An Improvement of the Lovász Local Lemma via Cluster Expansion

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Abstract

An old result by Shearer relates the Lovász Local Lemma with the independent set polynomial on graphs, and consequently, as observed by Scott and Sokal, with the partition function of the hard core lattice gas on graphs. We use this connection and a recent result on the analyticity of the logarithm of the partition function of the abstract polymer gas to get an improved version of the Lovász Local Lemma. As an application we obtain tighter bounds on conditions for the existence of latin transversal matrices.

1 Introduction

The main aim of this paper is to outline how techniques developed within statistical mechanics can be applied to improve combinatorial results proved by means of the Lovász Local Lemma (LLL). These improvements rely on three major contributions:

(i) Shearer's relation [14] between the LLL and independent-set polynomials;

(ii) Scott and Sokal's subsequent connection [16] with lattice-gas partition functions, and

(iii) relatively recent results in [7] on the convergence radius of logs of partition functions.

While point (iii) contains the main mathematical tool, the steps needed to exploit its combinatorial consequences by the way of the LLL, are far from obvious. Our note is intended, then, to fulfill a double role: First, to offer the stochastic-combinatorial community an improvement of the LLL whose power is illustrated in well known examples. Second, to present a simple and explicit road-map showing how improvements in cluster expansion estimations lead to improvements in the LLL. Thus, the results of our paper yield, on the one hand, an effective tool

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to solve some combinatorial problems, and, on the other hand, an illustration of how statistical mechanical methods can be profitably used in seemingly unrelated —not even probilistic— problems.

This short note is organized as follows. In the next section we state our results. In section 3 we recall the connections between LLL and statistical mechanics and prove our results. Applications are presented in section 4. Some technical aspects of the basic result of this paper are presented in the appendix.

2 Results

One of the more powerful tools used in the probabilistic method in combinatorics is the so-called *Lovász local Lemma*, first proved by Erdös and Lovász in [5]. To state this lemma we need several preliminary definitions.

Hereafter $|U|$ denotes the cardinality of a finite set U. Let X be a finite set and $\{A_x\}_{x\in X}$ be a family of events in some probability space with probabilities $\mathbb{P}(A_x)$ to occur. A graph G with vertex set $V(G) = X$ is a *dependency graph* for the family of events $\{A_x\}_{x\in X}$ if, for each $x \in X$, A_x is independent of all the events in the σ-algebra generated by $\{A_y : y \in X\backslash\Gamma^*_G(x)\}$, where $\Gamma_G(x)$ denotes the vertices of G adjacent to x and $\Gamma^*_G(x) = \Gamma_G(x) \cup \{x\}$. For any non empty $S \subseteq X$ we also denote by $\Gamma_G(S)$ the neighborhood of S, i.e., $\Gamma_G(S) = \{x \in X\backslash S : x \text{ adjacent to } y \text{ in } G \text{ for some } y \in S\}$.

Let $\bar{A}_x$ be the complement event of A_x and let $\bigcap_{x\in X} \bar{A}_x$ be the event that none of the events $\{A_x\}_{x\in X}$ occurs. The Lováz local lemma gives a sufficient criterion to guarantee that $\bigcap_{x\in X} \bar{A}_x$ has positive probability (and hence is non empty).

Theorem 1 (Lovász local lemma). *Suppose that G is a dependence graph for the family of events $\{A_x\}_{x\in X}$ with probability $\mathbb{P}(A_x) \leq p_x$ and there exists a sequence $\boldsymbol{\mu} = (\mu_x)_{x\in X}$ of real numbers in $[0, +\infty)$ such that, for each $x \in X$,*

$$p_x \;\leq\; R_x \;=\; \frac{\mu_x}{\varphi_x(\boldsymbol{\mu})} \tag{2.1}$$

with

$$\varphi_x(\boldsymbol{\mu}) \;=\; \prod_{y\in\Gamma^*_G(x)} (1 + \mu_y) \tag{2.2}$$

$$\;=\; \sum_{R\subseteq\Gamma^*_G(x)} \prod_{x\in R} \mu_x \tag{2.3}$$

Then

$$\mathbb{P}(\bigcap_{x\in X} \bar{A}_x) \;\geq\; \prod_{x\in X} \frac{1}{(1+\mu_x)} \;>\; 0 \tag{2.4}$$

Here and in the sequel the sum over all subsets in the r.h.s. of (2.3) includes the empty set and we agree that $\prod_{x\in\emptyset} \mu_x \equiv 1$

Remark. In the literature the Lovász local lemma above is usually written in terms of variables $r_x = \mu_x/(1 + \mu_x) \in [0, 1)$, so that condition (2.1) is replaced by $p_x \;\leq\; r_x \prod_{y\in\Gamma_G(x)}(1 - r_y)$ and bound (2.4) is replaced by $\mathbb{P}(\bigcap_{x\in X} \bar{A}_x) \geq \prod_{x\in X}(1 - r_x)$. The present formulation, however, shows more directly the improvements contained in Theorem 2 below.

