**MIDTERM MULTIDIMENSIONAL REAL ANALYSIS**

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— Solutions —

**Exercise 1.**

(a) $G$ inverts the distance of points to the origin. It preserves all radial rays and interchanges the sphere of radius $r$ centred at the origin with that of radius $\frac{1}{r}$.

(b) This can be done in several ways:

1. Let $x \in U$, then by Hadamard's lemma there exist continuous functions $\phi\colon U \to \mathrm{Lin}(\mathbb{R}^n, \mathbb{R})$ and $\Gamma\colon U \to \mathrm{Lin}(\mathbb{R}^n, \mathbb{R}^n)$ such that

$$f(y) = f(x) + \phi(y)(y - x) \qquad \text{and} \qquad G(y) = G(x) + \Gamma(y)(y - x)$$

for all $y \in U$. Moreover, $\phi(x) = Df(x)$ and $\Gamma(x) = DG(x)$.

Consequently, we find that

$$\begin{aligned}(fG)(y) &= f(y)\,(G(x) + \Gamma(y)(y - x)) \\ &= f(x)\,G(x) + \phi(y)(y - x)\,G(x) + f(y)\,\Gamma(y)(y - x) \\ &= fG(x) + H(y)(y - x),\end{aligned}$$

where $H\colon U \to \mathrm{Lin}(\mathbb{R}^n, \mathbb{R}^n)$ is the function given by

$$H(x) = f(x)\,\Gamma(y) + G(x)\,\phi(x).$$

Continuity of $H$ follows by application of the sum and product rules for continuous functions. By applying Hadamard's lemma again, we conclude that $fG$ is differentiable at $x$, with total derivative

$$D(fG)(x) = H(x) = f(x)\Gamma(x) + G(x)\phi(x) = f(x)DG(x) + G(x)Df(x).$$

2. Because $f$ and $G$ are differentiable by assumption, one can write

$$f(x + h) = f(x) + Df(x)h + R_f(x + h)$$

and

$$G(x + h) = G(x) + DG(x)h + R_G(x + h).$$

Here $R_f\colon U \to \mathbb{R}$ and $R_G\colon U \to \mathbb{R}^n$ satisfy

$$\lim_{h\to 0} \frac{R_f(x + h)}{\|h\|} = 0 \qquad \text{and} \qquad \lim_{h\to 0} \frac{R_G(x + h)}{\|h\|} = 0.$$

By working out the product of these two expressions, one obtains

$$(fG)(x + h) = f(x)G(x) + \big(f(x)DG(x)h + Df(x)h\,G(x)\big) + R_{fG}(x + h),$$

where the final term reads

$$R_{fG}(x+h) = Df(x)h\, DG(x)h + R_f(x+h)G(x) + f(x+h)\, R_G(x+h).$$

Since $h \mapsto G(x)$ and $h \mapsto f(x+h)$ are continuous functions, we obviously have

$$\lim_{h\to 0} \frac{R_f(x+h)}{\|h\|} G(x) = 0 \qquad \text{and} \qquad \lim_{h\to 0} f(x+h) \frac{R_G(x+h)}{\|h\|} = 0.$$

For the first term, we can make the estimate

$$\frac{|Df(x)h|\,\|DG(x)h\|}{\|h\|} \le \frac{\|Df(x)\|\,\|DG(x)\|\,\|h\|^2}{\|h\|} = \|Df(x)\|\,\|DG(x)\|\,\|h\|,$$

so this also vanishes in the limit for $h \to 0$. We conclude that

$$\lim_{h\to 0} \frac{R_{fG}(x+h)}{\|h\|} = 0.$$

Hence, $fG$ is differentiable and its total derivative is given by

$$\begin{aligned} D(fG)(x)h &= f(x)DG(x)h + Df(x)h\, G(x) \\ &= \big(f(x)DG(x) + G(x)Df(x)\big)h. \end{aligned}$$

3. One can use the fact that an $\mathbb{R}^n$-valued function is differentiable if and only if all of its components are.

