c^{\infty}(\Omega).$$

In what follows, when talking about $\mathcal{D}'(\Omega)$ in the local case, we will still be thinking of $\mathcal{D}'(\Omega)_{old}$ (which is more natural in the local context); one may choose any of the two models, but one should stil keep in mind the identification between the two. A good illustration of the need of being careful is the change of coordinates formula at the level of distributions (see Example 2.3.9).

$\underline{\mathcal{E}'(M,E)}$ (compactly supported generalized sections):

This is the space of compactly supported distributional sections of E, or the space of compactly supported generalized sections of E. It is defined as in the local case (but making again use of $E^{\vee}$), as

$$\mathcal{E}'(M,E) := (\mathcal{E}(M,E^{\vee}))^*.$$

Note that, by the same pairing as before, one obtains an inclusion

$$\mathcal{D}(M,E) \hookrightarrow \mathcal{E}'(M,E).$$

Hence, as in the local case, we obtain a diagram of inclusions

$$\begin{array}{ccc} \mathcal{D}(M,E) & \longrightarrow & \mathcal{E}(M,E) \\ \downarrow & & \downarrow \\ \mathcal{E}'(M,E) & \longrightarrow & \mathcal{D}'(M,E) \end{array} .$$

Example 2.3.5.

1. when $E = \mathbb{C}_M$ is the trivial line bundle over M, we have shortened the notations to $\mathcal{D}(M)$, $\mathcal{E}(M)$ etc. Hence, as vector spaces,
$$\mathcal{D}(M) = \Gamma_c(M),\ \mathcal{E}(M) = C^{\infty}(M),$$
while the elements of $\mathcal{D}'(M)$ will be called distributions on M.
2. staying with the trivial line bundle, but assuming now that $M = \Omega$ is an open subset of $\mathbb{R}^n$, we recover the spaces discussed in the previous section. Note that, in the case of distributions, we are using the identification of the density bundle with the trivial bundle induced by the section $|dx^1 \dots dx^n|$.
3. when $E = \mathbb{C}^p_M$ is the trivial bundle over M of rank p, then clearly
$$\mathcal{D}(M, \mathbb{C}^p_M) = \mathcal{D}(M)^p,\ \mathcal{E}(M, M \times \mathbb{C}^p_M) = \mathcal{E}(M)^p.$$
On the other hand, using the canonical identification between E^* and E, we also obtain
$$\mathcal{D}'(M, \mathbb{C}^p_M) = \mathcal{D}'(M)^p,\ \mathcal{E}'(M, \mathbb{C}^p_M) = \mathcal{E}'(M)^p.$$

Note that, as in the local theory, distributions $u \in \mathcal{D}'(M,E)$ can be restricted to arbitrary opens $U \subset M$, to give distributions $u|_U \in \mathcal{D}'(U, E|_U)$. More precisely, the restriction map

$$\mathcal{D}'(M,E) \to \mathcal{D}'(U, E|_U)$$

is defined as the dual of the map

$$\mathcal{D}'(U, E^{\vee}|_U) \to \mathcal{D}(M, E^{\vee})$$

which takes a compactly supported section defined on U and extends it by zero outside U.

Exercise 2.3.6. For a vector bundle E over M,

1. Show that $U \mapsto \mathcal{D}(U, E|_U)$ forms a sheaf over M.
2. Define the support of any $u \in \mathcal{D}(M,E)$.
3. Show that the injection
$$\mathcal{E}'(M,E) \hookrightarrow \mathcal{D}'(M,E)$$
identifies $\mathcal{E}'(M,E)$ with the space of compactly supported distributional sections (as a vector space only!).

Exercise 2.3.7. Show that $\mathcal{D}(M,E)$ is dense in $\mathcal{E}(M,E)$, $\mathcal{D}'(M,E)$ and $\mathcal{E}'(M,E)$ (you are allowed to use the fact that this is known for trivial line bundles over opens in $\mathbb{R}^n$).

Invariance under isomorphisms

Given two vector bundles, E over M and F over a manifold N, an isomorphism h between E and F is a pair (h, h_0), where $h_0 : M \to N$ is a diffeomorphism and $h : E \to F$ is a map which covers h_0 (i.e. sends the fiber E_x to $F_{h_0(x)}$ or, equivalently, the diagram below is commutative) and such that, for each $x \in M$, it restricts to a linear isomorphism between E_x and $F_{h_0(x)}$.

$$\begin{array}{ccc} E & \xrightarrow{h} & F \\ \downarrow & & \downarrow \\ M & \xrightarrow{h_0} & N \end{array} .$$

We now explain how such an isomorphism h induces isomorphisms between the four functional spaces of E and those of F (really an isomorphism between the diagrams they fit in). The four isomorphisms from the functional spaces of E to those of F will be denoted by the same letter h_*. At the level of smooth sections, this is simply

$$h_* : \mathcal{E}(M, E) \to \mathcal{E}(N, F),\ h_*(s)(y) = h(s(h_0^{-1}(y))),$$

which also restricts to the spaces $\mathcal{D}$. At the level of generalized sections,

$$h_* : \mathcal{D}'(M, E) \to \mathcal{D}'(N, F),$$

is the dual of the map

$$h^\vee : \mathcal{D}(F^\vee) \to \mathcal{D}(E^\vee)$$

defined by

$$h^\vee(u)(e_x) = h_0^*(u(h(e_x))), \quad (e_x \in E_x),$$

where we have used the pull-back of densities, $h_0^* : D_{N,h_0(x)} \to D_{M,x}$.

The same formula defines h_* on the spaces $\mathcal{E}'$.

Example 2.3.8. Given a rank p vector bundle E over M, one often has to choose opens $U \subset M$ which are domains of both a coordinate chart (U, κ) for M as well as the domains of a trivialization $\tau : E|_U \to U \times \mathbb{C}^p$ for E. We say that (U, κ, τ) is a total trivialization for E over U. Note that such a data defines an isomorphism h between the vector bundle $E|_U$ over U and the trivial bundle $\kappa(U) \times \mathbb{C}^p$:

$$h_0 = \kappa,\ h(e_x) = (\kappa(x), \tau(e_x)).$$

Hence any total trivialization (U, κ, τ) induces isomorphisms

$$h_{\kappa,\tau} : \mathcal{D}(U, E|_U) \to \mathcal{D}(\kappa(U))^p,\ h_{\kappa,\tau} : \mathcal{E}(U, E|_U) \to \mathcal{E}(\kappa(U))^p, \text{ etc.}$$

(see also Example 2.3.5).

Example 2.3.9. Special care has to be taken in the case when $M = \Omega$ is an open in $\mathbb{R}^n$, E is the trivial line bundle and we work with distributions. See first our discussion from Example 2.3.4. For a change of coordinates (diffeomorphism) $\chi : \Omega_1 \to \Omega_2$ between two opens in $\mathbb{R}^n$, if we choose to represent $\mathcal{D}'(\Omega_i)$ as the dual of $\mathcal{D}(\Omega_i) = C_c^\infty(\Omega_i)$ (and we will do so), going carefully through the idenitification of the density bundles with the trivial ones (see again Example 2.3.4), we find that the change of coordinates from this section becomes

$$\chi_* : \mathcal{D}'(\Omega_1) \to \mathcal{D}'(\Omega_2),\ \chi_*(u)(\phi) = u(|Jac(\chi)| \cdot \phi \circ \chi),$$

i.e. precisely the one from the local theory ((2.2) in Section 2.2 from this chapter).

2.4. General operators and kernels

Given two vector bundles, E over a manifold M and F over a manifold N, an operator from E to F is, roughly speaking, a linear map which associates to a "section of E" a "section of F". The quotes refer to the fact that there are several different choices for the meaning of sections: ranging from smooth sections to generalized sections, or versions with compact supports (or other types of sections). The most general type of operators are is following.