In this paper we report the following improved version of this lemma.

Theorem 2 (Improved Lovász lemma). *Suppose that G is a dependence graph for the family of events $\{A_x\}_{x\in X}$ each one with probability $\mathbb{P}(A_x) \le p_x$ and there exists a sequence $\boldsymbol{\mu} = \{\mu_x\}_{x\in X}$ of real numbers in $[0, +\infty)$ such that, for each $x \in X$,*

$$p_x \;\le\; R_x^* \;=\; \frac{\mu_x}{\varphi_x^*(\boldsymbol{\mu})} \tag{2.5}$$

where

$$\varphi_x^*(\boldsymbol{\mu}) = \sum_{\substack{R\subseteq\Gamma_G^*(x)\\ R \text{ indep in } G}} \prod_{x\in R} \mu_x \tag{2.6}$$

Then

$$\mathbb{P}\Big(\bigcap_{x\in X} \bar{A}_x\Big) \;\ge\; \prod_{x\in X} (1-p_x)^{\widetilde{\varphi}_x^*(\boldsymbol{\mu})} \;>\; 0 \tag{2.7}$$

with

$$\widetilde{\varphi}_x^*(\boldsymbol{\mu}) = \sum_{\substack{R\subseteq\Gamma_G(x)\\ R \text{ indep in } G}} \prod_{x\in R} \mu_x = \varphi_x^*(\boldsymbol{\mu}) - \mu_x \tag{2.8}$$

As $\varphi_x^*(\boldsymbol{\mu}) \le \varphi_x(\boldsymbol{\mu})$ [compare (2.3) and (2.6)], $R_x \le R_x^*$. Thus, (2.5) is less restrictive than the condition (2.1) appearing in the Lovász Local Lemma. Moreover, the later condition depends only on the cardinality of the neighborhood $\Gamma_G^*(x)$ of each vertex x of G. In contrast, condition (2.5) in Theorem 2 depends also on the edges between vertices of $\Gamma_G^*(x)$. In consequence, the improvement brought by Theorem 2 is greater when vertices in each neighborhood $\Gamma_G^*(x)$ are grouped into few cliques. In fact, if x is a vertex in a graph G such that $\Gamma_G^*(x)$ is the union of cliques $c_1, \ldots, c_k$, then

$$\varphi_x^*(\boldsymbol{\mu}) \le \prod_{i=1}^{k} \Big[1 + \sum_{x_i\in c_i} \mu_{x_i}\Big] \tag{2.9}$$

This is the expression we shall use in the applications below. Note also that when G is the complete graph Theorem 2 is tight. Indeed, in this case the events $\{A_x\}_{x\in X}$ could be disjoint so that $\mathbb{P}(\bigcap_{x\in X} \bar{A}_x) = 1 - \sum_{x\in X} p_x$, and hence $\sum_{x\in X} p_x < 1$ is required. But, when G is the complete graph we have $\varphi_x^*(\boldsymbol{\mu}) = 1 + \sum_{y\in X} \mu_y$ and condition (2.5) becomes $p_x \le \mu_x/(1 + \sum_{y\in X} \mu_y)$, which summed over all $x \in X$ yields $\sum_{x\in X} p_x < 1$ (in the limit when $\mu_x \to \infty$).

On the other hand the improvement is null when vertices of $\Gamma_G(x)$ form an independent set in the dependency graph. This occurs, for instance, in bipartite graphs and trees. In particular, if the dependency graph is a regular tree, the Lovász Local Lemma is tight (see [14], or [3] Theorem 1.18, or [16] Theorem 5.6).