   For $1 \le i \le n$, the $i$-th component of $fG$ is given by $(fG)_i(x) = f(x)G_i(x)$ and is a product of scalar functions. Both $f$ and $G_i$ are differentiable by assumption, so one may conclude from the product rule that their product is as well, with total derivative

$$D(fG)_i(x) = G_i(x)\, Df(x) + f(x)\, DG_i(x).$$

   Since each of its components are differentiable, the original function $fG$ is as well and its derivative is given by

$$D(fG)(x)h = \begin{pmatrix} D(fG)_1(x)h \\ \vdots \\ D(fG)_n(x)h \end{pmatrix} = \begin{pmatrix} G_1(x)\, Df(x)h + f(x)\, DG_1(x)h \\ \vdots \\ G_n(x)\, Df(x)h + f(x)\, DG_n(x)h. \end{pmatrix}$$

   More concisely, we read off that $D(fG)(x) = G(x)Df(x) + f(x)\, DG(x)$.

(b) In our specific case, we have that $f(x)\, G(x) = x$ for all $x \in \mathbb{R}^n \setminus \{0\}$, so $fG = \mathrm{id}$. From this, it follows that

$$D(fG) = G\, Df + f\, DG = D\mathrm{id} = \mathrm{id}.$$

We know the derivative of $f\colon x \mapsto \|x\|^2$ to be $Df(x)h = 2\,\langle x, h\rangle = 2\,x^{\mathrm{T}}h$, so the above identity tells us that

$$\begin{aligned} DG(x) &= f(x)^{-1}(\mathrm{id} - G(x)\cdot Df(x)) \\ &= \frac{1}{\|x\|^2}\left(\mathrm{id} - \frac{x}{\|x\|^2}\cdot 2\,x^{\mathrm{T}}\right) = \frac{1}{\|x\|^2}\, A(x), \end{aligned}$$

where for $x \in \mathbb{R} \setminus \{0\}$, $A(x)$ denotes the matrix

$$A(x) = I - 2\,\frac{x\,x^{\mathrm{T}}}{\|x\|^2}.$$

(c) We recognise $A(x)$ as the matrix representing a reflection in the plane perpendicular to $x$. We will verify that this is an orthogonal transformation.

Because $A^{\mathrm{T}}(x) = A(x)$, we see that

$$A^{\mathrm{T}}(x)A(x) = \left(I - 2\,\frac{x\,x^{\mathrm{T}}}{\|x\|^2}\right)^2 = I^2 - 4\,\frac{x\,x^{\mathrm{T}}}{\|x\|^2} + 4\,\frac{x\,x^{\mathrm{T}}x\,x^{\mathrm{T}}}{\|x\|^4}.$$

Because $x^{\mathrm{T}}x = \|x\|^2$, the last two terms cancel out and we may conclude that $A^{\mathrm{T}}(x)A(x) = I^2 = I$.

**Exercise 2.**

(a) Introduce $g : \mathbb{R}^3 \to \mathbb{R}$ by

$$g(x) = \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} + \frac{x_3^2}{c^2},$$

so that $M = \{x \in \mathbb{R}^3 : g(x) = 1\}$. A simple computation shows that the derivative of $g$ at $x \in \mathbb{R}^3$ reads

$$Dg(x) = \begin{pmatrix} \dfrac{2\,x_1}{a^2} & \dfrac{2\,x_2}{b^2} & \dfrac{2\,x_3}{c^2} \end{pmatrix}, \tag{1}$$

which is non-zero for all $x \neq 0$. Hence $g$ is a submersion at every point $x \in M$ and its geometric tangent space at $x$ is given by

$$\tilde{T}_x M = \{y \in \mathbb{R}^3 \mid Dg(x)(y - x) = 0\} = \{y \in \mathbb{R}^3 \mid Dg(x)y = 2\}.$$

For this have used that $Dg(x)x = \dfrac{2\,x_1}{a^2}x_1 + \dfrac{2\,x_2}{b^2}x_2 + \dfrac{2\,x_3}{c^2}x_3 = 2\,g(x) = 2.$

(b) The distance from the origin to the tangent plane at $x \in M$ can be found through either a geometric argument or by applying the method of Lagrange multipliers.

