

TUTORIAL IV:  
Two-parameter bifurcation analysis of equilibria and limit  
cycles with MATCONT

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This session is devoted to the numerical continuation of codim 1 bifurcations of equilibria and limit cycles in systems of autonomous ODEs depending on two parameter

$$\dot{x} = f(x, \alpha), \quad x \in \mathbb{R}^n, \alpha \in \mathbb{R}^2,$$

and detection of their codim 2 bifurcations. We will also switch at some of codim 2 equilibrium bifurcations to the continuation of codim 1 bifurcation curves rooted there.

## 1 Bifurcations of equilibria in the Bykov–Yablonskii–Kim model

We will use MATCONT to continue equilibria and their bifurcations in the following chemical model that describes CO oxidation on platinum:

$$\begin{cases} \dot{x} &= 2k_1z^2 - 2k_{-1}x^2 - k_3xy \\ \dot{y} &= k_2z - k_{-2}y - k_3xy \\ \dot{s} &= k_4z - k_{-4}s, \end{cases}$$

where  $z = 1 - x - y - s$ . The ratio

$$K = \frac{k_{-4}}{k_4}$$

and the parameter  $k_2$  will be used as two bifurcation parameters.

### 1.1 Specify the model in MATCONT

Specify a new ODE system in MATCONT with the coordinates  $(x, y, s)$  and time  $(t)$ :

```
z=1-x-y-s
x'=2*Q1*z^2-2*Q5*x^2-Q3*x*y
y'=Q2*z-Q6*y-Q3*x*y
s'=Q4*z-K*Q4*s
```

The reaction rates  $k_1, k_2, k_3, k_4, k_{-1}, k_{-2}$  are denoted by Q1, Q2, Q3, Q4, Q5, Q6, respectively, while K stands for  $K$ . Generate the derivatives of order 1, 2, and 3 symbolically.

### 1.2 Equilibrium continuation

Select **Type|Initial point|Equilibrium**. In the **Starter** window input the following numerical values, corresponding to a stable equilibrium in the system:

x	0.001137
y	0.891483
s	0.062345
Q1	2.5
Q2	2.204678
Q3	10
Q4	0.0675
Q5	1
Q6	0.1
K	0.4

Activate the parameter Q2.

In the **Continuer** window, decrease the maximal step size along the curve to

MaxStepsize	0.025
-------------	-------

Open a **2Dplot** window with  $(Q2, x)$  on the axes and input the following visibility limits:

Abscissa:            0.5                    2.0  
 Ordinate:            0.0                    0.16

Also open a **Numeric** window and make **EIGENVALUES** visible using **Window|Layout**.

Start the continuation of the equilibrium curve with the **Compute|Forward** command. Monitor the eigenvalues in the **Numeric** window. There will be four bifurcation points detected: two *Hopf points* (labeled by H) and two *fold points* (labeled by LP). Resume computations at each bifurcation point, and stop them when the curve leaves the window. You should get the central S-shaped curve presented in Figure 1. In a narrow interval of  $Q2$ -values (between two LP-points) the system has three equilibria.

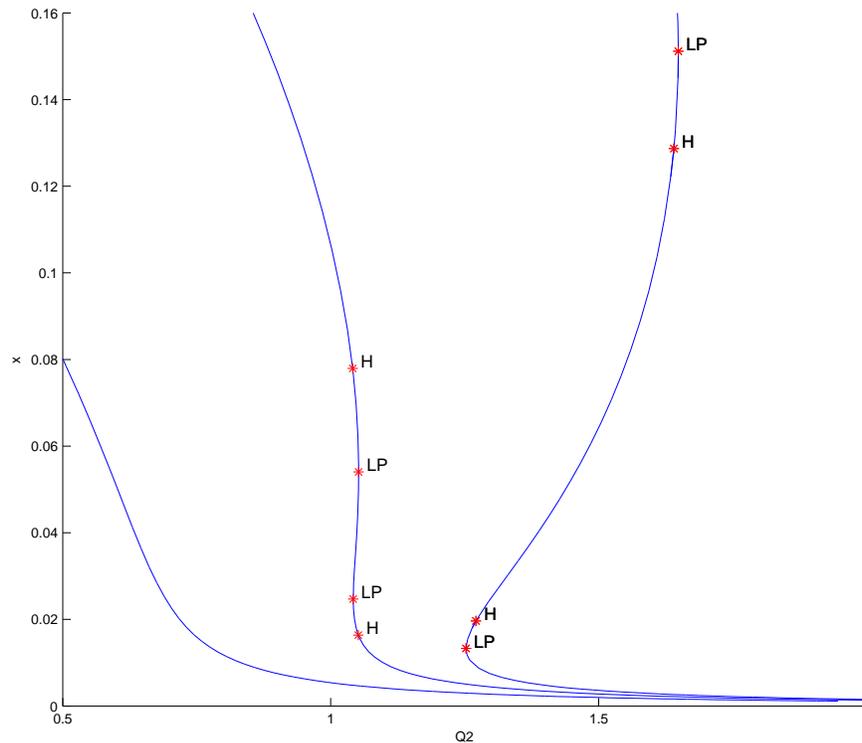


Figure 1: Equilibrium curves with  $K = 0.15, 0.4,$  and  $2.0$

The normal form coefficients for the Hopf and limit point bifurcations can be read in the MATLAB Command Window:

```
label = H , x = ( 0.016358 0.523971 0.328337 1.051557 )
First Lyapunov coefficient = 1.070086e+01
label = LP, x = ( 0.024717 0.450257 0.375018 1.042049 )
a=1.166509e-01
label = LP, x = ( 0.054030 0.302241 0.459807 1.052200 )
a=-1.346534e-01
label = H , x = ( 0.077929 0.233063 0.492149 1.040991 )
First Lyapunov coefficient = 4.332247e+00
```

Note that both first Lyapunov coefficients are positive (implying that unstable limit cycles bifurcate there), while the LP-coefficients are both nonzero.

