Analysis in several variables

Additional exercise

A real symmetric matrix $A \in \mathrm{Mat}(n, \mathbf{R})$ is said to be positive definite if $\langle v, A\,v \rangle > 0$ for all $v \in \mathbf{R}^n \setminus \{0\}$.

a) Show that for any $B \in \mathrm{Mat}(n, \mathbf{R})$ symmetric and positive definite, there exists a unique symmetric positive definite matrix $A \in \mathrm{Mat}(n, \mathbf{R})$ such that $A^2 = B^3$.

Hint: any symmetric matrix is diagonalisable.

From now on, let $A, B \in \mathrm{Mat}(n, \mathbf{R})$ be two fixed positive definite symmetric matrices such that $A^2 = B^3$.

The goal of this exercise is to use the implicit function theorem to show that for any matrix Y sufficiently close to B there exists a unique matrix X near A such that $X^2 = Y^3$ and that, moreover, this matrix X depends smoothly on Y. To this end, consider the function

$$F\colon\ \mathrm{Mat}(n, \mathbf{R}) \times \mathrm{Mat}(n, \mathbf{R}) \to \mathrm{Mat}(n, \mathbf{R}),$$

given by $F(X, Y) = X^2 - Y^3$.

By identifying $\mathrm{Mat}(n, \mathbf{R}) \simeq \mathbf{R}^{n^2}$, we can interpret F as a function from $\mathbf{R}^{n^2} \times \mathbf{R}^{n^2}$ to $\mathbf{R}^{n^2}$ and try to apply the implicit function theorem. In terms of the matrix entries X_{ij} and Y_{ij}, the function F is a polynomial and it will therefore in particular be C^∞.

b) Calculate the (total) derivative

$$\mathrm{D}F(A, B) \in \mathrm{Lin}(\mathrm{Mat}(n, \mathbf{R}) \times \mathrm{Mat}(n, \mathbf{R}), \mathrm{Mat}(n, \mathbf{R}))$$

by working out $F(A + H, B + K)$ for $(H, K) \in \mathrm{Mat}(n, \mathbf{R}) \times \mathrm{Mat}(n, \mathbf{R})$.

c) For $H \in \mathrm{Mat}(n, \mathbf{R})$ arbitrary and $M = \mathrm{D}F(A, B)(H, 0) \in \mathrm{Mat}(n, \mathbf{R})$, show that the trace of $H^T M$ can be expressed as

$$\mathrm{tr}(H^T M) = \mathrm{tr}(H^T A\,H) + \mathrm{tr}(H A\,H^T).$$

d) Use the fact that A is positive definite to show that $\mathrm{tr}(H^T A\,H) > 0$ whenever $H \neq 0$. Then prove that $\mathrm{D}_A F(A, B) := \mathrm{D}F(A, B)|_{\mathrm{Mat}(n,\mathbf{R}) \times \{0\}}$ is an invertible linear map from $\mathrm{Mat}(n, \mathbf{R}) \simeq \mathbf{R}^{n^2}$ to itself.

e) Prove that there exist open neighbourhoods $U \subseteq \mathrm{Mat}(n, \mathbf{R})$ of A and $V \subseteq \mathrm{Mat}(n, \mathbf{R})$ of B and a unique C^∞ function $f\colon V \to U$ such that for $(X, Y) \in U \times V$

$$X^2 = Y^3 \iff X = f(Y).$$