We also point out that the lower bound (2.7) is stronger than (2.4) in the region of validity of the latter. Indeed, the condition

$$\frac{1}{1+\mu_x} \;\le\; (1-p_x)^{\widetilde{\varphi}_x^*(\boldsymbol{\mu})} \tag{2.10}$$

holds as as long as

$$p_x \;\le\; \overline{R}_x \;\equiv\; 1 - \left(\frac{1}{1+\mu_x}\right)^{1/\widetilde{\varphi}_x^*(\boldsymbol{\mu})} \tag{2.11}$$

We have

$$R_x^* \ \geq \ \overline{R}_x \ \geq \ \widetilde{R}_x^* \ \equiv \ \frac{\mu_x}{(1+\mu_x)\,\widetilde{\varphi}_x^*(\boldsymbol{\mu})} \ \geq \ R_x \tag{2.12}$$

The first and second inequalities are a consequence of the elementary identity $(1+a)^b \leq 1+ab$ valid for $a \geq -1$ and $0 \leq b \leq 1$. The last inequality follows easily from the fact that $(1+\mu_x)\,\widetilde{\varphi}_x^* \leq \varphi_x(\boldsymbol{\mu})$. The firt two inequalities in (2.12) are strict unless $\widetilde{\varphi}_x^*(\boldsymbol{\mu}) = 1$, i.e. unless $\mu_y = 0$ for all $y \in \Gamma_G(x)$. The final inequality is an equality when $\Gamma_G(x)$ is an independent set. We conclude that the bound (2.7) improves (2.4) within the region $\{p_x \leq R_x\}$ and extends it to the larger region $\{p_x \leq \overline{R}_x\}$. The intermediate diameters $\widetilde{R}_x^*$ could be convenient for calculations.

We also establish an improved version of the Lopsided Lovász local lemma.

Theorem 3 (**Improved Lopsided Lovász local lemma).** *Let $\{A_x\}_{x\in X}$ be a family of events on some probability space and let G a graph with vertex set X . Suppose that $\boldsymbol{\mu} = (\mu_x)_{x\in X}$ are real numbers in $[0,+\infty)$ such that, all $x \in X$ and all $Y \subseteq X \setminus \Gamma_G^*(x)$ we have*

$$\mathbb{P}(A_x | \bigcap_{y\in Y} \bar{A}_y) \leq \frac{\mu_x}{\varphi_x^*(\boldsymbol{\mu})} \tag{2.13}$$

Then

$$\mathbb{P}(\bar{A}) \ \geq \ \prod_{x\in X} (1-p_x)^{\widetilde{\varphi}_x^*(\boldsymbol{\mu})} \ > \ 0 \tag{2.14}$$

Finally, we remark that in a very recent paper [12] Pegden has shown that the improvement of LLL presented in this note also holds for the Moser and Tardos's recent algorithmic version of the Local Lovász Lemma [10].

3 Proof of Theorems 2 and 3

Theorem 2 and 3 follow by exploiting the connection —discovered by Scott and Sokal in [16] (see also [17])— between an old theorem by Shearer [14] and the statistical mechanics of the hard-core gas. Scott and Sokal proved an "equivalence theorem" (see theorem 4.1 in [16] or theorem 3.1 in [17]) showing that bounds on lattice gases are equivalent to bounds in the dependency graph context. The proof of our new result relies only on the first part of this connection, namely on how bounds in the lattice gas context transfer to bounds in the combinatorial context. For the benefit of the reader we briefly review those ingredients of the Scott-Sokal equivalence needed for our proof. We start by quoting Shearer's theorem.

Theorem 4 (**Shearer).** *Let $\{A_x\}_{x\in X}$ be a family of events in some probability space. Let G be a dependence graph for the family $\{A_x\}_{x\in X}$ and let $\{p_x\}_{x\in X}$ numbers in $[0,1]$ such that, for all $x \in X$, $\mathbb{P}(A_x) \leq p_x$. Let $S \subseteq X$. Define*

$$P(S) = \sum_{\substack{U:\ S\subseteq U\subseteq X \\ U \text{ indep in } G}} (-1)^{|U|-|S|} \prod_{x\in U} p_x \tag{3.1}$$

If $P(S) \geq 0$ for all $S \subseteq X$ then

$$\mathbb{P}(\bigcap_{x\in X} \bar{A}_x) \ \geq \ P(\emptyset) \tag{3.2}$$

Furthermore these bounds are the best possible, i.e. there can be constructed a family of events $\{B_x\}_{x\in X}$ in a suitable probability space with probabilities $\mathbb{P}(B_x) = p_x$ and dependency graph G, such that $\mathbb{P}(\bigcap_{x\in X} \bar{B}_x) = P(\emptyset)$.

Remark. As shown by Scott and Sokal (see e.g. Theorem 3.1 in [17], bounds (3.12a)-(3.12c)), the original Shearer's proof of Theorem 4 works equally well even if the hypothesis: "*Let G be a dependence graph for the family $\{A_x\}_{x\in X}$ and let, for all $x \in X$, $\mathbb{P}(A_x) \le p_x$.*" is replaced by the weaker hypothesis: "*Let G be a graph with vertex set X and let $\{p_x\}_{x\in X}$ numbers in $[0,1]$ such that, for all $x \in X$ and all $Y \subseteq X \setminus \Gamma^*_G(x)$ $\mathbb{P}(A_x|\bigcap_{y\in Y}\bar{A}_y) \le p_x$*".

Theorem 4 can be rephrased in terms of the (multivariate) independent set polynomial, a.k.a. *partition function* $Z_G(\boldsymbol{w})$ of the *hard-core lattice gas* with complex activities $\boldsymbol{w} = \{w_x\}_{x\in X}$ (with $w_x \in \mathbb{C}$ for all $x \in X$) on the graph G, defined by

$$Z_G(\boldsymbol{w}) = \sum_{\substack{R\subseteq X \\ R \text{ indep in } G}} \prod_{x\in R} w_x \tag{3.3}$$

Indeed, it is immediate to see that

$$P(\emptyset) = \sum_{\substack{U:U\subseteq X \\ U \text{ indep in } G}} (-1)^{|U|} \prod_{x\in U} p_x = Z_G(-\boldsymbol{p}) \tag{3.4}$$

where $-\boldsymbol{p} = \{-p_x\}_{x\in X}$. Moreover, for any non-empty $S \subseteq X$,

$$P(S) = \sum_{\substack{U:\ S\subseteq U\subseteq X \\ U \text{ indep in } G}} (-1)^{|U|-|S|} \prod_{x\in U} p_x = \sum_{\substack{R\subseteq X\setminus(S\cup\Gamma_G(S)) \\ R \text{ indep in } G}} \prod_{x\in R}(-p_x) \prod_{y\in S} p_y = Z_G(-\boldsymbol{p}_S) \prod_{y\in S} p_y \tag{3.5}$$

where $-\boldsymbol{p}_S = \{-p^S_x\}_{x\in X}$ and

$$p^S_x = \begin{cases} 0 & x \in S \cup \Gamma_G(S) \\ p_x & \text{otherwise} \end{cases} \tag{3.6}$$

Formula (3.4) and (3.5) imply immediately that Shearer theorem 4 can be rephrased as follows.

Proposition 5. *Let $\{A_x\}_{x\in X}$ be a family of events in some probability space. Let G be a dependence graph for the family $\{A_x\}_{x\in X}$ and let $\{p_x\}_{x\in X}$ numbers in $[0,1]$ such that, for all $x \in X$, $\mathbb{P}(A_x) \le p_x$. Let $Z_G(\boldsymbol{w})$ be the partition function of of the hard-core lattice gas on G with complex activities $\boldsymbol{w} = \{w_x\}_{x\in X}$ (with $w_x \in \mathbb{C}$ for all $x \in X$).*
If $Z_G(\boldsymbol{w}) \neq 0$ in the polydisc $|w_x| \le p_x$, Then

$$\mathbb{P}(\bigcap_{x\in X} \bar{A}_x) \ \ge\ Z_G(-\boldsymbol{p}) \ >\ 0\,. \tag{3.7}$$

Furthermore these bounds are the best possible, i.e. there can be constructed a family of events $\{B_x\}_{x\in X}$ in a suitable probability space with probabilities $\mathbb{P}(B_x) = p_x$ and dependency graph G, such that $\mathbb{P}(\bigcap_{x\in X}\bar{B}_x) = Z_G(-\boldsymbol{p})$.

