RETAKE COMPLEX FUNCTIONS

August 24 2011, 9:00-12:00

- Put your name and studentnummer on every sheet you hand in.
- When you use a theorem, show that the conditions are met.

Exercise 1. ***(30 pt)*** Given complex numbers a_0, a_1, u_1, u_2, define a_n for $n \geq 2$ by the difference equation:

$$a_n = u_1 a_{n-1} + u_2 a_{n-2}. \tag{1}$$

Assume that the polynomial $z^2 - u_1 z - u_2$ has two roots $\alpha, \beta \in \mathbb{C}$ such that $|\alpha| < |\beta|$.

a. (*10 pt*) Show that $a_n = A\alpha^n + B\beta^n$ satisfies (1) with suitable $A, B \in \mathbb{C}$.

b. (*10 pt*) Determine the radius of convergence of the power series

$$\sum_{n=0}^{\infty} a_n z^n. \tag{2}$$

c. (*10 pt*) Find a rational function $f(z)$ for which (2) is the power series expansion at $z = 0$.

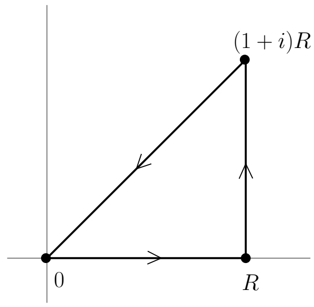
Exercise 2. ***(20 pt)*** Show that all solutions of the equation $z^4 = 6z + 3$ satisfy $|z| < 2$.

Exercise 3. ***(20 pt)*** Prove that

$$\int_0^\infty \sin(\pi x^2)\ dx = \int_0^\infty e^{-2\pi x^2} dx\ .$$

Hint: Consider the integral of an appropriate analytic function $f(z)$ over the following closed path:

Continued on p.2



Exercise 4. ***(30 pt)*** Let $a \in \mathbb{R} \setminus \mathbb{Z}$. Consider the function

$$f(z) = \frac{\cos(\pi z)}{(z-a)^2 \sin(\pi z)}.$$

a. (*10 pt*) For integer $n \geq 0$, introduce the rectangular closed path γ_n passing through the points

$$\pm\left(n + \frac{1}{2}\right) \pm in$$

and oriented couter-clockwise. Prove by integral estimates that

$$\left| \int_{\gamma_n} f(z)dz \right| \to 0 \quad \text{as} \quad n \to \infty.$$

b. (*10 pt*) Find poles of f and compute the residue of f at each of them.

c. (*5 pt*) Show that

$$\int_{\gamma_n} f(z)dz = 2\pi i \left(-\frac{\pi}{\sin^2(\pi a)} + \sum_{k=-n}^{n} \frac{1}{\pi(k-a)^2} \right).$$

d. (*5 pt*) Compute $\displaystyle\sum_{n=-\infty}^{\infty} \frac{1}{(n-a)^2}$.

Bonus Exercise. ***(20 pt)*** Is there an analytic function f on $\mathbb{C} \setminus [-1,1]$ that satisfies

$$e^{f(z)} = \frac{z+1}{z-1} \ ?$$