Rename the computed equilibrium curve via **Select|Curve** and **Actions|Rename** into

**Equilibrium1(+)**

Change the value of the parameter  $K$  in the **Starter** window:

K            0.15

and repeat the computations with **Compute|Forward**. You will get a monotone curve without bifurcation points (the left curve in Figure 1). There is only one equilibrium for all parameter values  $Q2$ .

Finally, change the value of  $K$  to

K            2.0

and click on **Compute|Forward** again. Another equilibrium curve will come out with the order of the singularities reversed (the right curve in Figure 1). There is now a big interval of the parameter values, in which there are three equilibria in the system.

In the MATLAB Command Window, the following messages appeared:

```
label = LP, x = ( 0.013310 0.771372 0.071773 1.252630 )
a=1.705788e+00
label = H , x = ( 0.019639 0.726320 0.084680 1.271108 )
Neutral saddle
label = H , x = ( 0.128699 0.384106 0.162399 1.640254 )
Neutral saddle
label = LP, x = ( 0.151168 0.344167 0.168222 1.648672 )
a=1.787355e+00
```

indicating that the LP-bifurcations are still nondegenerate ( $a \neq 0$ ), while – instead of two Hopf points – two *neutral saddles* have appeared, which are not bifurcation points. Note that they are still marked by H.

Store the last two computed curves permanently by renaming them into

**Equilibrium2(+)**

**Equilibrium3(+)**

via **Select|Curve** and **Actions|Rename**. Close the **Curves** window.

### 1.3 Continuation of fold and Hopf bifurcations

This section explains how to continue fold and Hopf bifurcations with respect to two control parameters. While doing so, you will see how MATCONT detects some codim 2 bifurcations and computes their normal form coefficients.

#### 1.3.1 Fold continuation

Select the second LP: **Limit point** in the first computed curve **Equilibrium1(+)** via **Select | Initial point** menu in the appearing **Initial points** window. The new **Starter** window corresponding to the continuation of a limit point curve **LP\_LP** is to appear, in which you should keep parameter **Q2** active and activate parameter **K**, see Figure 2. Compute the limit point curve in both directions to get Figure 3. Notice that both parameters (**Q2**,**K**) are varied along the curve. Each point in the curve is a limit point for the equilibrium curve at the corresponding value of **K**.

There were detected three points, corresponding to codim 2 bifurcations: two *Bogdanov-Takens* (BT) and one *cusp* (CP). At each BT point the system has an equilibrium with a double zero eigenvalue, while at the CP point there is an equilibrium with a simple zero eigenvalue but zero coefficient of the fold normal form. The normal form coefficients ( $a, b, c$ ) of these bifurcations are reported in the MATLAB Command Window and they are all nonzero:

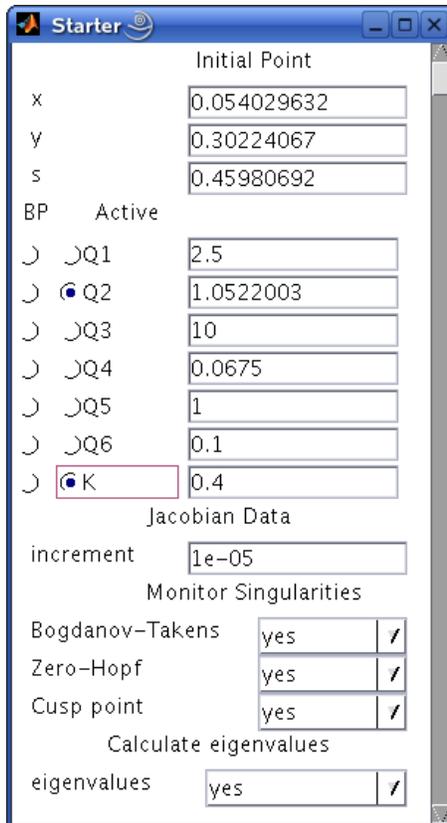


Figure 2: **Starter** window for the continuation of the limit point curve.

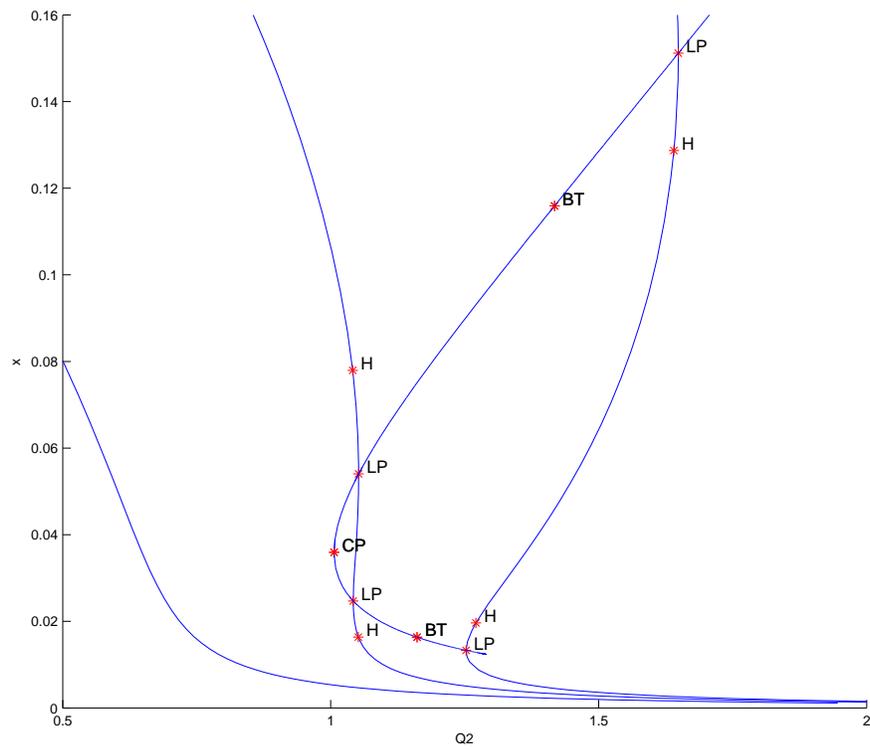


Figure 3: Limit point curve in Bykov's model: BT - Bogdanov-Takens points; CP - cusp.

```
label = BT , x = ( 0.115909 0.315467 0.288437 1.417628 0.971397 )
(a,b)=(8.378442e-02, 2.136280e+00)
```

```
label = CP , x = ( 0.035940 0.352008 0.451368 1.006408 0.355991 )
c=3.627844e-01
```

```
label = BT , x = ( 0.016337 0.638410 0.200456 1.161199 0.722339 )
(a,b)=(-4.822563e-02, -1.937632e+00)
```

Rename the obtained limit point branches as following:

Fold(+)

Fold(-)

### 1.3.2 Hopf continuation

The Bogdanov-Takens points are common points for the limit point curves and curves corresponding to equilibria with eigenvalues  $\lambda_1 + \lambda_2 = 0, \lambda_3 \neq 0$ . Actually, at each BT point, the Hopf bifurcation curve (with  $\lambda_{1,2} = \pm i\omega_0, \omega_0 > 0$ ) turns into the neutral saddle curve (with real  $\lambda_1 = -\lambda_2$ ). Thus, we can start a Hopf curve from a Bogdanov-Takens point.

Select the BT: **Bogdanov-Takens point** in the curve **Fold(+)** as initial. Then choose **Type** |**Curve**|**Hopf** to prepare for the continuation of the Hopf curve. The new **Starter** window for the continuation of a Hopf curve will appear automatically, see Figure 4. Check that the type of

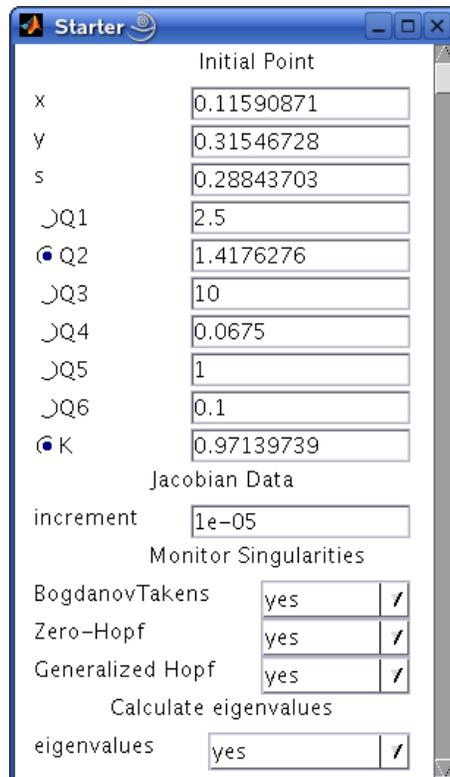


Figure 4: **Starter** window for the continuation of the Hopf curve from the BT-point.

the curve to be computed is changed in the **MatCont** window.

**Compute**|**Forward** with **Compute** |**Extend** produces a *closed* Hopf curve as in Figure 5. Actually, only the part of it to the left of two BT points corresponds to a Hopf bifurcation; the