Definition 2.4.1. If E is a vector bundle over M and F is a vector bundle over N, a general operator from E to F is a linear continuous map

$$P : \mathcal{D}(M, E) \to \mathcal{D}'(N, F).$$

Remark 2.4.2. Note that general operators are often described with different domains and codomains. For instance, if $\mathcal{F}_1$ and $\mathcal{F}_2$ is any of the symbols $\mathcal{E}$, $\mathcal{D}$, $\mathcal{E}'$ or $\mathcal{D}'$ (or any of the other functional spaces that will be discussed in the next lecture), one can look at continuous linear operators

$$P : \mathcal{F}_1(M, E) \to \mathcal{F}_2(M, F). \tag{2.3}$$

But since in all cases $\mathcal{D} \subset \mathcal{F}_1$ and $\mathcal{F}_2 \subset \mathcal{D}'$ (with continuous inclusions), P does induce a general operator

$$P_{\text{gen}} : \mathcal{D}(M, E) \to \mathcal{D}'(N, F).$$

Conversely, since $\mathcal{D}(M, E)$ is dense in all the other functional spaces that we have discussed (Exercise 2.3.7), P_{gen} determines P uniquely. Hence, saying that we have an operator (2.3) is the same as saying that we have a general operator P_{gen} with the property that it extends to $\mathcal{F}_1(M, E)$, giving rise to a continuous operator taking values in $\mathcal{F}_2(M, F)$.

On the other extreme, one has the so-called smoothing operators, i.e. operators which transforms generalized sections into smooth sections.

Definition 2.4.3. If E is a vector bundle over M and F is a vector bundle over F, a smoothing operator from E to F is a linear continuous map

$$P : \mathcal{E}'(M, E) \to \mathcal{E}(N, F).$$

We denote by $\Psi^{-\infty}(E, F)$ the space of all such smoothing operators. When E and F are the trivial line bundles, we will simplify the notation to $\Psi^{-\infty}(M)$.

In other words, a smoothing operator is a general operator $P : \mathcal{D}(M, E) \to \mathcal{D}'(N, F)$ which

1. takes values in $\mathcal{E}(N, F)$.
2. extends to a continuous linear map from $\mathcal{E}'(M, E)$ to $\mathcal{E}(N, F)$.

A very useful way of interpreting operators is in terms of their so-called "kernels". The idea of a kernel is quite simple- and to avoid (just some) notational complications, let us first briefly describe what happens when $M = U \subset \mathbb{R}^m$ and $N = V \subset \mathbb{R}^n$ are two open, and the bundles involved are the trivial line bundles. Then the idea is the following: any $K \in C^{\infty}(V \times U)$ induces an operator

$$P_K : \mathcal{D}(U) \to \mathcal{E}(V), \quad K(\phi)(y) = \int_U K(y,x)\phi(x)dx.$$

Even more: composing with the inclusion $\mathcal{E}(V) \hookrightarrow \mathcal{D}'(V)$, i.e. viewing P_K as an application

$$P_K : \mathcal{D}(U) \to \mathcal{D}'(V),$$

this map does not depend on K as a smooth function, but just on K as a distribution (i.e. on $u_K \in \mathcal{D}'(V \times U)$). Indeed, for $\phi \in \mathcal{D}(U)$, $P_K(\phi)$, as a distribution on V, is

$$u_{P_K(\phi)} : \psi \mapsto \int_V P_K(\phi)\psi = \int_{V \times U} K(y,x)\psi(y)\phi(x)dydx = u_K(\psi \otimes \phi),$$

where $\psi \otimes \phi \in C^{\infty}(V \times U)$ is the map $(y,x) \mapsto \psi(y)\phi(x))$. In other words, any

$$K \in \mathcal{D}'(V \times U)$$

induces a linear operator

$$P_K : \mathcal{D}(U) \to \mathcal{D}'(V), \ P_K(\phi)(\psi) = K(\psi \otimes \phi)$$

which can be shown to be continuous. Moreover, this construction defines a bijection between $\mathcal{D}'(V \times U)$ and the set of all general operators (even more, when equipped with the appropriate topologies, this becomes an isomorphism of l.c.v.s.'s).

The passing from the local picture to vector bundles over manifolds works as usual, with some care to make the construction independent of any choices. Here are the details. Given the vector bundles E over M and F over N, we consider the vector bundle over $N \times M$:

$$F \boxtimes E^{\vee} := \mathrm{pr}_1^*(F) \otimes \mathrm{pr}_2^*(E^{\vee}),$$

where pr_j is the projection on the j-th component. Hence, the fiber over $(y,x) \in N \times M$ is

$$(F \boxtimes E^{\vee})_{(y,x)} = F_y \otimes E_x^* \otimes D_{M,x}.$$

Note that the functional dual of this bundle is canonically identified with:

$$(F \boxtimes E^{\vee})^{\vee} \cong F^{\vee} \boxtimes E.$$

Exercise 2.4.4. Work out this isomorphism. (Hints: the density bundle of $N \times M$ is canonically identified with $D_N \otimes D_M$; $D_M^* \otimes D_M \cong \underline{\mathrm{Hom}}(D_M, D_M)$ is canonically isomorphic to the trivial line bundle.)

As before, one may decide to use a fixed positive density on M and one on N, and then replace $F \boxtimes E^{\vee}$ by $F \boxtimes E^*$ and $F^{\vee} \boxtimes E$ by $F^* \boxtimes E$.

Fix now a distribution

$$K \in \mathcal{D}'(N \times M, F \boxtimes E^{\vee}).$$

We will associate to K a general operator

$$P_K : \mathcal{D}(M, E) \to \mathcal{D}'(N, F).$$

Due to the definition of the space of distributions, and to the identification mentioned above, K will be a continuous function

$$K : \mathcal{D}(N \times M, F^\vee \boxtimes E) \to \mathbb{C}.$$

For $\psi \in \mathcal{D}(M, F^\vee)$ and $\phi \in \mathcal{D}(M, E)$ we denote by

$$\psi \otimes \phi \in \mathcal{D}(N \times M, F^\vee \boxtimes E)$$

the induced section $(y, x) \mapsto \psi(y) \otimes \phi(x)$. To describe P_K, let $\phi \in \mathcal{D}(M, E)$ and we have to specify $P_K(\phi) \in \mathcal{D}'(N, F)$, i.e. the continuous functional

$$P_K(\phi) : \mathcal{D}(N, F^\vee) \to \mathbb{C}.$$

We define

$$P_K(\phi)(\psi) := K(\psi \otimes \phi).$$

The general operator P_K is called the general operator associated to the kernel K. Highly non-trivial is the fact that any general operator arises in this way (and then K will be called the kernel of P_K).

Theorem 2.4.5. *The correspondence $K \mapsto P_K$ defines a 1-1 correspondence between*

1. *distributions $K \in \mathcal{D}'(N \times M; F \boxtimes E^\vee)$.*
2. *general operators $P : \mathcal{D}(M, E) \to \mathcal{D}'(N, F)$.*

Moreover, in this correspondence, one has

$$K \in \mathcal{E}(N \times M; F \boxtimes E^\vee) \Longleftrightarrow P \text{ is smoothing.}$$

Note (for your curiosity): the 1-1 correspondence actually defines an isomorphism of l.c.v.s.'s between

1. $\mathcal{D}'(N \times M; F \boxtimes E^\vee)$ with the strong topology.
2. the space $\mathcal{L}(\mathcal{D}(M, E), \mathcal{D}'(N, F))$ of all linear continuous maps, endowed with the strong topology.

Exercise 2.4.6. Let

$$P = \frac{d}{dx} : \mathcal{E}(\mathbb{R}) \to \mathcal{E}(\mathbb{R}).$$

Compute its kernel, and show that this is not a smoothing operator.