Proof. We just need to prove that $Z_G(\boldsymbol{w}) \neq 0$ in the polydisc $|w_x| \le p_x$ implies that $P(S) \ge 0$ for all $S \subset X$. The restriction of $Z_G(\boldsymbol{w})$ to $K_{\boldsymbol{p}} = \prod_{x\in X}[-p_x, p_x]$ is a real polynomial function and it is positive in $K^+_{\boldsymbol{p}} = \prod_{x\in X}[0, p_x]$. Moreover, since $Z_G(\boldsymbol{w}) \neq 0$ in the polydisc $|w_x| \le p_x$, $Z_G(\boldsymbol{w})$ has no zeroes in $K_{\boldsymbol{p}} = \prod_{x\in X}[-p_x, p_x]$. Therefore $Z_G(\boldsymbol{w})$ is positive in the whole $K_{\boldsymbol{p}}$ and in particular $Z_G(-\boldsymbol{p}) > 0$ and so, by (3.4) $P(\emptyset) > 0$. On the other hand, for all $S \subset X$, the set $K_{\boldsymbol{p}^S} = \prod_{x\in X}[-p^S_x, p^S_x] \subset K_{\boldsymbol{p}}$. So we also have that $Z_G(\boldsymbol{w})$ is positive in $K_{\boldsymbol{p}^S}$ and in particular $Z_G(-\boldsymbol{p}^S) > 0$. Hence, by (3.5), $P(S) \ge 0$. □

Estimating the region of the partition function $Z_G(\boldsymbol{w})$ (or, equivalently, of the analyticity radius of $\log Z_G(\boldsymbol{w})$) that is free of zeros is a classical subject in statistical mechanics. Until 2007 the best estimate was due to Dobrushin [4], which, as observed by Scott and Sokal, yields, via proposition 5, exactly the usual the Lovász local lemma. However, Dobrushin criterion has been recently improved by two of us [7] via the following theorem.

Theorem 6. *Let $\boldsymbol{\mu} = \{\mu_x\}_{x\in X}$ be a collection of nonnegative numbers. Then $Z_G(\boldsymbol{w}) \neq 0$ for all $\boldsymbol{w}$ such that*

$$|w_x| \;\leq\; R^*_x \;\equiv\; \frac{\mu_x}{\varphi^*_x(\boldsymbol{\mu})} \qquad\qquad \forall x \in X \tag{3.8}$$

with

$$\varphi^*_x(\boldsymbol{\mu}) = \sum_{\substack{R\subseteq\Gamma^*_G(x)\\ R \text{ indep in } G}} \prod_{x\in R} \mu_x \tag{3.9}$$

Furthermore,

$$Z_G(-|\boldsymbol{w}|) \;\geq\; \prod_{x\in X} (1-|w_x|)^{\widetilde{\varphi}^*_x(\boldsymbol{\mu})} \tag{3.10}$$

where $\widetilde{\varphi}^*_x(\boldsymbol{\mu})$ is defined in (2.8).

Remark . The lower bound (3.10) is not explicitly given in [7]. It can be proven, however, in a straightforward way from an upper bound on the (positive) quantity $\frac{-\partial}{\partial|\boldsymbol{w}_i|}\{\ln Z_G(-|\boldsymbol{w}|)\}$ presented in this reference. For completeness we include the corresponding argument in the appendix.

Theorem 2 is proven by plugging Theorem 6 in Proposition 5. Theorem 3 follows from the remark after Theorem 4.

4 Applications

We present below two examples showing how bounds previously obtained via the the LLL can be easily improved using the new versions of LLL (i.e. theorem 2 for application 1 and theorem 3 for application 2). Of course these two examples do not exhaust the list of possible applications of Theorems 2 and 3 in combinatorial problems. E.g., Theorem 2 and Theorem 3 have already been applied to improve some bounds on graph colorings in [11] and [2] respectively.

Application 1.

Proposition 7. *Let $G = (V_G, E_G)$ be a graph with maximum degree Δ and $V_G = V_1\cup V_2\cup \ldots \cup V_n$ a partition of vertices set V_G into n pairwise disjoints sets. Suppose that for each set V_i we have $|V_i| \geq 4\Delta$. Then, there exists an independent set $W \subseteq V_G$ of cardinality n that contains exactly one vertex from each V_i.*

We remark that the same statement, with the $2e$ replacing 4, has been proved by Alon and Spencer in [1], page 70, using the Lovász local Lemmma. On the other hand, by a completely different method, Haxell has observed that the constant can be lowered to 2 (see [8] and references therein). So even if Proposition 7 does not give the optimal bound obtained by Haxell, still improves on old Lovász local Lemmma.