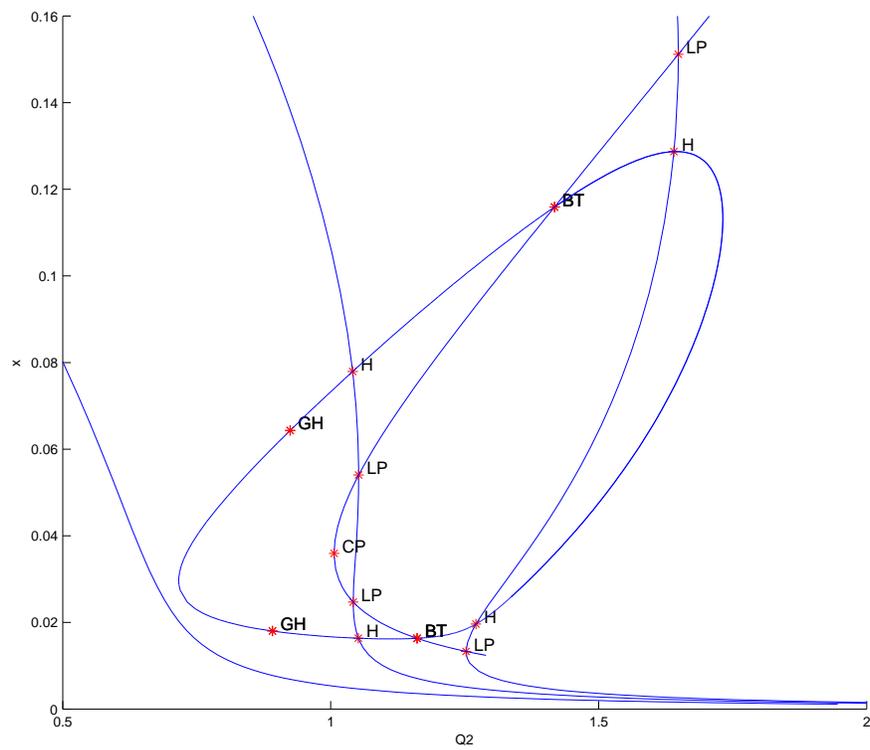


Figure 5: The parameter plot with the added Hopf curve in Bykov's model: GH - generalized Hopf points

other part represents a neutral saddle. As it has been mentioned, this transition happens at the Bogdanov-Takens points. There are two more codim 2 bifurcation points at the Hopf point: Two *Bautin* or *generalized Hopf* (GH) points, where the first Lyapunov coefficient  $l_1$  vanishes.

The Command Window of MATLAB shows the following messages related to the GH points:

```
label = GH, x = ( 0.018022 0.368238 0.497968 0.891319 0.232487 0.003324 )
l2=-7.768996e+02
label = GH, x = ( 0.064311 0.211095 0.554870 0.924255 0.305879 0.003512 )
l2=-2.401234e+02
```

indicating that both GH points are nondegenerate, since the second Lyapunov coefficients  $l_2$  are nonzero.

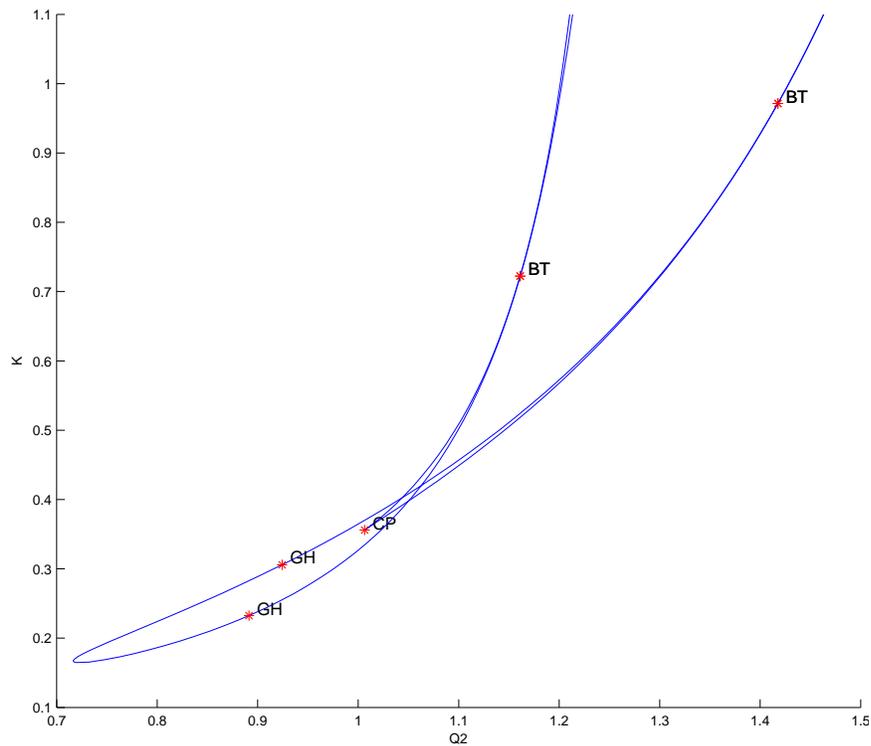


Figure 6: Fold and Hopf curves in Bykov's model: BT- Bogdanov-Takens, CP -cusp, GH - generalized Hopf points

### 1.3.3 Two-parameter bifurcation diagram

Rename the obtained Hopf curve into

Hopf (+)

and delete all computed equilibrium curves.

Open a new **2Dplot** window to plot the bifurcation diagram in the (Q2,K)-plane with the visibility limits

Q2	0.7	1.5
K	0.1	1.1

Upon redrawing the diagram in the (Q2, k)-window, you will get Figure 6.

The edge between two fold curves is a parameter region where the system has three equilibria. Crossing a Hopf curve results in the appearance of either stable or unstable periodic orbit. There are other bifurcations in the model.

## 2 Fold and torus bifurcations of cycles in the Steinmetz–Larter model

Consider the following chemical model:

$$\begin{cases} \dot{A} &= -k_1ABX - k_3ABY + k_7 - k_{-7}A, \\ \dot{B} &= -k_1ABX - k_3ABY + k_8, \\ \dot{X} &= k_1ABX - 2k_2X^2 + 2k_3ABY - k_4X + k_6, \\ \dot{Y} &= -k_3ABY + 2k_2X^2 - k_5Y. \end{cases}$$

We will study bifurcations of limit cycles of this model when parameters  $(k_7, k_8)$  vary<sup>1</sup>.