1. The distance from the origin to the plane will be equal to the length of the component of $x \in \tilde{T}_x M$ orthogonal to it. Since we know that $\operatorname{grad} g(x) = [Dg(x)]^{\mathrm{T}}$ is orthogonal to the tangent space $T_x M$, this length will be given by

$$d(0, \tilde{T}_x M) = \frac{\langle x, \operatorname{grad} g(x)\rangle}{\|\operatorname{grad} g(x)\|} = \frac{Dg(x)x}{\|Dg(x)\|}.$$

We have already computed the numerator $Dg(x)x = 2$, and the denominator can be read off from equation (1). We thus obtain

$$d(0, \tilde{T}_x M) = \left(\frac{x_1^2}{a^4} + \frac{x_2^2}{b^4} + \frac{x_3^2}{c^4}\right)^{-\frac{1}{2}}.$$

2. One may also arrive at this answer through the method of Lagrange multipliers. The distance $d(0, \tilde{T}_x M)$ is then obtained by minimising the function $f\colon x \mapsto \|x\|^2$ on the geometric tangent plane $\tilde{T}_x M$. Since the plane $\tilde{T}_x M \subseteq$ is a closed subset of $\mathbb{R}^3$, $f$ assumes a minimum on it at some point $y_0 \in \tilde{T}_x M$ and the distance from the origin to the plane will be the square root of this minimum. (NB: The intersection $\tilde{T}_x M \cap \overline{B(0,R)}$ is compact and non-empty for an appropriately chosen $R > 0$. The norm assumes a minimum on it, which is in fact a global minimum.)

   The point $y_0 \in \tilde{T}_x M$ will necessarily be a critical point for $f$, which means that $\operatorname{grad} f(y_0) = 2\,y_0$ is orthogonal to $\tilde{T}_x M$, hence parallel to $\operatorname{grad} g(x)$. Let $\lambda \in \mathbb{R}$ be such that $y_0 = \lambda \operatorname{grad} g(x)$, then we see that (since $y_0 \in \tilde{T}_x M$)

   $$Dg(x)y_0 = \langle \operatorname{grad} g(x), \lambda \operatorname{grad} g(x)\rangle = \lambda \,\|\operatorname{grad} g(x)\|^2 = 2.$$

   We derive that $\lambda = 2\,\|\operatorname{grad} g(x)\|^{-2}$ and that therefore

   $$\|y_0\| = |\lambda|\,\|\operatorname{grad} g(x)\| = \frac{2}{\|\operatorname{grad} g(x)\|} = \left(\frac{x_1^2}{a^4} + \frac{x_2^2}{b^4} + \frac{x_3^2}{c^4}\right)^{-\frac{1}{2}}.$$

   This confirms our earlier conclusion.

3. The critical point described in part 2 also corresponds to a critical point for the Lagrange function

   $$L\colon \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}, \qquad (y,\lambda) \mapsto f(y) - \lambda\, h(y),$$

   where $f(y) = \|y\|^2$ and $h(y) = Dg(x)y - 2$.

   Since $Df(y) = 2\,y^{\mathrm{T}}$ and $Dh(y) = Dg(x)$, the equation $DL(y,\lambda) = 0$ becomes

   $$DL(y) = (Df(y) - \lambda\, Dh(y), h(y)) = (2\,y^{\mathrm{T}} - \lambda\, Dg(x), Dg(x)y - 2) = 0.$$

   Solving this system of equations essentially comes down to following the steps from option 2.

**Exercise 3.**

(a) The function $\Phi$ is clearly $C^\infty$, and we can explicitly compute its derivative

$$\begin{aligned} D\Phi(\theta,t) &= (\partial_\theta \Phi(\theta,t) \quad \partial_t \Phi(\theta,t)) \\ &= \begin{pmatrix} -\frac{1}{2}t\sin(\frac{1}{2}\theta)\cos\theta - (2 + t\cos(\frac{1}{2}\theta))\sin\theta & \cos(\frac{1}{2}\theta)\cos\theta \\ -\frac{1}{2}t\sin(\frac{1}{2}\theta)\sin\theta + (2 + t\cos(\frac{1}{2}\theta))\cos\theta & \cos(\frac{1}{2}\theta)\sin\theta \\ \frac{1}{2}t\cos(\frac{1}{2}\theta) & \sin(\frac{1}{2}\theta) \end{pmatrix} \end{aligned}$$

There are at least three ways to verify that $D\Phi(\theta,t)$ is injective for all $(\theta,t) \in D$, so that $\Phi$ is an immersion.

1. One can compute the determinant of the upper $2 \times 2$ block of $D\Phi(\theta, t)$. This determinant equals

$$-(2 + t\cos(\tfrac{1}{2}\theta))\cos(\tfrac{1}{2}\theta).$$

This is non-zero for all $(\theta, t) \in D$, meaning that $D\Phi(\theta, t)$ has rank 2 and that $\Phi$ is an immersion.