LECTURE 3
Functional spaces on manifolds

The aim of this section is to introduce Sobolev spaces on manifolds (or on vector bundles over manifolds). These will be the Banach spaces of sections we were after (see the previous lectures). To define them, we will take advantage of the fact that we have already introduced the very general spaces of sections (the generalized sections, or distributions), and our Banach spaces of sections will be defined as subspaces of the distributional spaces.

It turns out that the Sobolev-type spaces associated to vector bundles can be built up from smaller pieces and all we need to know are the Sobolev spaces H_r of the Euclidean space $\mathbb{R}^n$ and their basic properties. Of course, it is not so important that we work with the Sobolev spaces themselves, but only that they satisfy certain axioms (e.g. invariance under changes of coordinates). Here we will follow an axiomatic approach and explain that, starting with a subspace of $\mathcal{D}'(\mathbb{R}^n)$ satisfying certain axioms, we can extend it to all vector bundles over manifolds. Back to Sobolev spaces, there is a subtle point: to have good behaved spaces, we will first have to replace the standard Sobolev spaces H_r, by their "local versions", denoted $H_{r,\mathrm{loc}}$. Hence, strictly speaking, it will be these local versions that will be extended to manifolds. The result will deserve the name "Sobolev space" (without the adjective "local") only on manifolds which are compact.

3.1. General functional spaces

When working on $\mathbb{R}^n$, we shorten our notations to

$$\mathcal{E} = \mathcal{E}(\mathbb{R}^n), \mathcal{D} = \mathcal{D}(\mathbb{R}^n), \mathcal{D}' = \dots$$

and, similarly for the Sobolev space of order r:

$$H_r = H_r(\mathbb{R}^n) = \{u \in \mathcal{D}' : \partial^\alpha u \in L^2 : \quad \forall\, |\alpha| \leq r\}.$$

Definition 3.1.1. A functional space on $\mathbb{R}^n$ is a l.c.v.s. space $\mathcal{F}$ satisfying:

1. $\mathcal{D} \subset \mathcal{F} \subset \mathcal{D}'$ and the inclusions are continuous linear maps.
2. for all $\phi \in \mathcal{D}$, multiplication by ϕ defines a continuous map $m_\phi : \mathcal{F} \to \mathcal{F}$.

Similarly, given a vector bundle E over a manifold M, one talks about functional spaces on M with coefficients in E (or just functional spaces on (M, E)).

As in the case of smooth functions, one can talk about versions of $\mathcal{F}$ with supports. Given a functional space $\mathcal{F}$ on $\mathbb{R}^n$, we define for any compact $K \subset \mathbb{R}^n$,

$$\mathcal{F}_K = \{u \in \mathcal{F},\ \mathrm{supp}(u) \subset K\},$$

endowed with the topology induced from $\mathcal{F}$, and we also define

$$\mathcal{F}_{\mathrm{comp}} := \cup_K \mathcal{F}_K$$

(union over all compacts in $\mathbb{R}^n$), endowed with the inductive limit topology. In terms of convergence, that means that a sequence (u_n) in $\mathcal{F}_{\mathrm{comp}}$ converges to $u \in \mathcal{F}_{\mathrm{comp}}$ if and only if there exists a compact K such that

$$\mathrm{supp}(u_n) \subset K \ \ \forall n, \ \ u_n \to u \ \text{ in } \mathcal{F}.$$

Note that $\mathcal{F}_{\mathrm{comp}}$ is itself a functional space.

Finally, for any functional space $\mathcal{F}$, one has another functional space (dual in some sense to $\mathcal{F}_{\mathrm{com}}$), defined by:

$$\mathcal{F}_{\mathrm{loc}} = \{u \in \mathcal{D}'(\mathbb{R}^n) : \phi u \in \mathcal{F} \ \ \forall\, \phi \in C_c^\infty(\mathbb{R}^n)\}.$$

This has a natural l.c. topology so that all the multiplication operators

$$m_\phi : \mathcal{F}_{\mathrm{loc}} \to \mathcal{F},\ u \mapsto \phi u \quad (u \in \mathcal{D})$$

are continuous- namely the smallest topology with this property. To define it, we use a family P of seminorms defining the l.c. topology on $\mathcal{F}$ and, for every $p \in P$ and $\phi \in C_c^\infty(\mathbb{R}^n)$ we consider the seminorm $q_{p,\phi}$ on $\mathcal{F}_{\mathrm{loc}}$ given by

$$q_{p,\phi}(u) = p(\phi u).$$

The l.c. topology that we use on $\mathcal{F}_{\mathrm{loc}}$ is the one induced by the family $\{q_{p,\phi} : p \in P, \phi \in \mathcal{D}$. Hence, $u_n \to u$ in this topology means $\phi u_n \to \phi u$ in $\mathcal{F}$, for all ϕ's.

Exercise 3.1.2. Show that, for any l.c.v.s. V, a linear map

$$A : V \to \mathcal{F}_{\mathrm{loc}}$$

is continuous if and only if , for any test function $\phi \in \mathcal{D}$, the composition with the multiplication m_ϕ by ϕ is a continuous map $m_\phi \circ A : V \to \mathcal{F}$.

Example 3.1.3. The four basic functional spaces $\mathcal{D}$, $\mathcal{E}$, $\mathcal{E}'$, $\mathcal{D}'$ are functional spaces and

$$\mathcal{D}_{\mathrm{comp}} = \mathcal{E}_{\mathrm{comp}} = \mathcal{D},\ (\mathcal{D}^{'})_{\mathrm{comp}} = (\mathcal{E}^{'})_{\mathrm{comp}} = \mathcal{E}',$$

$$\mathcal{D}_{\mathrm{loc}} = \mathcal{E}_{\mathrm{loc}} = \mathcal{E},\ (\mathcal{D}^{'})_{\mathrm{loc}} = (\mathcal{E}^{'})_{\mathrm{loc}} = \mathcal{D}'.$$

The same holds in the general setting of vector bundles over manifolds.

Regarding the Sobolev spaces, they are functional spaces as well, but the inclusions

$$H_{r,\mathrm{com}} \hookrightarrow H_r \hookrightarrow H_{r,\mathrm{loc}}$$

are strict (and the same holds on any open $\Omega \subset \mathbb{R}^n$).

The local nature of the spaces $\mathcal{F}_K$ is indicated by the following partition of unity argument which will be very useful later on.

Lemma 3.1.4. *Assume that $K \subset \mathbb{R}^n$ is compact, and let $\{\eta_j\}_{j\in J}$ a finite partition of unity over K, i.e. a family of compactly supported smooth functions on $\mathbb{R}^n$ such that $\sum_j \eta_j = 1$ on K. Let $K_j = K \cap supp(\eta_j)$. Then the linear map*

$$I : \mathcal{F}_K \to \Pi_{j\in J}\mathcal{F}_{K_j},\ u \mapsto (\eta_j u)_{j\in J}$$

is a continuous embedding (i.e. it is an isomorphism between the l.c.v.s. $\mathcal{F}_K$ and the image of I, endowed with the subspace topology) and the image of I is closed.

Proof The fact that I is continuous follows from the fact that each component is multiplication by a compactly supported smooth function. The main observation is that there is a continuous map R going backwards, namely the one which sends $(u_j)_{j\in J}$ to $\sum_j u_j$, such that $R \circ I = \mathrm{Id}$. The rest is a general fact about t.v.s.'s: if $I : X \to Y$, $R : Y \to X$ are continuous linear maps between two t.v.s.'s such that $R \circ I = \mathrm{Id}$, then I is an embedding and $D(X)$ is closed in Y. Let's check this. First, I is open from X to $D(X)$: if $B \subset X$ is open then, remarking that

$$I(B) = I(X) \cap R^{-1}(B)$$

and using the continuity of R, we see that $I(B)$ is open in $I(X)$. Secondly, to see that $I(X)$ is closed in Y, one remarks that

$$I(X) = \mathrm{Ker}(\mathrm{Id} - I \circ R).$$

□

Similar to Lemma 3.1.4, we have the following.

Lemma 3.1.5. *Let $\mathcal{F}$ be a functional space on $\mathbb{R}^n$ and let $\{\eta_i\}_{i\in I}$ be a partition of unity, with $\eta_i \in \mathcal{D}$. Let K_i be the support of η_i. Then*

$$I : \mathcal{F}_{loc} \to \Pi_{i\in I}\mathcal{F}_{K_i},\quad u \mapsto\ (\mu_i u)_{i\in I}$$

is a continuous embedding with closed image.

Proof This is similar to Lemma 3.1.4 and the argument is identical. Denoting by X and Y the domain and codomain of I, we have $I : X \to Y$. On the other hand, we can consider $R : Y \to X$ sending $(u_i)_{i\in I}$ to $\sum_i u_i$ (which clearly satisfies $R \circ I = \mathrm{Id}$). What we have to make sure is that, if $\{K_i\}_{i\in I}$ is a locally finite family of compact subsets of $\mathbb{R}^n$, then one has a well-defined continuous map

$$R : \Pi_i\mathcal{F}_{K_i} \to \mathcal{F}_{\mathrm{loc}},\quad (u_i)_{i\in I} \mapsto \sum_i u_i.$$

First of all, $u = \sum_i u_i$ makes sense as a distribution: as a linear functional on test functions,

$$u(\phi) := \sum_i u_i(\phi)$$

(this is a finite sum whenever $\phi \in \mathcal{D}$). Even more, when restricted to $\mathcal{D}_K$, one finds I_K finite such that the previous sum is a sum overall $i \in I_K$ for all $\phi \in \mathcal{D}_K$. This shows that $u \in \mathcal{D}'$. To check that it is in $\mathcal{F}_{\mathrm{loc}}$, we look at ϕu for $\phi \in \mathcal{D}$ (and want to check that it is in $\mathcal{F}$). But, again, we will get a finite sum of ϕu_i's, hence an element in $\mathcal{F}$. Finally, to see that the map is continuous, we have to check (see Exercise 3.1.2) that $m_\phi \circ A$ is continuous as a map to $\mathcal{F}$, for all ϕ. But, again, this is just a finite sum of the projections composed with m_ϕ. □

3.2. The Banach axioms

Regarding the Sobolev spaces H_r on $\mathbb{R}^n$, one of the properties that make them suitable for various problems (and also for the index theorem) is that they are Hilbert spaces. On the other hand, as we already mentioned, we will have to use variations of these spaces for which this property is lost when we deal with manifolds which are not compact. So, it is important to realize what remains of this property.

Definition 3.2.1. (**Banach axiom**) Let $\mathcal{F}$ be a functional space on $\mathbb{R}^n$. We say that:

1. $\mathcal{F}$ is Banach if the topology of $\mathcal{F}$ is a Banach topology.
2. $\mathcal{F}$ is locally Banach if, for each compact $K \subset \mathbb{R}^n$, the topology of $\mathcal{F}_K$ is a Banach topology.

Similarly, we talk about "Frechet", "locally Frechet", "Hilbert" and "locally Hilbert" functional spaces on $\mathbb{R}^n$, or, more generally, on a vector bundle E over a manifold M.

Of course, if $\mathcal{F}$ is Banach then it is also locally Banach (and similarly for Frechet and Hilbert). However, the converse is not true.

Example 3.2.2. $\mathcal{E}$ is Frechet (but not Banach- not even locally Banach). $\mathcal{D}$ is not Frechet, but it is locally Frechet. The Sobolev spaces H_r are Hilbert. Their local versions $H_{r,\mathrm{loc}}$ are just Frechet and locally Hilbert. The same applies for the same functional spaces on opens $\Omega \subset \mathbb{R}^n$.

Proposition 3.2.3. *A functional space $\mathcal{F}$ is locally Banach if and only if each $x \in \mathbb{R}^n$ admits a compact neighborhood K_x such that $\mathcal{F}_{K_x}$ has a Banach topology (similarly for Frechet and Hilbert).*

Proof From the hypothesis it follows that we can find an open cover $\{U_i : i \in I\}$ of $\mathbb{R}^n$ such that each $\overline{U}_i$ is compact and $\mathcal{F}_{\overline{U}_i}$ is Banach. It follows that, for each compact K inside one of these opens, $\mathcal{F}_K$ is Banach. We choose a partition of unity $\{\eta_i\}_{i\in I}$ subordinated to this cover. Hence each $\mathrm{supp}(\eta)_i$ is compact inside U_i, $\{\mathrm{supp}(\eta)_i\}_{i\in I}$ is locally finite and $\sum_i \eta_i = 1$.

Now, for an arbitrary compact K, $J := \{j \in I : \eta_j|_K \neq 0\}$ will be finite and then $\{\eta_j\}_{j\in J}$ will be a finite partition of unity over K hence we can apply Lemma 3.1.4. There K_j will be inside U_j, hence the spaces $\mathcal{F}_{K_j}$ have Banach topologies. The assertion follows from the fact that a closed subspace of a Banach space (with the induced topology) is Banach. □

Remark 3.2.4. If $\mathcal{F}$ is of locally Banach (or just locally Frechet), then $\mathcal{F}_{\mathrm{comp}}$ (with its l.c. topology) is a complete l.c.v.s. which is not Frechet (hint: Theorem 2.1.11).

3.3. Invariance axiom

In general, a change of coordinates (diffeomorphism) $\chi : \mathbb{R}^n \to \mathbb{R}^n$ induces a topological isomorphism

$$\chi_* : \mathcal{D}' \to \mathcal{D}'$$

(See (2.2) and Example 2.3.9 for the precise formula and the explanations). To be able to pass to manifolds, we need invariance of $\mathcal{F}$ under changes of coordinates. In order to have a notion of local nature, we also consider a local version of invariance.

Definition 3.3.1. Let $\mathcal{F}$ be a functional space on $\mathbb{R}^n$. We say that:

1. $\mathcal{F}$ is invariant if for any diffeomorphism χ of $\mathbb{R}^n$, χ_* restricts to a topological isomorphism
$$\chi_* : \mathcal{F} \xrightarrow{\sim} \mathcal{F}.$$
2. $\mathcal{F}$ is locally invariant if for any diffeomorphism χ of $\mathbb{R}^n$ and any compact $K \subset \mathbb{R}^n$, χ_* restricts to a topological isomorphism
$$\chi_* : \mathcal{F}_K \xrightarrow{\sim} \mathcal{F}_{\chi(K)}.$$

Similarly, we talk about invariance and local invariance of functional spaces on vector bundles over manifolds.

Clearly, invariant implies locally invariant (but not the other way around).

Example 3.3.2. Of course, the standard spaces $\mathcal{D}$, $\mathcal{E}$, $\mathcal{D}'$, $\mathcal{E}'$ are all invariant (in general for vector bundles over manifolds). However, H_r is not invariant but, fortunately, it is locally invariant (this is a non-trivial result which will be proved later on using pseudo-differential operators). As a consequence (see also below), the spaces $H_{r,\text{loc}}$ are invariant.