### 2.1 System specification

Specify a new ODE system – say `STLAR` – in `MATCONT`

```
A' = -k1*A*B*X - k3*A*B*Y + k7 - km7*A
B' = -k1*A*B*X - k3*A*B*Y + k8
X' = k1*A*B*X - 2*k2*X^2 + 2*k3*A*B*Y - k4*X + k6
Y' = -k3*A*B*Y + 2*k2*X^2 - k5*Y
```

where  $(A, B, X, Y)$  are the coordinates and  $(k_1, k_2, k_3, k_4, k_5, k_6, k_7, km_7, k_8)$  are the parameters. Use (default) `t` for time and generate symbolically partial derivatives of order 1, 2, and 3.

### 2.2 Preliminary one-parameter analysis

#### 2.2.1 Continuation of an equilibrium

To begin with, we continue an equilibrium of the model and detect its Hopf bifurcation.

Input **Type|Initial point|Equilibrium** in the main `MatCont` window.

Input the following numerical data in the appearing **Starter** window:

```
A          31.78997
B           1.45468
X           0.01524586
Y           0.1776113
k1          0.1631021
k2          1250
k3          0.046875
k4          20
k5          1.104
k6          0.001
k7          4.235322
km7         0.1175
k8          0.5
```

<sup>1</sup>If you have worked out TUTORIAL III, go directly to Section 2.3.

These values correspond to an equilibrium  $(A, B, X, Y)$  in the system. Activate the parameter  $k_7$ .

Open **Window|Numeric** and change its appearance via the **Window|Layout** command. Namely, select **EIGENVALUES** to be shown in the window.

Start **Compute|Forward**. The equilibrium curve will be continued and you get a Hopf bifurcation. The message in the MATLAB Command Window

```
label = H , x = ( 34.808899 1.328517 0.015246 0.177611 4.590046 )
First Lyapunov coefficient = 1.527549e-02
```

at  $k_7 = 4.590046\dots$  indicates a *subcritical* Hopf bifurcation. Indeed, there are two eigenvalues of the equilibrium with  $\text{Re } \lambda_{1,2} \approx 0$  at this parameter value visible in the **Numeric** window. The critical frequency  $\text{Im } \lambda_1 \neq 0$ , while the first Lyapunov coefficient is positive. Thus, there should exist an unstable limit cycle, bifurcating from the equilibrium. Stop computations and rename the computed curve via **Select|Curve** and **Actions|Rename** into

**Equilibrium(+)**

**Compute|Backward** and **Compute|Extend** once until you get a *Hopf bifurcation* at  $k_7 = 0.712475\dots$  with the message

```
label = H , x = ( 1.808301 25.573303 0.015246 0.177611 0.712475 )
First Lyapunov coefficient = -2.371880e-02
```

in the MATLAB Command Window. The first Lyapunov coefficient is negative now. This means that a *stable* limit cycle bifurcates from the equilibrium, when it loses stability. Stop computations and rename the computed curve into **Equilibrium(-)**.

## 2.2.2 Cycle continuation

Select the **H: Hopf** point in the curve **Equilibrium(-)** as initial. **MATCONT** will prepare to continue a limit cycle curve from the Hopf point (curve type **H\_LC**). Choose the **yes**-option to monitor all singularities in the **Starter** window. Set

```
MaxStepsize          1
MaxNumPoints         25
```

in the **Continuer** window. Change the appearance of the **Numeric** window via the **Window|Layout** command by selecting **MULTIPLIERS** to be shown.

Click **Compute|Forward** to start the continuation of the limit cycle. At  $k_7 = 0.716434\dots$  the message **Neimark-Sacker** indicates a *torus* bifurcation. In the MATLAB Command Window, the following message appears:

```
Neimark-Sacker (period = 1.091213e+01, parameter = 7.164336e-01)
Normal form coefficient = -1.405969e-06
```

Indeed, there are two complex multipliers with (approximately)  $|\mu| = 1$  and one trivial multiplier (approximately) equal to 1. This can be seen in the **Numeric** window. The normal form coefficient is small but negative, indicating that a stable two-dimensional *invariant torus* bifurcates from the limit cycle.

Rename the computed curve into **cycle**.

## 2.3 Two-parameter analysis

### 2.3.1 Continuation of the Hopf bifurcation curve

Select the **H: Hopf** point in the computed **Equilibrium(+)** curve using the **Select|Initial point** command. Select **Type|Curve|Hopf**. Activate one more parameter, namely  $k_8$ , in the **Starter** window.

Open a new **2Dplot** window with the variables  $k_7$  and  $k_8$  as abscissa and ordinate, respectively, and the visibility limits

Abscissa	0	7
Ordinate	0.3	1

Continue the Hopf bifurcation curve with **Compute|Forward** followed by **Compute|Extend**.  
At

$$(k_7, k_8) = (6.336044 \dots, 0.413039 \dots)$$

a *generalized Hopf* bifurcation will be found, where the first Lyapunov coefficient vanishes (label **GH**). Resume the continuation and terminate it when the Hopf curve leaves the window. In the MATLAB Command Window, the following message appears:

```
label = GH, x = ( 50.408556 0.979006 0.013424 0.131840 6.336044 0.413039 0.594360 )
12=3.326157e-03
```

from which follows that this codim 2 bifurcation is nondegenerate (the second Lyapunov coefficient 12 is nonzero). Rename the computed curve in **Hopf(+)**.

The backward continuation of the Hopf bifurcation curve brings in one more generalized Hopf point at

$$(k_7, k_8) = (0.999480 \dots, 0.645896 \dots).$$

Resume and extend until the window is left. You should get Figure 7.

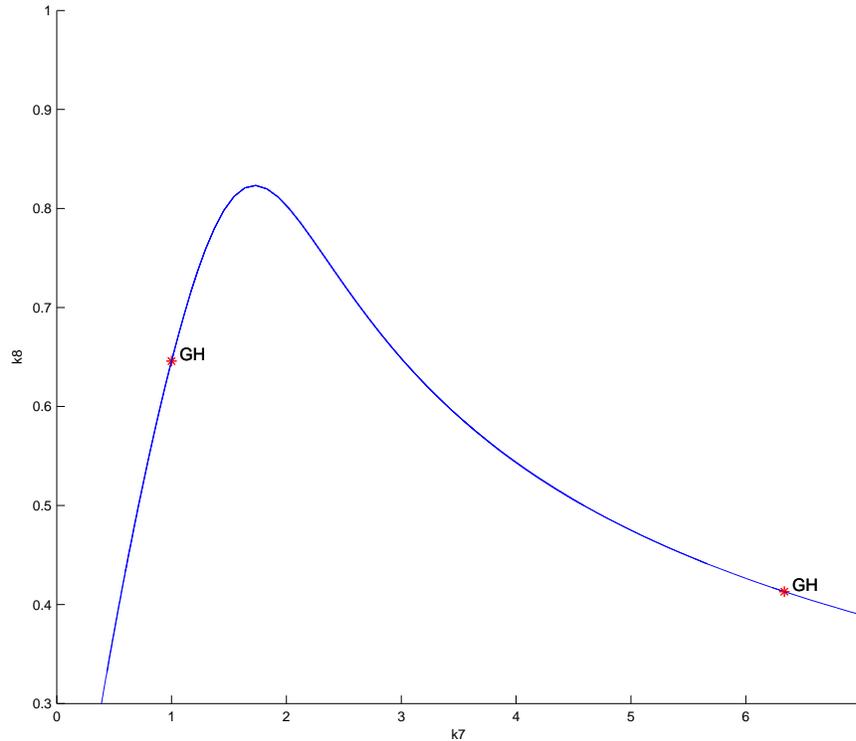


Figure 7: The Hopf bifurcation curve: GH's - two generalized Hopf points.

The MATLAB Command Window contains the message:

```
label = GH, x = ( 3.009221 14.184417 0.017978 0.260260 0.999480 0.645896 0.292299 )
12=-4.044044e-03
```

indicating that this GH is also nondegenerate. Rename this curve in **Hopf(-)**.

### 2.3.2 Continuation of the LPC-bifurcation starting at the GH-point

Click **Select|Initial point** and select GH: Generalized Hopf point in the Hopf(+) curve as initial. Since it is known that an LPC curve originates there, select **Type|Curve|Limit point of cycles** to continue. The **Starter** and **Continuer** windows for the continuation of the fold bifurcation of cycles from the generalized Hopf point will appear.

In the **Starter** window, set **yes** to monitor for all singularities except the **Cusp point** of cycles and decrease the **amplitude** of the predicted LPC to 0.01. Check that **ntst** is 20 and **ncol** is 4. In the **Continuer** window, set **MaxStepsize** equal to 2.0 and **MaxNumPoints** equal to 300. The **Starter** and **Continuer** windows should look like in Figure 8.

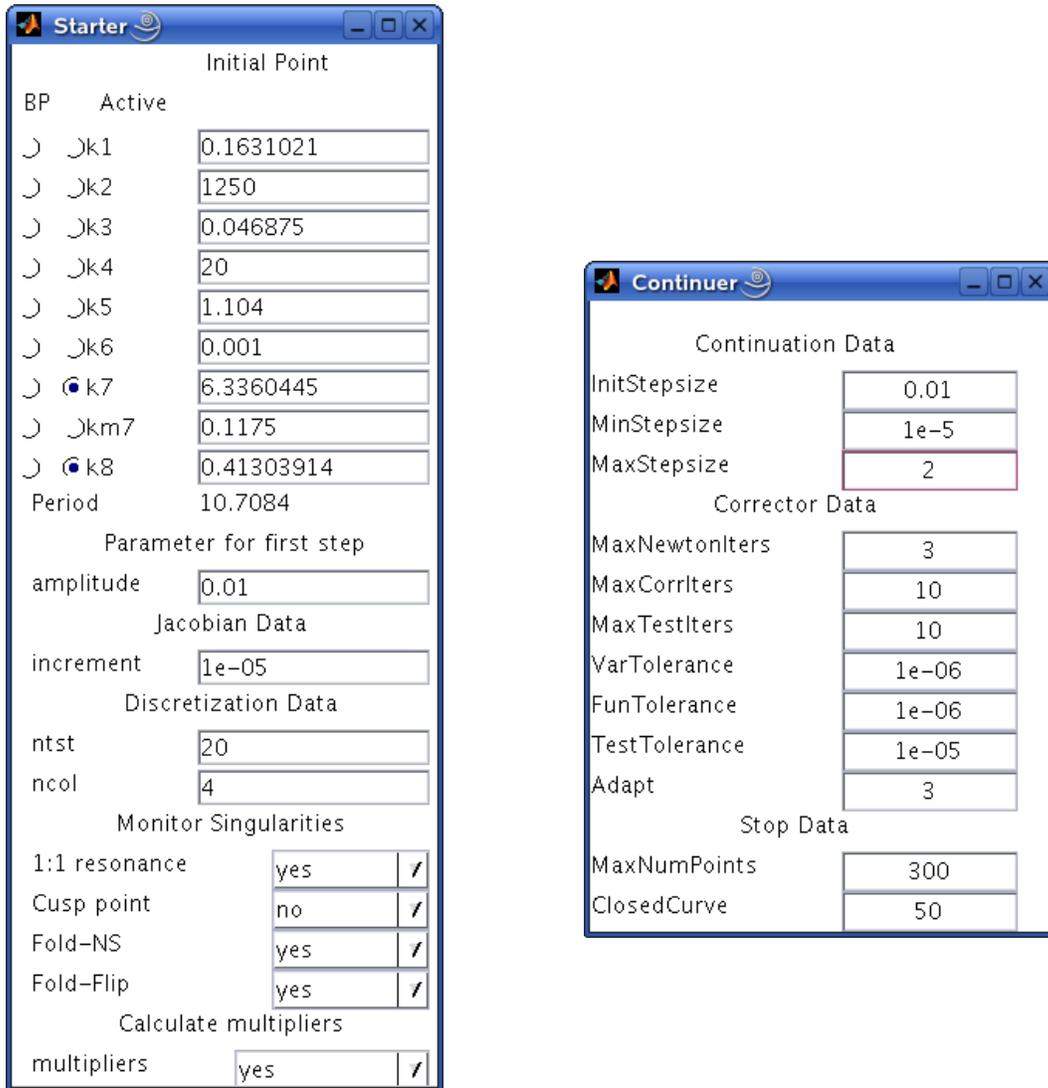


Figure 8: The **Starter** and **Continuer** windows for the LPC-continuation from the GH-point.

**Compute|Forward** the LPC-bifurcation curve (with **Compute|Extend**) and observe that it actually connects the two found GH points in the Hopf curve, see Figure 9. The continuation (which takes some time) stops with the message **Step size too small** near the left GH point.

The computed curve will also contains two extra codim 2 points labeled by R1. These are **strong resonance 1:1** points, where the cycle has a triple multiplier 1 (counting the trivial

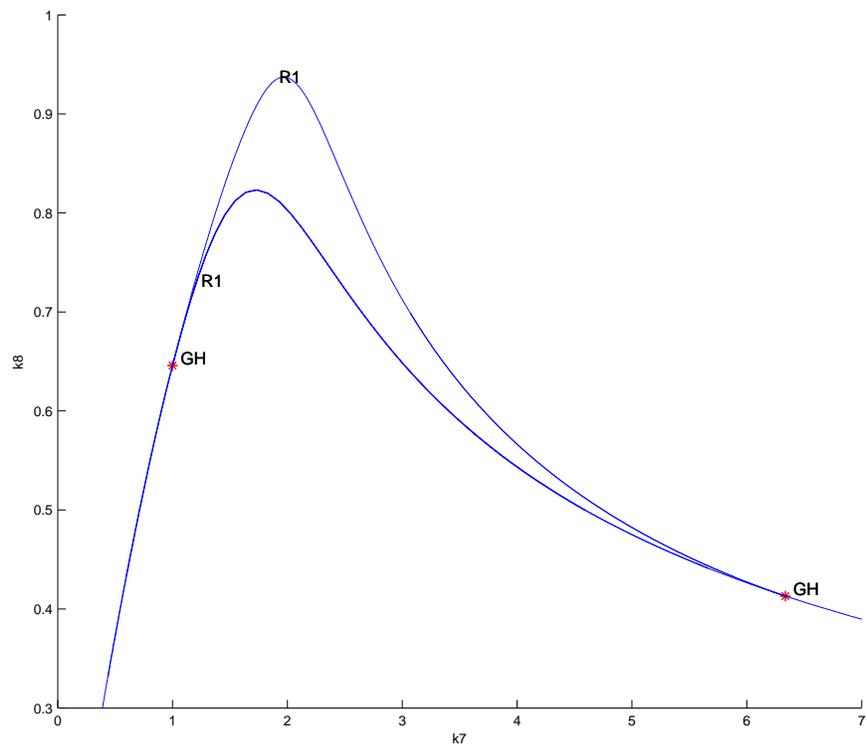


Figure 9: Hopf and limit point of cycles curves connecting the GH-points.

multiplier). In the **Numeric** window, you can read the corresponding values of  $(k_7, k_8)$ . The MATLAB Command Window gives the following messages:

```
label = R1 (period = 1.400190e+01, parameters = 1.857676e+00, 9.304221e-01)
label = R1 (period = 1.239456e+01, parameters = 1.179554e+00, 7.239571e-01)
```

with the parameter values also given.

Rename the computed GH\_LPC curve into `cyclefold`.

### 2.3.3 Continuation of the NS-curve in two parameters

Take the NS: Neimark-Sacker point in the limit cycle curve `cycle` as initial. To speed up the continuation, set `no` to monitor all singularities and compute the multipliers.

Change the plotting region of the **2Dplot** window to

```
Abscissa      1          2.2
Ordinate      0.6      1
```

and redraw the diagram.

Click **Compute|Forward** and wait until the computed NS-curve enters the window (this takes some time). Extend the computation several times to obtain Figure 10.