2. One can also decompose

$$D\Phi(\theta,t) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -\frac{1}{2}t\sin(\frac{1}{2}\theta) & \cos(\frac{1}{2}\theta) \\ 2 + t\cos(\frac{1}{2}\theta) & 0 \\ \frac{1}{2}t\cos(\frac{1}{2}\theta) & \sin(\frac{1}{2}\theta) \end{pmatrix}.$$

Since $2 + t\cos(\frac{1}{2}\theta) > 0$ for $(\theta, t) \in D$, the two columns of the $3 \times 2$-matrix on the second line are linearly independent. Because the square matrix that was factored out is invertible, we conclude that $D\Phi(\theta, t)$ is injective and that $\Phi$ is therefore an immersion.

3. Another option is calculating the cross product $\partial_\theta\Phi(\theta, t) \times \partial_t\Phi(\theta, t)$. The third component of this cross product is

$$-(2 + t\cos(\tfrac{1}{2}\theta))\cos(\tfrac{1}{2}\theta)(\sin^2\theta + \cos^2\theta) = -(2 + t\cos(\tfrac{1}{2}\theta))\cos(\tfrac{1}{2}\theta).$$

This is non-zero for all $(\theta, t) \in D$, which means that the columns of $D\Phi(\theta, t)$ are linearly independent. We conclude that $D\Phi(\theta, t)$ has rank 2 and that $\Phi$ is an immersion.

(b) For $(x, y) \in \mathbb{R}^2$ of the form $(x, y) = \rho(\cos\phi, \sin\phi)$ with $\rho > 0$ and $\phi \in ]-\pi, \pi[$, one can recover $\rho = \sqrt{x^2 + y^2}$ and $\phi = 2\arctan(\frac{y}{\rho + x})$. We therefore define

$$\rho\colon \mathbb{R}^2 \setminus \{(0,0)\} \to ]0, \infty[, \qquad \text{and} \qquad \phi\colon \mathbb{R}^2 \setminus \{(x, 0) \mid x \leq 0\} \to ]-\pi, \pi[$$

by setting

$$\rho(x, y) = \sqrt{x^2 + y^2} \qquad \text{and} \qquad \phi(x, y) = 2\arctan\left(\frac{y}{\rho(x, y) + x}\right).$$

Since all functions involved are smooth on their domain, $\rho$ and $\phi$ are $C^\infty$ as well.

If $(x, y, z) = \Phi(\theta, t)$, then we see that $\theta = \phi(x, y)$ and $2 + t\cos(\frac{1}{2}\theta) = \rho(x, y)$, from which $t$ can also be obtained since $\cos(\frac{1}{2}\theta) \neq 0$. This leads us to conclude that the map $\Psi\colon \mathbb{R}^3 \setminus \{(x, 0, z) \in \mathbb{R}^3 \mid x \leq 0\} \to ]-\pi, \pi[\times\mathbb{R}$ such that

$$\Psi(x, y, z) = \begin{pmatrix} \phi(x, y) \\ \frac{\rho(x,y) - 2}{\cos(\frac{1}{2}\phi(x,y))} \end{pmatrix}$$

is a left-inverse of $\Phi$, i.e. $\Psi \circ \Phi = \mathrm{id}\colon D \to D$. We deduce that $\Phi$ is injective and that its inverse is the restriction $\Psi|_{\Phi(D)}\colon \Phi(D) \to D$.

Since we have described it as a composition of continuous functions, $\Psi$ is also continuous, as is the restriction $\Psi|_{\Phi(D)}\colon \Phi(D) \to D$. We conclude that $\Phi$ is a $C^\infty$ embedding and that its image $\Phi(D)$ is therefore a 2-dimensional $C^\infty$ submanifold of $\mathbb{R}^3$.

(c) Notice that each term in $g$ has factor $(2+t\cos(\frac{1}{2}\theta))$. This implies

$$\begin{aligned} g &= \big(2+t\cos(\tfrac{1}{2}\theta)\big)\left[4\sin\theta+4t\cos\theta\sin(\tfrac{1}{2}\theta)-\sin\theta\left(4+4t\cos(\tfrac{1}{2}\theta)+t^2\right)\right. \\ &\qquad\left. +\,2t\sin(\tfrac{1}{2}\theta)\left(2+t\cos(\tfrac{1}{2}\theta)\right)\right] \\ &= \big(2+t\cos(\tfrac{1}{2}\theta)\big)\left[4t\left(\cos\theta\sin(\tfrac{1}{2}\theta)-\sin\theta\cos(\tfrac{1}{2}\theta)\right)\right. \\ &\qquad\left. -\,t^2\sin\theta+4t\sin(\tfrac{1}{2}\theta)+2t^2\sin(\tfrac{1}{2}\theta)\cos(\tfrac{1}{2}\theta)\right] \\ &= \big(2+t\cos(\tfrac{1}{2}\theta)\big)\left[-4t\sin(\tfrac{1}{2}\theta)-t^2\sin\theta+4t\sin(\tfrac{1}{2}\theta)+t^2\sin\theta\right] \;=\; 0, \end{aligned}$$

since

$$2\sin(\tfrac{1}{2}\theta)\cos(\tfrac{1}{2}\theta)=\sin\theta$$

and

$$\cos\theta\sin(\tfrac{1}{2}\theta)-\sin\theta\cos(\tfrac{1}{2}\theta)=\sin(\tfrac{1}{2}\theta-\theta)=-\sin(\tfrac{1}{2}\theta).$$

We conclude that $g(\Phi(\theta,t))=0$ for all $(\theta,t)\in D$.

(d) One can parametrise the circle $S$ by $f\colon ]-\pi,\pi]\to\mathbb{R}^3, \theta\mapsto(2\cos\theta, 2\sin\theta, 0)$. Note that $f(]-\pi,\pi[)\subseteq\Phi(D)$ because $f(\theta)=\Phi(\theta,0)$ for $\theta\in]-\pi,\pi[$.

The fact that $f$ is continuous then tells us that

$$f(]-\pi,\pi])=f\left(\overline{]-\pi,\pi[}\right)\subseteq\overline{f(]-\pi,\pi[)}\subseteq\overline{V}=M,$$

where $\overline{]-\pi,\pi[}=]-\pi,\pi]$ denotes the closure of $]-\pi,\pi[$ in $]-\pi,\pi]$.

One way to derive this is by writing $\pi=\lim_{n\to\infty}a_n$ for some sequence $(a_n)_{n\in\mathbb{N}}$ with $a_n\in]-\pi,\pi[$, so that $f(\pi)=\lim_{n\to\infty 0}f(a_n)$ by the continuity of $f$. From this we conclude that $f(\pi)$ is a limit point of $f(]-\pi,\pi[)\subseteq V$ and is therefore in the closure $M=\overline{V}$.

The gradient of $g$ can easily be computed, and reads

$$\operatorname{grad} g(x)=\begin{pmatrix} 4x_3-2\,x_1\,x_2+4\,x_1\,x_3 \\ 4-(x_1^2+x_2^2+x_3^2)-2\,x_2^2+4\,x_3\,x_2 \\ 4\,x_1-2\,x_2\,x_3+2\,(x_1^2+x_2^2) \end{pmatrix}$$

By plugging in $x=f(\theta)$, we obtain the expression

$$\operatorname{grad} g(f(\theta)=\begin{pmatrix} -8\cos\theta\sin\theta \\ 4-4-8\sin^2\theta \\ 8\cos\theta+8 \end{pmatrix}=4\begin{pmatrix} -\sin(2\,\theta) \\ \cos(2\theta)-1 \\ 2\,(\cos\theta+1) \end{pmatrix}.$$

The last component is non-zero for all $\theta\in]\pi,\pi[$, while for $\theta=\pi$ all components vanish. Thus, $g$ is a submersion at every point of $S$ except for $f(\pi)=(-2,0,0)$.

This shows that $V$ is a submanifold at every point in $S\cap V$, corroborating the conclusion from part (b).

(e) Here again several approaches are possible.

1. Since we have shown that the function $g$ is a submersion at $x=\Phi(\theta,0)=f(\theta)$ for $\theta\in]-\pi,\pi[$ and $V\subseteq g^{-1}(\{0\})$, we also know that the gradient

grad $g(x)$ is normal to the tangent space $T_{\Phi(\theta,0)}V$. Because grad $g(f(0)) = (0,0,16)$, it follows that also $n_0 = (0,0,1)$ is orthogonal to $T_{\Phi(0,0)}V$.

