EXAM COMPLEX FUNCTIONS

APRIL 19 2010

- You may do this exam either in English or Dutch.
- Put your name and studentnummer on every sheet you hand in.
- Give only reasoned solutions, but try to be concise.

1. (*3 pt*) Let $U := \{z \in \mathbb{C} | \mathrm{Re}(z) > 0\}$. Suppose f is holomorphic on U and satisfies $f(1) = 0$ and $\mathrm{Re}(f(z)) = \log|z|$. Show that f is unique and determine it.

2. (*5 pt*) Let $d \in \mathbb{N}$. Consider the series $f_d(z) = \sum_{n=0}^{\infty} n^{d-1} z^n$. Show that its radius of convergence is 1 and prove (by induction) that there exists a polynomial p_d of degree at most $d-1$ such that for $|z| < 1$
$$f_d(z) = \frac{p_d(z)}{(1-z)^d}$$
Use the method of generating functions to prove that
$$\sum_{n=0}^{d} \binom{d}{n} (-1)^n n^{d-1} = 0$$

3. (*5 pt*) Let $n \in \mathbb{N}_{\geq 2}$ and find the roots of $z^n + 1$. Show that:
$$\int_0^{\infty} \frac{dx}{x^n + 1} = \frac{\pi/n}{\sin \pi/n}$$
Hint: Use the chain which lies on the boundary of the circular sector with vertices $0, R$ and $Re^{\frac{2\pi i}{n}}$. Show that the integral of $1/(z^n+1)$ over the circular arc approaches 0 as $R \to \infty$. (See page 2 for a picture.)

4. (*3 pt*) Prove the (global) maximum modulus principle.

5. (*4 pt*) Let $U := \{z \in \mathbb{C} | \mathrm{Re}(z) > 0\}$ and assume $f : U \to \mathbb{C}$ is analytic. Suppose that $f(z) = g(z) f(z+1)$ for all $z \in U$ for some analytic function $g : \mathbb{C} \to \mathbb{C}$. Prove there exists an analytic continuation of f to $\mathbb{C}$.

6. *(2 pt Bonus)* Find all analytic functions f on $\mathbb{C}$ with $|f(z)| = |f(|z|)|$.

The chain for exercise 3: