ENDTERM COMPLEX FUNCTIONS

JUNE 29 2011, 9:00-12:00

- Put your name and studentnummer on every sheet you hand in.
- When you use a theorem, show that the conditions are met.

Exercise 1. ***(5 pt)*** Let $U = \{z \in \mathbb{C} : z \neq x,\ x \leq 0\}$. Prove that any continuous closed path γ in U is homologous to 0 in U.

Exercise 2. ***(10 pt)*** Show that equation $z^6 + 4z^2 = 1$ has exactly two simple roots with $|z| < 1$.

Exercise 3. ***(10 pt)*** Let $a > b > 0$. Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{x^3 \sin x}{(x^2 + a^2)(x^2 + b^2)}\, dx.$$

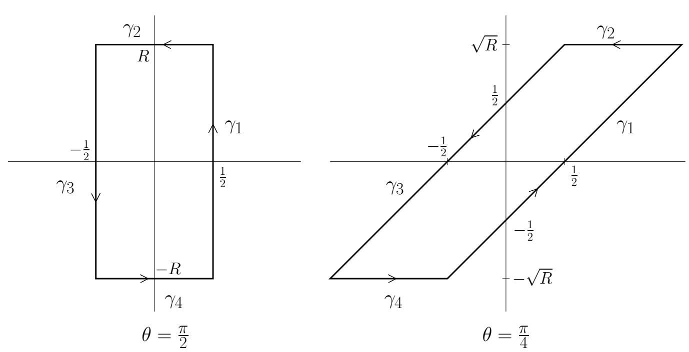
Exercise 4. ***(20 pt)*** Consider the function $f(z) = \dfrac{e^{i\pi z^2}}{\sin(\pi z)}$.

a. (*2 pt*) Find all singular points of f in $\mathbb{C}$ and determine their type. Compute $\mathrm{Res}_0 f$.

b. (*5 pt*) Show that $f(z) - f(z-1) = 2ie^{i\pi z(z-1)}$ for $z \neq 0, \pm 1, \pm 2, \ldots$.

c. (*3 pt*) Let γ be a closed path composed by four straight line segments $\gamma_k, k = 1, 2, 3, 4$, connecting the points

$$\pm\frac{1}{2} \pm Re^{i\theta}, \quad 0 < \theta < \pi,$$



and oriented counterclockwise. Let γ_1 and γ_3 be the right and left sides of γ, respectively. Prove that

$$\int_{\gamma_3} f(z)dz = -\int_{\gamma_1} f(z-1)dz.$$

d. (*5 pt*) Let γ_2 and γ_4 be the top and bottom sides of γ, respectively. Show for $\theta = \frac{\pi}{2}$ and $\frac{\pi}{4}$ that

$$\lim_{R\to\infty} \left| \int_{\gamma_2} f(z)dz + \int_{\gamma_4} f(z)dz \right| = 0.$$

e. (*5 pt*) Use the information obtained for $\theta = \frac{\pi}{2}$ and $\frac{\pi}{4}$ to evaluate the integrals

$$\int_{-\infty}^{\infty} \cos(\pi x^2)dx \qquad \text{and} \qquad \int_{-\infty}^{\infty} \mathrm{e}^{-\pi x^2} dx,$$

respectively.

Bonus Exercise. (*10 pt*) Let $f(z) = \dfrac{1}{z^2 \sin z}$.

a. (*5 pt*) Find poles of f and compute the residue of f at each of them.

b. (*5 pt*) Evaluate

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}.$$