Proposition 3.3.3. *A functional space $\mathcal{F}$ is locally invariant if and only if for any diffeomorphism χ of $\mathbb{R}^n$, any $x \in \mathbb{R}^n$ admits a compact neighborhood K_x such that χ_* restricts to a topological isomorphism*

$$\chi_* : \mathcal{F}_{K_x} \xrightarrow{\sim} \mathcal{F}_{\chi(K_x)}.$$

Proof We will use Lemma 3.1.4, in a way similar to the proof of Proposition 3.2.3. Let K be an arbitrary compact and χ diffeomorphism. We will check the condition for K and χ. As in the proof of Proposition 3.2.3, we find a finite partition of unity $\{\eta_j\}_{j\in J}$ over K such that each $K_j = K \cap \text{supp}(\eta_j)$ has the property that

$$\chi_* : \mathcal{F}_{K_j} \xrightarrow{\sim} \mathcal{F}_{\chi(K_j)}.$$

We apply Lemma 3.1.4 to K and the partition $\{\eta_j\}$ (with map denoted by I) and also to $\chi(K)$ and the partition $\{\chi_*(\eta_i) = \eta_j \circ \chi^{-1}\}$ (with the map denoted I_χ). Once we show that $\chi_*(\mathcal{F}_K) = \mathcal{F}_{\chi(K)}$ set-theoretically, the lemma clearly implies that this is also a topological equality. So, let $u \in \mathcal{F}_K$. Then $\eta_j u \in \mathcal{F}_{K_j}$ hence

$$\chi_*(\eta_i) \cdot \chi_*(u) = \chi_*(\eta_j \cdot u) \in \mathcal{F}_{\chi(K_j)} \subset \mathcal{F}_K,$$

hence also

$$\chi_*(u) = \sum_j \chi_*(\eta_i) \cdot \chi_*(u) \in \mathcal{F}_K.$$

□

3.4. Density axioms

We briefly mention also the following density axioms.

Definition 3.4.1. Let $\mathcal{F}$ be a functional space on $\mathbb{R}^n$. We say that:

1. $\mathcal{F}$ is normal if $\mathcal{D}$ is dense in $\mathcal{F}$.
2. $\mathcal{F}$ is locally normal if, for any compact $K \subset \mathbb{R}^n$, $\mathcal{F}_K$ is contained in the closure of $\mathcal{D}$ in $\mathcal{F}$.

Similarly, we talk about normal and locally normal functional spaces on vector bundles over manifolds.

Again, normal implies locally normal and one can prove a characterization of local normality analogous to Proposition 3.2.3 and Proposition 3.3.3.

Example 3.4.2. All the four basic functional spaces $\mathcal{D}$, $\mathcal{E}$, $\mathcal{D}'$ and $\mathcal{E}'$ are normal (also with coefficients in vector bundles). Also the space H_r is normal. However, for arbitrary opens $\Omega \subset \mathbb{R}^n$, the functional spaces $H_r(\Omega)$ (on Ω) are in general not normal (but they are locally normal). The local spaces $H_{r,\mathrm{loc}}$ are always normal (see also the next section).

The normality axiom is important especially when we want to consider duals of functional spaces. Indeed, in this case a continuous linear functional $\xi : \mathcal{F} \to \mathbb{C}$ is zero if and only if its restriction to $\mathcal{D}$ is zero. It follows that the canonical inclusions dualize to continuous injections

$$\mathcal{D} \hookrightarrow \mathcal{F}^* \hookrightarrow \mathcal{D}'.$$

The duality between $\mathcal{F}_{\mathrm{loc}}$ and $\mathcal{F}_{\mathrm{comp}}$ can then be made more precise- one has:

$$(\mathcal{F}_{\mathrm{loc}})^* = (\mathcal{F}^*)_{\mathrm{comp}}, (\mathcal{F}_{\mathrm{comp}})^* = (\mathcal{F}^*)_{\mathrm{loc}}$$

(note: all these are viewed as vector subspaces of $\mathcal{D}'$, each one endowed with its own topology, and the equality is an equality of l.c.v.s.'s).

3.5. Locality axiom

In general, the invariance axiom is not enough for passing to manifolds. One also needs a locality axiom which allows us to pass to opens $\Omega \subset \mathbb{R}^n$ without loosing the properties of the functional space (e.g. invariance).

Definition 3.5.1. (**Locality axiom**) We say that a functional space $\mathcal{F}$ is local if, as l.c.v.s.'s,

$$\mathcal{F} = \mathcal{F}_{\mathrm{loc}}.$$

Similarly we talk about local functional spaces on vector bundles over manifolds.

Note that this condition implies that $\mathcal{F}$ is a module not only over $\mathcal{D}$ but also over $\mathcal{E}$.

Example 3.5.2. From the four basic examples, $\mathcal{E}$ and $\mathcal{D}'$ are local, while $\mathcal{D}$ and $\mathcal{E}'$ are not. Unfortunately, H_r is not local- and we will soon replace it with $H_{r,\mathrm{loc}}$ (in general, for any functional space $\mathcal{F}$, $\mathcal{F}_{\mathrm{loc}}$ is local).

With the last example in mind, we also note that, in general, when passing to the localized space, the property of being of Banach (or Hilbert, or Frechet) type does not change.

Exercise 3.5.3. Show that, for any functional space $\mathcal{F}$ and for any compact $K \subset \mathbb{R}^n$

$$(\mathcal{F}_{\text{loc}})_K = \mathcal{F}_K,$$

as l.c.v.s.'s. In particular, $\mathcal{F}$ is locally Banach (or locally Frechet, or locally Hilbert), or locally invariant, or locally normal if and only if $\mathcal{F}_{\text{loc}}$ is.

With the previous exercise in mind, when it comes to local spaces we have the following:

Theorem 3.5.4. *Let $\mathcal{F}$ be a local functional space on $\mathbb{R}^n$. Then one has the following equivalences:*

1. *$\mathcal{F}$ is locally Frechet if and only if it is Frechet.*
2. *$\mathcal{F}$ is locally invariant if and only if it is invariant.*
3. *$\mathcal{F}$ is locally normal if and only if it is normal.*

Note that in the previous theorem there is no statement about locally Banach. As we have seen, this implies Frechet. However, local spaces cannot be Banach.

Proof (of Theorem 3.5.4) In each part, we still have to prove the direct implications. For the first part, if $\mathcal{F}$ is locally Frechet, choosing a countable partition of unity and applying the previous lemma, we find that $\mathcal{F}_{\text{loc}}$ is Frechet since it is isomorphic to a closed subspace of a Frechet space (a countable product of Frechet spaces is Frechet!). For the second part, the argument is exactly as the one for the proof of Proposition 3.3.3, but using Lemma 3.1.5 instead of Lemma 3.1.4. For the last part, let us assume that $\mathcal{F}$ is locally normal. It suffices to show that $\mathcal{E}$ is dense in $\mathcal{F}$: then, for any open $U \subset \mathcal{F}$, $U \cap \mathcal{E} \neq \emptyset$; but $U \cap \mathcal{E}$ is an open in $\mathcal{E}$ (because $\mathcal{E} \hookrightarrow \mathcal{F}$ is continuous) hence, since $\mathcal{D}$ is dense in $\mathcal{E}$ (with its canonical topology), we find $U \cap \mathcal{D} \neq \emptyset$.

To show that $\mathcal{E}$ is dense in $\mathcal{F}$, we will need the following variation of Lemma 3.1.5. We choose a partition of unity η_i as there, but with $\eta_i = \mu_i^2$, $\mu_i \in \mathcal{D}$. Let

$$A = \mathcal{F}, \quad X = \Pi_{i \in I} \mathcal{F},$$

(X with the product topology). We define

$$i : A \to X,\ u \mapsto (\mu_i u)_i, \quad p : X \to A,\ (u_i)_i \mapsto \sum \mu_i u_i.$$

As in the lemma, these make A into a closed subspace of X. Hence we can place ourselves into the setting that we have a subspace $A \subset X$ of a l.c.v.s. X, which has a projection into A, $p : X \to A$ (not that we will omit writing i from now on). Consider the *subset*

$$Y = \Pi_{i \in I} \mathcal{D} \subset X.$$

Modulo the inclusion $A \hookrightarrow X$, $B = A \cap Y$ becomes $\mathcal{E}$ and $p(Y) = B$. Also, since A (or its image by i) is inside the closed subspace of X which is $\Pi_i \mathcal{F}_{K_i}$, we see that the hypothesis of local normality implies that $A \subset \overline{Y}$ (all closures are w.r.t. the topology of X). We have to prove that A is in the closure of B. Let

$a \in A$, V an open neighborhood of a in X. We have to show that $V \cap B \neq \emptyset$. From V we make $V' = p^{-1}(A \cap V)$- another open neighborhood of a in X. Since $A \subset \overline{Y}$, we have $V' \cap Y \neq \emptyset$. It now suffices to remark that $p(V') \subset V \cap B$. □

Finally, let us point out the following corollary which shows that, in the case of locally Frechet spaces, locality can be checked directly, using test functions and without any reference to $\mathcal{F}_{\text{loc}}$.

