**RETAKE COMPLEX FUNCTIONS**

August 22 2012, 9:00-12:00

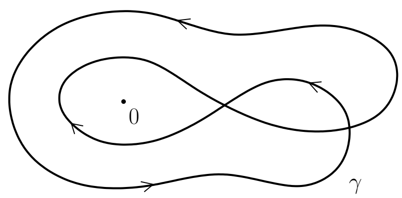
- Put your name and studentnummer on every sheet you hand in.
- When you use a theorem, show that the conditions are met.

**Exercise 1.** ***(20 pt)*** Let $k > 0$ with $k \neq 1$ and $\alpha, \beta \in \mathbb{C}$ with $\alpha \neq \beta$. Prove that the set

$$\{z \in \mathbb{C} : |z - \alpha| = k|z - \beta|\}$$

is a circle and find its center and radius.

**Exercise 2.** ***(10 pt)*** Compute $\displaystyle\int_\gamma \frac{e^z - e^{-z}}{z^4} dz$ along the path shown below.




**Exercise 3.** ***(20 pt)*** Let $w = f(z)$ be an analytic function in a neighbourhood of $z = 0$ with $f(0) = 0$ but $f'(0) \neq 0$.

**a.** (*15 pt*) Suppose that $f$ has a series representation

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

near $z = 0$. Derive explicit expressions for the coeffcients $b_n$ with $n \leq 5$ in the series representation of the inverse function $g = f^{-1}$ near $w = 0$:

$$g(w) = \sum_{n=0}^{\infty} b_n w^n.$$

**b.** (*5 pt*) Prove that

$$\arcsin w = w + \frac{1}{6}w^3 + \frac{3}{40}w^5 + O(|w|^7).$$

**Turn the page!**

**Exercise 4.** ***(20 pt)*** Let $f(z) = z^6 - 5z^4 + 10$.

**a.** (*15 pt*) Prove that $f$ has

(i) no zeroes with $|z| < 1$;
(ii) 4 zeroes with $|z| < 2$;
(iii) 6 zeroes with $|z| < 3$;

**b.** (*5 pt*) For cases (ii) and (iii), show that all zeroes are different.

**Exercise 5.** ***(30 pt)*** Let $a \in \mathbb{R} \setminus \mathbb{Z}$. Consider the function

$$f(z) = \frac{\cos(\pi z)}{(z-a)^2 \sin(\pi z)}.$$

**a.** (*10 pt*) For integer $n \geq 0$, introduce the rectangular closed path $\gamma_n$ passing through the points

$$\pm\left(n + \frac{1}{2}\right) \pm in$$

and oriented couter-clockwise. Prove by integral estimates that

$$\left|\int_{\gamma_n} f(z)dz\right| \to 0 \quad \text{as} \quad n \to \infty.$$

**b.** (*10 pt*) Find poles of $f$ and compute the residue of $f$ at each of them.

**c.** (*5 pt*) Show that

$$\int_{\gamma_n} f(z)dz = 2\pi i \left(-\frac{\pi}{\sin^2(\pi a)} + \sum_{k=-n}^{n} \frac{1}{\pi(k-a)^2}\right).$$

**d.** (*5 pt*) Compute $\displaystyle\sum_{n=-\infty}^{\infty} \frac{1}{(n-a)^2}$.

**Bonus Exercise.** ***(20 pt)*** Suppose that $f$ is an entire function and for every point $a \in \mathbb{C}$ at least one of the coefficients $c_n$ in the Taylor series

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$$

vanishes. Prove that $f$ is a polynomial.