MIDTERM COMPLEX FUNCTIONS

APRIL 18 2012, 9:00-12:00

- Put your name and studentnummer on every sheet you hand in.
- When you use a theorem, show that the conditions are met.

Exercise 1 ***(7 pt)*** Let $a, b, c \in \mathbb{C}$ be located on the unit circle and let $a + b + c = 0$. Prove that the corresponding points are the vertices of an equilateral triangle.

Exercise 2 ***(10 pt)*** Write the Cauchy-Riemann equations in polar coordinates (r, θ). Then show that the function $\log z = \log r + i\theta,\ z = re^{i\theta}$, is holomorphic in the region $r > 0,\ -\pi < \theta < \pi$.

Exercise 3 ***(10 pt)*** Suppose $f : U \to \mathbb{C}$ is a non-constant holomorphic function on an open set $U \subset \mathbb{C}$ containing the closed unit disc $\overline{D(0,1)}$. Suppose that $|f(z)| = 1$ for all $z \in \mathbb{C}$ with $|z| = 1$. Prove that the equation $f(z) = 0$ has a solution in the open unit disc $D(0, 1)$.

Exercise 4 ***(8 pt)*** Compute

$$\int_\gamma \frac{\sin z}{z^2}\, dz \quad \text{and} \quad \int_\gamma \frac{\cos z}{z^3}\, dz,$$

where γ is the unit circle $|z| = 1$ oriented counter-clockwise and traced once.

Exercise 5 ***(10 pt)*** Suppose that a complex function f has a power series representation near the origin, i.e. there is a power series $\sum_{n=0}^{\infty} a_n z^n$ that converges absolutely to $f(z)$ in an open disc centered at $z = 0$.

(i) Assuming that $a_0 \neq 0$, prove that the function

$$g(z) = \frac{1}{f(z)}$$

also has has a power series representation near the origin.

(ii) Derive explicit formulas for the coefficients b_0, b_1, b_2, and b_3 in the series $\sum_{n=0}^{\infty} b_n z^n$ representing the function g near the origin.