Corollary 3.5.5. *If $\mathcal{F}$ is a functional space which is locally Frechet, then $\mathcal{F}$ is local if and only if the following two (test-)conditions are satisfied:*

1. *$u \in \mathcal{D}'(\mathbb{R}^n)$ belongs to $\mathcal{F}$ if and only if $\phi u \in \mathcal{F}$ for all $\phi \in C_c^\infty(\mathbb{R}^n)$.*
2. *$u_n \to u$ in $\mathcal{F}$ if and only if $\phi u_n \to \phi$ in $\mathcal{F}$, for all $\phi \in C_c^\infty(\mathbb{R}^n)$.*

Proof The direct implication is clear. For the converse, assume that $\mathcal{F}$ is a functional space which satisfies these conditions. The first one implies that $\mathcal{F} = \mathcal{F}_{\text{loc}}$ as sets and we still have to show that the two topologies coincide. Let $\mathcal{T}$ be the original topology on $\mathcal{F}$ and let $\mathcal{T}_{\text{loc}}$ be the topology coming from $\mathcal{F}_{\text{loc}}$. Since $\mathcal{F} \hookrightarrow \mathcal{F}_{\text{loc}}$ is always continuous, in our situation, this tells us that $\mathcal{T}_{\text{loc}} \subset \mathcal{T}$. On the other hand, $\text{Id} : (\mathcal{F}, \mathcal{T}_{\text{loc}}) \to (\mathcal{F}, \mathcal{T})$ is clearly sequentially continuous hence, since $\mathcal{F}_{\text{loc}}$ is metrizable, the identity is also continuous, hence $\mathcal{T} \subset \mathcal{T}_{\text{loc}}$. This concludes the proof. □

3.6. Restrictions to opens

The main consequence of the localization axiom is the fact that one can restrict to opens $\Omega \subset \mathbb{R}^n$. The starting remark is that, for any such open Ω, there is a canonical inclusion

$$\mathcal{E}'(\Omega) \subset \mathcal{E}'(\mathbb{R}^n) \subset \mathcal{D}'(\mathbb{R}^n)$$

which should be thought of as "extension by zero outside Ω", which comes from the inclusion $\mathcal{E}'(\Omega) \subset \mathcal{E}'(\mathbb{R}^n)$ (obtained by dualizing the restriction map $C^\infty(\mathbb{R}^n) \to C^\infty(\Omega)$). In other words, any compactly supported distribution on Ω can be viewed as a (compactly supported) distribution on $\mathbb{R}^n$. On the other hand,

$$\phi u \in \mathcal{E}'(\Omega), \quad \forall\, \phi \in C_c^\infty(\Omega),\ u \in \mathcal{D}'(\Omega).$$

Hence the following makes sense:

Definition 3.6.1. Given a <u>local</u> functional space $\mathcal{F}$, for any open $\Omega \subset \mathbb{R}^n$, we define

$$\mathcal{F}(\Omega) := \{u \in \mathcal{D}'(\Omega) : \phi u \in \mathcal{F} \ \ \forall\, \phi \ \in C_c^\infty(\Omega)\},$$

endowed with the following topology. Let P be a family of seminorms defining the l.c. topology on $\mathcal{F}$ and, for every $p \in P$ and $\phi \in C_c^\infty(\Omega)$ we consider the seminorm $q_{p,\phi}$ on $\mathcal{F}$ given by

$$q_{p,\phi}(u) = p(\phi u).$$

We endow $\mathcal{F}(\Omega)$ with the topology associated to the family $\{q_{p,\phi} : p \in P, \phi \in C_c^\infty(\Omega)$.

Theorem 3.6.2. *For any local functional space $\mathcal{F}$ and any open $\Omega \subset \mathbb{R}^n$, $\mathcal{F}(\Omega)$ is a local functional space on Ω and, as such,*

1. *$\mathcal{F}(\Omega)$ is locally Banach (or Hilbert, or Frechet) if $\mathcal{F}$ is.*
2. *$\mathcal{F}(\Omega)$ is invariant if $\mathcal{F}$ is.*
3. *$\mathcal{F}(\Omega)$ is normal if $\mathcal{F}$ is.*

Proof For the first part, one remarks that $\mathcal{F}(\Omega)_K = \mathcal{F}_K$. For the second part, applying Theorem 3.5.4 to the local functional space $\mathcal{F}(\Omega)$ on Ω, it suffices to show local invariance. I.e., it suffices to show that for any $\chi : \Omega \to \Omega$ diffeomorphism and $x \in \Omega$, we find a compact neighborhood $K = K_x$ such that χ_* is an isomorphism between $\mathcal{F}(\Omega)_K (= \mathcal{F}_K)$ and $\mathcal{F}(\Omega)_{\chi(K)} (= \mathcal{F}_{\chi(K)})$. The difficulty comes from the fact that χ is not defined on the entire $\mathbb{R}^n$. Fix χ and x. Then we can find a neighborhood Ω_x of $x \in \Omega$ and a diffeomorphism $\tilde{\chi}$ on $\mathbb{R}^n$ such that

$$\tilde{\chi}|_{\Omega_x} = \chi|_{\Omega_x}$$

(this is not completely trivial, but it can be done using flows of vector fields, on any manifold). Fix any compact neighborhood $K \subset \Omega_x$. Using the invariance of $\mathcal{F}$, it suffices to show that $\chi_*(u) = \tilde{\chi}_*(u)$ for all $u \in \mathcal{F}_K$. But

$$\chi_*(u),\ \tilde{\chi}_*(u) \in \mathcal{F} \subset \mathcal{D}'$$

are two distributions whose restriction to $\chi(\Omega_x)$ is the same and whose restrictions to $\mathbb{R}^n - \chi(K)$ are both zero. Hence they must coincide. For the last part, since we deal with local spaces, it suffices to show that $\mathcal{F}(\Omega)$ is locally normal, i.e that for any compact $K \subset \Omega$, $\mathcal{F}_K(\Omega) = \mathcal{F}_K \subset \mathcal{F}(\Omega)$ is contained in the closure of $\mathcal{D}(\Omega)$. I.e., for any $u \in \mathcal{F}_K$ and any open $U \subset \mathcal{F}(\Omega)$ containing u, $U \cap \mathcal{D}(\Omega) =\neq \emptyset$. But since $\mathcal{F}$ is locally normal and the restriction map $r : \mathcal{F} \to \mathcal{F}(\Omega)$ is continuous, we have $r^{-1}(U) \cap \mathcal{D} \neq \emptyset$ and the claim follows. □

Exercise 3.6.3. By a sheaf of distributions $\hat{\mathcal{F}}$ on $\mathbb{R}^n$ we mean an assignment

$$\Omega \mapsto \hat{\mathcal{F}}(\Omega)$$

which associates to an open $\Omega \subset \mathbb{R}^n$ a functional space $\hat{\mathcal{F}}(\Omega)$ on Ω (local or not) such that:

1. if $\Omega_2 \subset \Omega_1$ and $u \in \hat{\mathcal{F}}(\Omega_1)$, then $u|_{\Omega_2}$ is in $\hat{\mathcal{F}}(\Omega_2)$. Moreover, the map
$$\hat{\mathcal{F}}(\Omega_1) \to \hat{\mathcal{F}}(\Omega_2),\ u \mapsto u|_{\Omega_1}.$$
is continuous.
2. If $\Omega = \cup_{i\in I}\Omega_i$ with $\Omega_i \subset \mathbb{R}^n$ opens (I some index set), then the map
$$\hat{\mathcal{F}}(\Omega) \to \Pi_{i\in I}\hat{\mathcal{F}}(\Omega_i),\ u \mapsto (u|_{\Omega_i})_{i\in I}$$
is a topological embedding which identifies the l.c.v.s. on the left with the closed subspace of the product space consisting of elements $(u_i)_{i\in I}$ with the property that $u_i|_{\Omega_i\cap\Omega_j} = u_j|_{\Omega_i\cap\Omega_j}$ for all i and j.

Show that

1. If $\mathcal{F}$ is a local functional space on $\mathbb{R}^n$ then $\Omega \mapsto \mathcal{F}(\Omega)$ is a sheaf of distributions.
2. Conversely, if $\hat{\mathcal{F}}$ is a sheaf of distributions on $\mathbb{R}^n$ then
$$\mathcal{F} := \hat{\mathcal{F}}(\mathbb{R}^n)$$
is a local functional space on $\mathbb{R}^n$ and $\hat{\mathcal{F}}(\Omega) = \mathcal{F}(\Omega)$ for all Ω's.

Below, for diffeomorphisms $\chi : \Omega_1 \to \Omega_2$ between two opens, we consider the induced $\chi_* : \mathcal{D}'(\Omega_1) \to \mathcal{D}'(\Omega_2)$ (see (2.2) and Example 2.3.9).

Corollary 3.6.4. *Let $\mathcal{F}$ be a local functional space. If $\mathcal{F}$ is invariant then, for any diffeomorphism $\chi : \Omega_1 \to \Omega_2$ between two opens in $\mathbb{R}^n$, χ_* induces a topological isomorphism*

$$\chi_* : \mathcal{F}(\Omega_1) \to \mathcal{F}(\Omega_2).$$

Proof The proof of 2. of Theorem 3.6.2, when showing invariance under diffeomorphisms $\chi : \Omega \to \Omega$ clearly applies to general diffeomorphism between any two opens. □

3.7. Passing to manifolds

Throughout this section we fix

$$\mathcal{F} = \underline{\text{local, invariant functional space on } \mathbb{R}^n}.$$

and we explain how to induce functional spaces $\mathcal{F}(M, E)$ (of "generalized sections of E of type $\mathcal{F}$") for any vector bundle E over an n-dimensional manifold M.

To define them, we will use local total trivializations of E, i.e. triples (U, κ, τ) consisting of a local chart (U, κ) for M and a trivialization $\tau : E|_U \to U \times \mathbb{C}^p$ of E over U. Recall (see Example 2.3.8) that any such total trivialization induces an isomorphism

$$h_{\kappa,\tau} : \mathcal{D}'(U, E|_U) \to \mathcal{D}'(\Omega_\kappa)^p \quad (\text{where } \Omega_\kappa = \kappa(U) \subset \mathbb{R}^n)^1.$$

Definition 3.7.1. We define $\mathcal{F}(M, E)$ as the space of all $u \in \mathcal{D}'(M, E)$ with the property that for any domain U of a total trivialization of E, $h_{\kappa,\tau}(u|_U) \in \mathcal{F}(\Omega_\kappa)^p$.

We still have to define the topology on $\mathcal{F}(M, E)$, but we make a few remarks first. Since the previous definition applies to all n-dimensional manifolds:

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[1] Let us make this more explicit. The total trivialization induces

1. a local frame $s_1, \ldots, s_p$ for E over U. Then, for any $s \in \Gamma(E)$ we find (local) coefficients $f_s^i \in C^\infty(\Omega_\kappa)$, i.e. satisfying
$$s(x) = \sum_i f_s^i(\kappa(x)) s_i(x) \quad \text{for } x \in U.$$
2. the local dual frame $s^1, \ldots, s^p$ of E^* and a local frame (i.e. non-zero section on U) of the density bundle of M, $|dx_\kappa^1 \wedge \ldots \wedge x_\kappa^n|$. Then, for any $\xi \in \Gamma(E^\vee)$ we find (local) coefficients $\xi_i \in C^\infty(\Omega_\kappa)$, i.e. satisfying
$$\xi(x) = \sum_i \xi_i(\kappa(x)) s^i(x) |dx_\kappa^1 \wedge \ldots \wedge x_\kappa^n|_x \quad (x \in U).$$
3. any $u \in \mathcal{D}'(M, E)$ has coefficients $u^i \in \mathcal{D}'(\Omega_\kappa)$, i.e. satisfying
$$u(\xi) = \sum_i u_i(\xi_i) \quad (\xi \in \Gamma_c(U, E^\vee)).$$

The map $h_{\kappa,\tau}$ sends u to $(u_1, \ldots, u_n)$.

1. when applied to an open $\Omega \subset \mathbb{R}^n$ and to the trivial line bundle $\mathbb{C}_\Omega$ over Ω, one recovers $\mathcal{F}(\Omega)$- and here we are using the invariance of $\mathcal{F}$.

2. it also applies to all opens $U \subset M$, hence we can talk about the spaces $\mathcal{F}(U, E|_U)$. From the same invariance of $\mathcal{F}$, when U is the domain of a total trivialization chart (U, κ, τ), to check that $u \in \mathcal{D}'(U, E|_U)$ is in $\mathcal{F}(U, E|_U)$, it suffices to check that $h_{\kappa,\tau}(u) \in \mathcal{F}(\Omega_\kappa)^p$- i.e. we do not need to check the condition in the definition for <u>all</u> total trivialization charts.

3. If $\{U_i\}_{i\in I}$ is one open cover of M and $u \in \mathcal{D}'(M, E)$, then
$$u \in \mathcal{F}(M, E) \Longleftrightarrow u|_{U_i} \in \mathcal{F}(U_i, E|_{U_i}) \ \ \forall\, i \in I.$$
This follows from the similar property of $\mathcal{F}$ on opens in $\mathbb{R}^n$.

Exercise 3.7.2. Given a vector bundle E over M and $U \subset M$, $\mathcal{F}$ induces two subspaces of $\mathcal{D}'(U, E|_U)$:
1. $\mathcal{F}(U, E|_U)$ just defined.
2. thinking of $\mathcal{F}(M, E)$ as a functional space on M (yes, we know, we still have to define the topology, but that is irrelevant for this exercise), we have an induced space:
$$\{u \in \mathcal{D}'(U, E|_U) : \phi u \in \mathcal{F}(M, E), \ \ \forall\, \phi \in \mathcal{D}(U)\}.$$

Show that the two coincide.

Next, we take an open cover $\{U_i\}_{i\in I}$ by domains of total trivialization charts (U_i, κ_i, τ_i). It follows that we have an inclusion
$$h : \mathcal{F}(M, E) \to \Pi_{i\in I}\mathcal{F}(\Omega_{\kappa_i})^p.$$
We endow $\mathcal{F}(M, E)$ with the induced topology.

Exercise 3.7.3. Show that the topology on $\mathcal{F}(M, E)$ does not depend on the choice of the cover and of the total trivialization charts.

Theorem 3.7.4. *For any vector bundle E over an n dimensional manifold M, $\mathcal{F}(M, E)$ is a local functional space on (M, E).*

Moreover, if $\mathcal{F}$ is locally Banach, or locally Hilbert, or Frechet (= locally Frechet since $\mathcal{F}$ is local), or normal (= locally normal), then so is $\mathcal{F}(M, E)$.

