# Around Rubik's Cube

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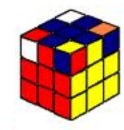
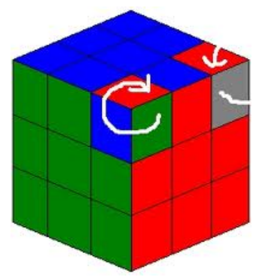
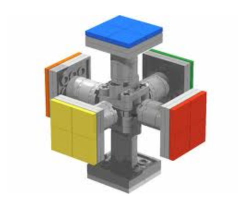
A corner cube in a Rubik's cube can be sitting in its own position, but twisted:



There is a routine consisting of about a dozen turns which twists one corner cube clockwise, another one counter clockwise, returning everything else to its old state.



One may also twist a single corner cube [monopoles exist], but this requires cheating: Take the cube apart, but do not remove the stickers.

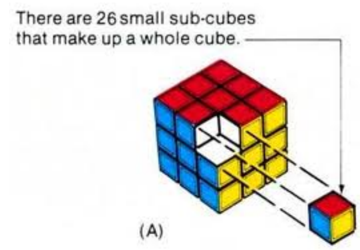


Put it back together, in the desired configuration.

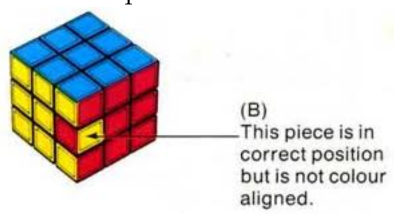
Notice there are six stickers on the cross. Remember their colors and decide on a way to hold the cross in space. We use the cross as a fixed reference frame.

How many cube shaped configurations are there? For the eight corner cubes there are eight corners. Once one has chosen at which corner a corner cube is to be placed, there are still three orientations. For the twelve edge cubes there are twelve edges, and once an edge is chosen there are still two orientations. Together $8!3^8 12!2^{12} = 519024039293878272000$. But without cheating only 43252003274489856000 of these configurations are reachable, so only one out of twelve may be reached.

The permutations of the subcubes is always an even permutation




and so is the permutation of the stickers on the edge cubes.



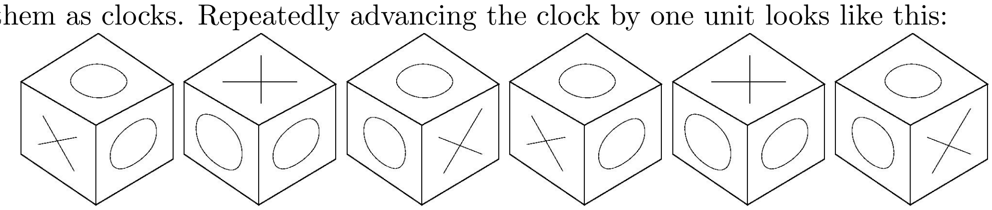

That explains a factor four.

Remains finding out why monopoles are out of reach. That will explain a factor three, and three times four is twelve.

For inspiration we consider the tic tac toe cube.

Paint all stickers on corner cubes and edge cubes white. Then mark the stickers on the corner cubes with one cross and two zeroes.

So now all corner cubes look the same, but they still have three orientations. Think of them as clocks. Repeatedly advancing the clock by one unit looks like this:

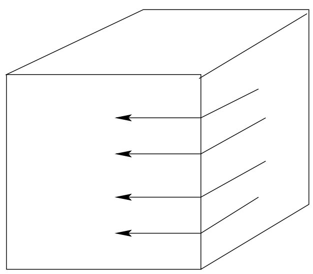


After three steps the clock repeats. The "no monopole" rule states: Under admissible operations the total advance of the clocks is always a multiple of three.

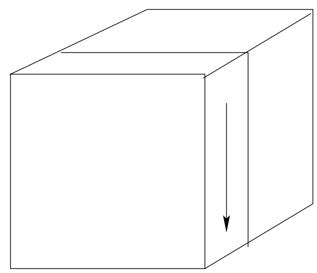
It suffices to check this for generators. Imagine you have memorized the position of the eight clocks. Hand your cube to a reliable person and ask to give one face a quarter turn. Do not look. Then ask the cube back, and hold the cross in the same orientation as before. Make a tally of the apparent time differences. Repeat four times. Ask turning the same face the same way each time. Each time the apparent total advance is the same, but after four times it is nothing modulo three. So when turning the face in question the rule is satisfied. The face was arbitrary. Any legal permutation is a composition of face turns. The rule follows.

## Some routines

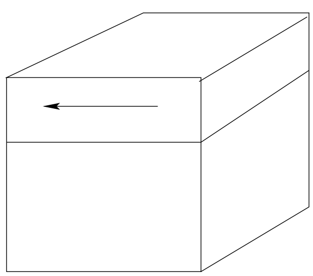
We use the letters $C$, $F$, $U$, $D$, $L$, $R$ to indicate a quarter turn of the full Cube, the Front face, the Upper layer, the Down layer, the Left face, the Right face respectively, in the direction indicated in the picture.



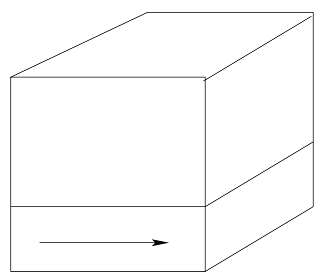
C



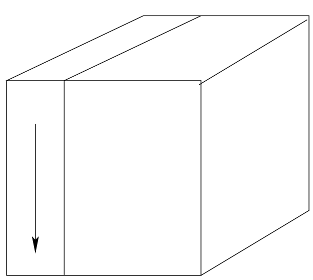
F



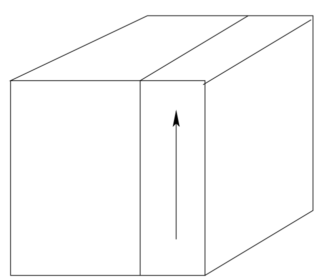
U



D

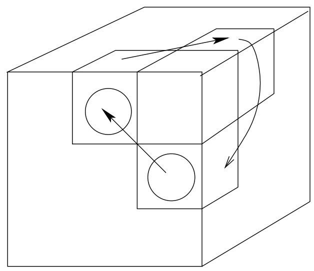


L



R

The procedure *pealing an apple* is defined as $RCRCRCRUR^{-1}C^{-1}R^{-1}C^{-1}R^{-1}C^{-1}R^{-1}U^{-1}$. It is to be read from left to right. So we start with turning the right face. Its effect is



There is also its left-right mirror image, starting with $L^{-1}C^{-1}$. Useful variants are obtained by substituting for $U$ a power of $U$.

## Supports

If $\sigma$ permutes the set $X$, then the support of $\sigma$ is defined by

$$\mathrm{supp}(\sigma) = \{x \in X \mid \sigma x \neq x\}.$$

One has $\sigma(\mathrm{supp}(\sigma)) = \mathrm{supp}(\sigma) = \mathrm{supp}(\sigma^{-1})$, and the size of $\mathrm{supp}(\sigma\tau\sigma^{-1}\tau^{-1})$ is at most three times the size of the intersection of $\mathrm{supp}(\sigma)$ with $\mathrm{supp}(\tau)$.

### Exercises

- (Twisting) Try the commutator of $R^{-1}DRFDF^{-1}$ with a power of $U$. What is the intersection of the supports when one takes for $X$ the set of stickers?
- (Flipping) Put $M = C^{-1}UD^{-1}$. Determine $(MR)^8$ and try the commutator of $(MR)^4$ with a power of $U$. What is the intersection of the supports when one takes for $X$ the set of stickers?
- (Three cycle on corners) Try the commutator of $U^{-1}L^{-1}U$ with a power of $R$. What is the intersection of the supports when one takes for $X$ the set of subcubes?
- Try the commutator of $U^{-1}L^2U$ with a power of $R$. What is the intersection of the supports when one takes for $X$ the set of subcubes?
- In what sense is *pealing an apple* a different kind of commutator?
- Show that 43252003274489856000 configurations are reachable.