Proof. Without loss of generality, we can assume that for all sets V_i we have $|V_i| = s$ for some integer s. The general case follows from this using the graph induced by G on a union of n subsets of cardinality s, each of them a subset of one V_i. Following [1], we take a random set W of n vertices as follows: we pick up from each set V_i, randomly and independently, a unique vertex according to a uniform distribution, that is, in each V_i the probability of a vertex to be chosen is $1/s$.

Let now $\tilde{E}_G = \{e \in E_G : |e \cap V_k| \leq 1, \forall k = 1, \dots, n\}$. Namely, $\tilde{E}_G \subset E_G$ contains all edges whose end-points are contained in two distinct sets V_i and V_j of the partition of V_G. For $e = \{a, b\} \in \tilde{E}_G$, let W_e be the event "$\{a, b\} \subset W$". Clearly, $\mathbb{P}(W_e) = 1/s^2 \equiv p$ if $e \in \tilde{E}_G$. Moreover, if $e = \{a, b\} \in \tilde{E}_G$ is such that $a \in V_i$ and $b \in V_j$, then the event W_e is mutually independent of all the events involving edges whose endpoints do not lie in $V_i \cup V_j$.

So the graph H with vertex set $V_H = \tilde{E}_G$ and edge set

$$E_H = \{\{e, e'\} \subset \tilde{E}_G : \ |e \cap V_k| + |e' \cap V_k| > 1 \text{ for some } k = 1, \dots, n\}$$

is a dependency graph for the family of events $\{W_e\}_{e \in \tilde{E}_G}$. The maximum degree of the graph H is less than $2s\Delta$. Indeed, for each of the two vertices of an edge $e \in \tilde{E}_G$ there are at most s vertices in the set V_i which contains that vertex, and since G has degree at most Δ, there are at most Δ edges containing a fixed vertex. It is also clear that $\Gamma^*_H(e)$ (i.e. the neighborhood of e in H) is the union of 2 cliques, each of cardinality at most $s\Delta$.

We may now apply Theorem 2 taking $\mu_e = \mu > 0$ for all $e \in V_H$ so that, recalling (2.9), we have $\varphi^*_e(\mu) \leq [1 + s\Delta\mu]^2$ and hence

$$\frac{\mu}{\varphi^*_e(\mu)} \ \geq \ \frac{\mu}{[1 + s\Delta\mu]^2} \ \equiv \ f(\mu)$$

As the right side assumes its maximum value at $\mu_0 = [s\Delta]^{-1}$ we may use Theorem 2 in the region $\frac{1}{s^2} \leq \frac{1}{4s\Delta} = f(\mu_0)$, a condition equivalent to say $s \geq 4\Delta$. This guarantees that there is a positive probability that none of the events $\{W_e\}_{e \in \tilde{E}_G}$ occur. In other words, we obtain that there is a positive probability that W is an independent set that contains one vertex of each V_i.
□

Application 2: Latin Transversals.

Let A be a $n \times n$ matrix with entries a_{ij}. Suppose that a_{ij} is integer for all $i, j = 1, \dots, n$. A permutation $\sigma : \{1, 2, \dots, n\} \to \{1, 2, \dots, n\} : i \mapsto \sigma(i)$ is called a Latin transversal of A if the entries $a_{i\sigma(i)}$ with $i = 1. \dots, n$ are all distinct.

Proposition 8. *Suppose that $k \leq (n-1)/(256/27)$ and suppose that no integer appears in more than k entries of A. Then A has a Latin transversal.*

The same proposition with $4e$ replacing $256/27$ is proved in [1], using the Lovász local Lemmma. We are not aware of any better bound for latin transversals.

Proof. We follow here the strategy described in [1]. Let σ be a permutation of $\{1, 2, \dots, n\}$ chosen at random with uniform distribution. Denote by T the set of all ordered four-tuples (i, j, i', j') such that $i < i'$, $j \neq j'$ and $a_{ij} = a_{i'j'}$. For each $(i, j, i', j') \in T$, let $A_{iji'j'}$ be the event that $\sigma(i) = j$ and $\sigma(i') = j'$. Clearly $A_{iji'j'}$ has a probability $\frac{1}{n(n-1)}$ to occur and if σ is not a Latin Transversal then some $A_{iji'j'}$ occurs. Hence, a Latin transversal of A exists if there is a non zero probability that none of the events $A_{iji'j'}$ occurs. We define a graph G with vertex set