Finally, if F is a vector bundle over another n-dimensional manifold N and (h, h_0) is an isomorphism between the vector bundles E and F, then $h_* : \mathcal{D}'(M, E) \to \mathcal{D}'(N, F)$ restricts to an isomorphism of l.c.v.s.'s
$$h_* : \mathcal{F}(M, E) \to \mathcal{F}(N, F).$$

Proof We just have to put together the various pieces that we already know (of course, here we make use of the fact that all proofs that we have given so far work for vector bundles over manifolds). To see that $\mathcal{F}(M, E)$ is a functional space we have to check that we have continuous inclusions
$$\mathcal{D}(M, E) \hookrightarrow \mathcal{F}(M, E) \hookrightarrow \mathcal{D}'(M, E).$$
We just have to remark that the map h used to define the topology of $\mathcal{F}(M, E)$ also describes the topology for $\mathcal{D}$ and $\mathcal{D}'$. To show that $\mathcal{F}(M, E)$ is local, one

uses the sheaf property of $\mathcal{F}(M,E)_{\text{loc}}$ (see Exercise 3.6.3) where the U_i's there are chosen as in the construction of h above. This reduces the problem to a local one, i.e. to locality of $\mathcal{F}$.

All the other properties follow from their local nature (i.e. Proposition 3.2.3 and the similar result for normality, applied to manifolds) and the fact that, for $K \subset M$ compact inside a domain U of a total trivialization chart (U, κ, τ), $\mathcal{F}_K(M,E)$ is isomorphic to $\mathcal{F}_K(\Omega_\kappa)^p$. □

Corollary 3.7.5. *If $\mathcal{F}$ is locally Banach (or Hilbert) and normal then, for any vector bundle E over a compact n dimensional manifold M, $\mathcal{F}(M,E)$ is a Banach (or Hilbert) space which contains $\mathcal{D}(M,E)$ as a dense subspace.*

Definition 3.7.6. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be local, invariant functional spaces on $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively, and assume that $\mathcal{F}_1$ is normal. Let M be an m-dimensional manifold and N an n-dimensional one, and let E and F be vector bundles over M and N, respectively. We say that a general operator

$$P : \mathcal{D}(M,E) \to \mathcal{D}'(N,F)$$

is of type $(\mathcal{F}_1, \mathcal{F}_2)$ if it takes values in $\mathcal{F}_2$ and extends to a continuous linear operator

$$P_{\mathcal{F}_1,\mathcal{F}_2} : \mathcal{F}_1(M,E) \to \mathcal{F}_2(N,E).$$

Note that, due to the normality axiom, the extension $P_{\mathcal{F}_1,\mathcal{F}_2}$ will be unique, hence the notation is un-ambiguous.

3.8. Back to Sobolev spaces

We apply the previous constructions to the Sobolev spaces H_r on $\mathbb{R}^n$. Let us first recall some of the standard properties of these spaces:

1. they are Hilbert spaces.
2. $\mathcal{D}$ is dense in H_r.
3. if $s > n/2 + k$ then $H_s \subset C^k(\mathbb{R}^n)$ (with continuous injection) (Sobolev's lemma).
4. for all $r > s$ and all $K \subset \mathbb{R}^n$ compact, the inclusion $H_{s,K} \hookrightarrow H_{s,K}$ is compact (Reillich's lemma).

Also, as we shall prove later, H_r are locally invariant (using pseudo-differential operators and Proposition 3.3.3). Assuming all these, we now consider the associated local spaces

$$H_{r,\text{loc}} = \{u \in \mathcal{D}' : \phi u \in H_r, \ \ \forall\ \phi \in \mathcal{D}\}$$

and the theory we have developed imply that:

1. $H_{r,\text{loc}}$ is a functional space which is locally Hilbert, invariant and normal.
2. $\cap_r H_{r,\text{loc}} = \mathcal{E}$. Even better, for $r > n/2 + k$, any $s \in H_{r,\text{loc}}$ is of class C^k.

Hence these spaces extend to manifolds.

Definition 3.8.1. For a vector bundle E over an n-dimensional manifold M,

1. the resulting functional spaces $H_{r,\text{loc}}(M,E)$ are called the local r-Sobolev spaces of E.

2. for $K \subset M$ compact, the resulting K-supported spaces are denoted $H_{r,K}(M,E)$.
3. the resulting compactly supported spaces are denoted $H_{r,\text{comp}}(M,E)$ (hence they are $\cup_K H_{r,K}(M,E)$ with the inductive limit topology).

If M is compact, we define the r-Sobolev space of E as

$$H_r(M,E) := H_{r,\text{loc}}(M,E)(= H_{r,\text{comp}}(M,E)).$$

Corollary 3.8.2. *For any vector bundle E over a manifold M,*

1. *$H_{r,loc}(M,E)$ are Frechet spaces.*
2. *$\mathcal{D}(M,E)$ is dense in $H_{r,loc}(M,E)$.*
3. *if a distribution $u \in \mathcal{D}'(M,E)$ belongs to all the spaces $H_{r,loc}(M,E)$, then it is smooth.*
4. *for $K \subset M$ compact, $H_{r,K}(M,E)$ has a Hilbert topology and, for $r > s$, the inclusion*

$$H_{r,K}(M,E) \hookrightarrow H_{s,K}(M,E)$$

is compact.

Proof The only thing that may still need some explanation is the compactness of the inclusion. But this follows from the Reillich's lemma and the partition of unity argument, i.e. Lemma 3.1.4 . □

Corollary 3.8.3. *For any vector bundle E over a compact manifold M, $H_r(M,E)$ has a Hilbert topology, contains $\mathcal{D}(M,E)$ as a dense subspace,*

$$\cap_r H_{r,loc}(M,E) = \Gamma(E)$$

and, for $r > s$, the inclusion

$$H_r(M,E) \hookrightarrow H_s(M,E)$$

is compact.

Finally, we point out the following immediate properties of operators. The first one says that differential operators of order k are also operators of type (H_r, H_{r-k}).

Proposition 3.8.4. *For $r \geq k \geq 0$, given a differential operator $P \in \mathcal{D}_k(E,F)$ between two vector bundles over M, the operator*

$$P : \mathcal{D}(M,E) \to \mathcal{D}(M,F)$$

admits a unique extension to a continuous linear operator

$$P_r : H_{r,loc}(M,E) \to H_{r-k,loc}(M,F).$$

The second one says that smoothing operators on compact manifolds, viewed as operators of type (H_r, H_s), are compact.

Proposition 3.8.5. *Let E and F be two vector bundles over a compact manifold M and consider a smoothing operator $P \in \Psi^{-\infty}(E,F)$. Then for any r and s, P viewed as an operator*

$$P : H_r(M,E) \to H_s(M,F)$$

is compact.

Remark 3.8.6. Back to our strategy of proving that the index of an elliptic differential operator $P \in \mathcal{D}_k(E, F)$ (over a compact manifold) is well-defined, our plan was to use the theory of Fredholm operators between Banach spaces. We have finally produced our Banach spaces of sections on which our operator will act:

$$P_r : H_r(M, E) \to H_{r-k}(M, F).$$

To prove that P_r is Fredholm, using Theorem 1.4.5 on the characterization of Fredholm operators and the fact that all smoothing operators are compact, we would need some kind of "inverse of P_r modulo smoothing operators", i.e. some kind of operator "of order $-k$" going backwards, such that $PQ - \mathrm{Id}$ and $QP - \mathrm{Id}$ are smoothing operators. Such operators "of degree $-k$" cannot, of course, be differential. What we can do however is to understand what make differential operators behave well w.r.t. (e.g.) Sobolev spaces- and the outcome is: it is not important that their total symbols are polynomials (of some degree k) in ξ, but only their symbols have a certain k-polynomial-like behaviour (in terms of estimates). And this is a property which makes sense even for k-negative (and that is where we have to look for our Q). This brings us to pseudo-differential operators ...

Exercise 3.8.7. Show that if such an operator $Q : H_{r-k}(M, F) \to H_r(M, E)$ is found (i.e. with the property that $PQ - \mathrm{Id}$ and $QP - \mathrm{Id}$ are smoothing), then the kernel of the operator $P : \Gamma(E) \to \Gamma(F)$ is finite dimensional. What about the cokernel?