The function $n$ described in the exercise is obtained by normalising the vectors grad $g(f(\theta))$ for $\theta \in ]-\pi,\pi[$ and setting

$$n(\theta) = \frac{\operatorname{grad} g(f(\theta))}{\|\operatorname{grad} g(f(\theta))\|} = \frac{1}{4\,|\cos(\frac{1}{2}\theta)|}\begin{pmatrix} -\sin(2\,\theta) \\ \cos(2\,\theta) - 1 \\ 2\,(\cos\theta + 1) \end{pmatrix}.$$

A few trigonometric identities have been applied to obtain the final, simplified expression:

$$\begin{aligned}
&\sin^2(2\,\theta) + (\cos(2\,\theta) - 1)^2 + 4\,(\cos\theta + 1)^2 \\
&\qquad = \sin^2(2\theta) + \cos^2(2\theta) - 2\cos(2\theta) + 1 + 4\cos^2\theta + 8\cos\theta + 4 \\
&\qquad = 6 - 2(\cos^2\theta - \sin^2\theta) + 4\cos^2\theta + 8\cos\theta \\
&\qquad = 8 + 8\cos\theta = 16\cos^2(\tfrac{1}{2}\theta).
\end{aligned}$$

We note that $|\cos(\frac{1}{2}\theta)| = \cos(\frac{1}{2}\theta)$ for $-\pi \le \theta \le \pi$, so that the limits $\lim_{\theta\to\pm\pi} n(\theta)$ can be obtained by applying l'Hôpital's rule:

$$\begin{aligned}
\lim_{\theta\to\pm\pi} n(\theta) &= \lim_{\theta\to\pm\pi} \frac{1}{4\cos(\frac{1}{2}\theta)}\begin{pmatrix} -\sin(2\,\theta) \\ \cos(2\,\theta) - 1 \\ 2\,(\cos\theta + 1) \end{pmatrix} \\
&= \lim_{\theta\to\pm\pi} \frac{1}{\frac{\mathrm{d}}{\mathrm{d}\theta}4\cos(\frac{1}{2}\theta)} \frac{\mathrm{d}}{\mathrm{d}\theta}\begin{pmatrix} -\sin(2\,\theta) \\ \cos(2\,\theta) - 1 \\ 2\,(\cos\theta + 1) \end{pmatrix} \\
&= \lim_{\theta\to\pm\pi} \frac{1}{-2\sin(\frac{1}{2}\theta)}\begin{pmatrix} -2\cos(2\,\theta) \\ -2\sin(2\,\theta) \\ -2\,\sin\theta \end{pmatrix}.
\end{aligned}$$

This is just the limit of a continuous function, so we read off that

$$\lim_{\theta\to\pi} n(\theta) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \text{and} \qquad \lim_{\theta\to-\pi} n(\theta) = -\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

2. A somewhat different approach involves the cross product $\partial_\theta\Phi(\theta,t) \times \partial_t\Phi(\theta,t)$ of the partial derivatives of part (a). Because $\Phi$ is an immersion, this cross-product is non-vanishing for every $(\theta,t) \in D$, and is orthogonal to the tangent space $T_{\Phi(\theta,t)}$.

   Since at $\partial_\theta\Phi(0,0) = (0,2,0)$ and $\partial_t\Phi(0,0) = (1,0,0)$, we have $\partial_\theta\Phi(0,0) \times \partial_t\Phi(0,0) = (0,0,-2)$ and we can again conclude that $n_0 = (0,0,1)$ is orthogonal to $T_{\Phi(0,0)}V$.

   Because $\partial_\theta\Phi(0,0) \times \partial_t\Phi(0,0)$ and $n_0$ are pointing in opposite directions, an additional minus sign needs to be introduced in the definition of $n$, so that

$$n(\theta) = \frac{-\partial_\theta\Phi(\theta,0) \times \partial_t\Phi(\theta,0)}{\|\partial_\theta\Phi\theta,0) \times \partial_t\Phi(\theta,0)\|}.$$

   This will lead to the same answer.

(f) The Möbius strip $M$ is a smooth 2-dimensional connected manifold with boundary in $\mathbb{R}^3$. It is similar to a cylinder in the sense that it can be described as the union of a continuous family of line segments over the circle, but these line segments are gradually twisted as one goes around the circle. This happens in such a way that if one follows a line segment around the circle once, its end points are interchanged. (It is a non-trivial fibre bundle.)

The Möbius strip is non-orientable, which can be expressed by saying that it has only 'one side'. This was demonstrated in part (e), where a vector normal to the surface was continuously transported around the loop once and ended up on the 'other side'.