T and two vertices (i,j,i',j') and (p,q,p',q') are adjacent if and only if $\{i,i'\}\cap\{p,p'\}\neq\emptyset$ or $\{j,j'\}\cap\{q,q'\}\neq\emptyset$. This graph has maximum degree less that $4nk$. Indeed, for a fixed (i,j,i',j'), we can choose (s,t) in $4n$ different ways with $s\in\{i,i'\}$ or $t\in\{j,j'\}$ for a given (i,j,i',j') and once (s,t) has been chosen we have less that k choices for (s',t') distinct from (s,t) such that $a_{st}=a_{s't'}$, since by assumption there are at most k entries of A with the same value. So we have less than $4nk$ four-tuples (s,t,s',t') such that $s=i$, or $s=i'$, or $t=j$ or $t=j'$ and such that $a_{st}=a_{s't'}$. Now, to each of those four-tuples (s,t,s',t') we can associate uniquely the four-tuple $(p,q,p',q')=(s,t,s',t')$ if $s<s'$ or the four-tuple $(p,q,p',q')=(s',t',s,t)$ if $s'<s$.

Alon and Spencer showed (see [1] pag. 79 or [6]) that

$$\mathbb{P}\Bigg(A_{iji'j'}|\bigcap_{(p,q,p',q')\in Y}\overline{A}_{pqp'q'}\Bigg)\leq\frac{1}{n(n-1)}$$

for any $(i,j,i',j')\in T$ and any set Y of members of T that are nonadjacent in G to (i,j,i',j'). Therefore we can apply Theorem 3 for the graph G with vertex set T described above. We take $\mu_x=\mu>0$, for all $x\in T$ and we observe that the set of vertices in $\Gamma^*_G((i,j,i',j'))$ is, by the previous construction, the union of 4 cliques each of cardinality at most nk. Thus, recalling (2.9), for such a graph G

$$\varphi^*_{(i,j,i',j')}(\mu)\leq(1+nk\mu)^4$$

and

$$\frac{\mu}{\varphi^*_{(i,j,i',j')}(\mu)}\geq\frac{\mu}{(1+nk\mu)^4}\equiv f(\mu)$$

As the righthand side assumes its maximum value at $\mu_0=\frac{1}{3nk}$, we may use Theorem 3 on the region $p=\frac{1}{n(n-1)}\leq 27/(256nk)=f(\mu_0)$, a condition equivalent to say $k\leq(n-1)/(256/27)$. □

Appendix. Proof of bound (2.7)

The partition functions of hard-core gases satisfy, in the sense of formal power series, the identities (see e.g. [13], [7])

$$-\frac{\partial}{\partial\rho_x}\log Z_G(-\boldsymbol{\rho})=\Pi_x(\boldsymbol{\rho})\qquad\qquad\forall x\in X$$

where $\Pi_x(\boldsymbol{\rho})$ are (G-dependent) formal power series on $\boldsymbol{\rho}=\{\rho_x\}_{x\in X}$, all whose terms are positive if $\rho_y\geq 0$, $y\in X$. The simultaneous convergence for some $\boldsymbol{\rho}$ of all the series $\Pi_y(\boldsymbol{\rho})$, $y\in X$, implies the analyticity of $\log Z_G(\boldsymbol{w})$ in the polydisc $\{|w_x|\leq\rho_x\}_{x\in X}$. Furthermore, within the region of convergence,

$$|\log Z_G(\boldsymbol{w})|\;\leq\;-\log Z_G(-|\boldsymbol{w}|)\tag{A.1}$$

and

$$-\log Z_G(-\boldsymbol{\rho})+\log Z_{G\setminus\{x\}}(-\boldsymbol{\rho})\;=\;\Sigma_x(\boldsymbol{\rho})\tag{A.2}$$

where $G\setminus\{x\}$ denotes the restriction of G to $X\setminus\{x\}$ and

$$\Sigma_x(\boldsymbol{\rho})\;=\;\rho_x\int_0^1\Pi_x(\boldsymbol{\rho}(\alpha))d\alpha\tag{A.3}$$

with

$$\rho_y(\alpha)\;=\;\begin{cases}\rho_y & \text{if } y\neq x\\ \alpha\rho_y & \text{if } y=x\end{cases}$$

As a consequence

$$-\log Z_G(-|\boldsymbol{w}|) \;\leq\; \sum_{x\in X} \Sigma_x(\boldsymbol{\rho}) \tag{A.4}$$

The lower bound (2.7) is proven from upper bounds for $\Sigma_x(\boldsymbol{\rho})$. They follow from the following proposition proved in [7].

Proposition 9. *Let $\boldsymbol{\mu} \equiv \{\mu_x \geq 0\}_{x\in X}$ and let, for all $x \in X$, $\rho_x \leq R^*_x$ (with R^* defined in (3.8)). Then there exists a convergent positive term series $\Pi^*_x(\boldsymbol{\rho})$ monotonic in $\boldsymbol{\rho}$ such that*

i) If $\boldsymbol{\rho}\boldsymbol{\Pi}^*(\boldsymbol{\rho}) = \{\rho_x\Pi^*_x(\boldsymbol{\rho})\}_{x\in X}$ and $\boldsymbol{\varphi}^*(\boldsymbol{\mu}) = \{\varphi^*_x(\boldsymbol{\mu})\}_{x\in X}$, then $\boldsymbol{\rho}\Pi^*(\boldsymbol{\rho})$ is fixed point of the map

$$\begin{aligned} \boldsymbol{T}^{\boldsymbol{\rho}} \equiv \{T^{\boldsymbol{\rho}}_x\}_{x\in X} : [0,\infty)^X &\;\longrightarrow\; [0,\infty)^X \\ T^{\boldsymbol{\rho}}_x(\cdot) &\;=\; \rho_x\varphi^*_x(\cdot) \end{aligned} \tag{A.5}$$

i.e.

$$\rho_x\Pi^*_x(\boldsymbol{\rho}) = T^{\boldsymbol{\rho}}_x\Big(\boldsymbol{\rho}\boldsymbol{\Pi}^*(\boldsymbol{\rho})\Big) \tag{A.6}$$

ii) The following bounds hold

$$\rho_x\Pi_x(\boldsymbol{\rho}) \;\leq\; \rho_x\Pi^*_x(\boldsymbol{\rho}) \;\leq\; \mu_x \tag{A.7}$$

Item *ii)* of Proposition 9 immediately implies that $\log Z_G(\boldsymbol{w})$ is analytic in the polydisk $\{|w_x| \leq R_x\}_{x\in X}$. To obtain an upper bound for $\Sigma_x(\boldsymbol{\rho})$ we make use of item *i)*. Indeed, by (A.5) and (A.6)

$$\begin{aligned} \Pi^*_x(\boldsymbol{\rho}) \;=\; \varphi^*_x(\boldsymbol{\rho}\Pi^*(\boldsymbol{\rho})) &\;=\; \rho_x\Pi^*_x(\boldsymbol{\rho}) + \widetilde{\varphi}^*_x(\boldsymbol{\rho}\Pi^*(\boldsymbol{\rho})) \\ &\;\leq\; \rho_x\Pi^*_x(\boldsymbol{\rho}) + \widetilde{\varphi}^*_x(\boldsymbol{\mu}) \end{aligned}$$

Thus, since $\Pi_x(\boldsymbol{\rho}) \leq \Pi^*_x(\boldsymbol{\rho})$,

$$\Pi_x(\boldsymbol{\rho}) \;\leq\; \frac{\widetilde{\varphi}^*_x(\boldsymbol{\mu})}{1-\rho_x}$$

By equation (A.3) this implies

$$\Sigma_x(\boldsymbol{\rho}) \;\leq\; \rho_x \int_0^1 \frac{\widetilde{\varphi}^*_x(\boldsymbol{\mu})}{1-\alpha\rho_x} d\alpha \;=\; -\ln\,[1-\rho_x]^{\widetilde{\varphi}^*_x(\boldsymbol{\mu})}$$

which, by (A.4), implies that for all $\rho_x \leq R^*_x$

$$Z_G(-\boldsymbol{\rho}) \geq \prod_{x\in X} [1-\rho_x]^{\widetilde{\varphi}^*_x(\boldsymbol{\mu})}$$

Acknowledgments

It is a pleasure to thank Yoshiharu Kohayakawa for helpful comments. This work has been partially supported by Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq), CAPES (Coordenação de Aperfeiçoamento de Pessoal de Nível Superior, Brasil) and FAPEMIG (Fundacão de Amparo à Pesquisa do Estado de Minas Gerais). RB thanks the hospitality of Laboratoire de Mathématiques Raphaël Salem in the Université de Rouen during his PhD work. RF and AP thank The Newton Institute for the generous support during the Combinatorics and Statistical Mechanics programme and the Mathematics Department of Università di Roma Tor Vergata (AP).

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