MIDTERM COMPLEX FUNCTIONS

APRIL 20 2011, 9:00-12:00

- Put your name and studentnummer on every sheet you hand in.
- When you use a theorem, show that the conditions are met.

Exercise 1. ***(8 pt)*** Consider the series $\sum_{n=1}^{\infty} \sin(n\phi) z^n$ for some $\phi \in (0, \pi)$.

a. (*3 pt*) Determine it's radius of convergence ρ and show that it's sum equals a rational function f on $|z| < \rho$.

b. (*4 pt*) Prove that for all non-negative integers n we have

$$2 \sum_{k=0}^{n} \sin(k\phi) \sin(k\phi - n\phi) = (n+1)\cos(n\phi) - \frac{\sin(n\phi + \phi)}{\sin(\phi)}$$

Hint: Use the method of generating functions, i.e. consider the series

$$\sum_{n=0}^{\infty} a_n z^n \text{ where } a_n = -4 \sum_{k=0}^{n} \sin(k\phi) \sin(n\phi - k\phi).$$

c. (*1 pt*) Prove that for all integers $n > 2$

$$\sum_{k=0}^{n} \sin^2\left(\frac{2\pi k}{n}\right) = \frac{n}{2}.$$

Exercise 2. ***(8 pt)*** Let $z_1, z_2, \ldots, z_n$ be points on the unit circle in $\mathbb{C}$. Prove that there exists a point z on the unit circle such that

$$|z - z_1| \cdot |z - z_2| \cdots |z - z_n| > 1.$$

Hint: Use the Maximum Modulus Principle.

Exercise 3. ***(10 pt)*** Let U be an open subset of $\mathbb{C}$ and let $f : U \to \mathbb{C}$ be a function satisfying $(f(z))^2 = z$ for all $z \in U$.

a. *(4 pt)* Show there exist $\alpha, \beta : U \to \{-1, 1\}$ such that for all $z \in U \setminus \mathbb{R}$

$$\mathrm{Re} f(z) = \frac{\alpha(z)}{\sqrt{2}} \sqrt{|z| + \mathrm{Re}(z)} \text{ and } \mathrm{Im} f(z) = \frac{\beta(z)}{\sqrt{2}} \sqrt{|z| - \mathrm{Re}(z)}$$

b. *(3 pt)* Show that if on $U \setminus \mathbb{R}$ the Cauchy-Riemann equations are satisfied then $|\mathrm{Im}(z)|\alpha(z) = \mathrm{Im}(z)\beta(z)$ for all $z \in U \setminus \mathbb{R}$.

c. *(3 pt)* Suppose C is a circle of radius R in U centered at the origin. Prove that f is not analytic.

Hint: Why should α be constant on $C \setminus \{-R\}$?

Exercise 4 ***(6 pt)*****.** Let R be a real positive number and let n be a non-negative integer. Calculate

$$\int_0^{2\pi} e^{R\cos(t)} \cos(R\sin(t) - nt)\, dt.$$

Exercise 5 ***(8 pt)*****.** Suppose the power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

has radius of convergence $\rho > 0$. Prove that f is analytic on the open disk $D(0, \rho)$, without using the equivalence "holomorphic" $\Leftrightarrow$ "analytic".

Hint: Use the binomial formula.