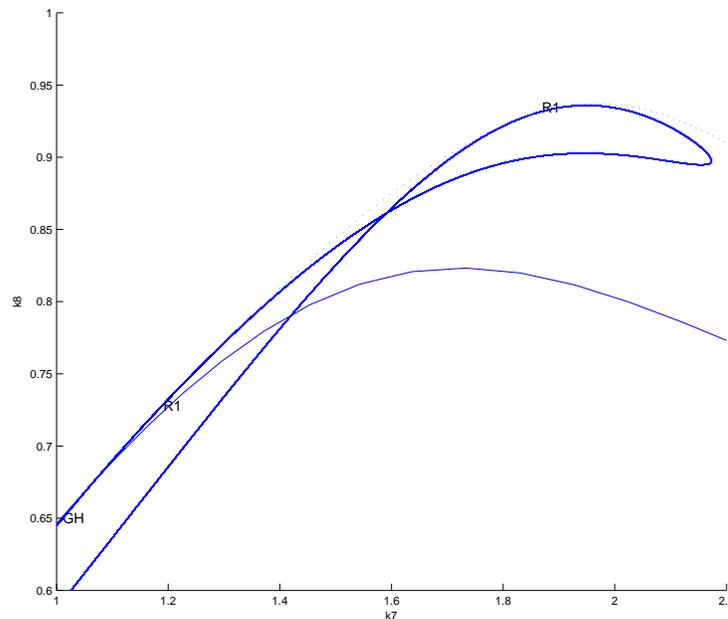


Figure 10: Hopf, limit point of cycles and Neimark-Sacker curves: The NS-curve between two R1-points corresponds to neutral saddle cycles.

The computed NS-curve passes through the R1 points of strong resonance 1:1, where it tangentially meets the LPC-curve and turns from the Neimark-Sacker bifurcation curve into the neutral saddle cycle curve (both are characterised by the presence of two multipliers with  $\mu_1\mu_2 = 1$ ).

**Warning:** There are many other bifurcations in the model and Figure 10 shows only few of them.

### 3 Additional Problems

A. Consider the following system by Lorenz [1984]:

$$\begin{cases} \dot{x} &= -y^2 - z^2 - ax + aF, \\ \dot{y} &= xy - bxz - y + G, \\ \dot{z} &= bxy + xz - z, \end{cases} \quad (1)$$

where  $(a, b, F, G)$  are parameters.

1. Using MATCONT, compute fold and Hopf bifurcation curves for equilibria of (1) with  $a = \frac{1}{4}, b = 4$ , in the parameter domain

$$\{(G, F) : 0 \leq G \leq 3, 0 \leq F \leq 3\}.$$

Find numerical parameter values  $(G_{\text{ZH}}, F_{\text{ZH}})$  at which (1) exhibits a *fold-Hopf bifurcation* (i.e., has an equilibrium with eigenvalues  $\lambda_1 = 0, \lambda_{2,3} = \pm i\omega_0, \omega_0 > 0$ ). Find numerical parameter values  $(G_{\text{CP}}, F_{\text{CP}})$  at which (1) exhibits a *cusp bifurcation* (i.e., has a triple equilibrium).

2. Derive analytical expressions for  $(G_{\text{ZH}}, F_{\text{ZH}})$  and  $(G_{\text{CP}}, F_{\text{CP}})$  as functions of  $(a, b)$  and verify the numerical results above.

*Hints:* At the fold-Hopf both the trace and the determinant of the Jacobian matrix of (1) vanish. To find a triple equilibrium, reduce the equilibrium system for (1) to one cubic equation for the  $x$ -coordinate of the equilibrium and look for its triple root.

B. Consider the *adaptive control system of Lur'e type*

$$\begin{cases} \dot{x} &= y, \\ \dot{y} &= z, \\ \dot{z} &= -\alpha z - \beta y - x + x^2. \end{cases} \quad (2)$$

1. Continue the equilibrium  $(x, y, z) = (0, 0, 0)$  of (2) with respect to parameter  $\alpha$  starting at  $\alpha = 2, \beta = 1$  and detect its Hopf bifurcation.
2. Compute the Hopf bifurcation curve in the  $(\alpha, \beta)$ -plane and plot it using the visibility limits

alpha	0	1.5
beta	0	2

3. Find the period-doubling (PD) of the cycle bifurcating from the Hopf point and continue it in two parameters.
4. Find the period-doubling of the *doubled* cycle and continue it in the  $(\alpha, \beta)$ -plane.

C. Consider the following prey-predator system from mathematical ecology by Bazykin and Khibnik [1981]:

$$\begin{cases} \dot{x} &= \frac{x^2(1-x)}{n+x} - xy, \\ \dot{y} &= -y(m-x), \end{cases} \quad (3)$$

where  $n > 0, 0 < m < 1$ .

1. Compare its bifurcation diagrams with respect to parameter  $m$  for  $n_1 = \frac{1}{4}$  and  $n_2 = \frac{1}{16}$  using MATCONT.
2. Compute the two-parameter bifurcation diagram in the  $(m, n)$ -plane and sketch all qualitatively different phase portraits.

*Hint:* The diagram should include a Hopf bifurcation curve (H) and a cycle fold curve (LPC) that connects a *generalized Hopf point* (GH) in H with point  $(m, n) = (0, 0)$ .

3. Derive an analytic expression for the Hopf bifurcation curve in (3). *Hints:* Consider the orbitally-equivalent polynomial system

$$\begin{cases} \dot{x} &= x^2(1-x) - xy(n+x), \\ \dot{y} &= -y(m-x)(n+x). \end{cases} \quad (4)$$

4. Verify the numerically found GH-point by proving that the first Lyapunov coefficient  $l_1$  vanish at  $(m_{\text{GH}}, n_{\text{GH}}) = (\frac{1}{4}, \frac{1}{8})$ . *Hints:*

- (a) At Hopf parameter values, translate the origin in (4) to the equilibrium and scale the variables to obtain a system in the form

$$\begin{cases} \dot{\xi} &= -\omega\eta + P(\xi, \eta), \\ \dot{\eta} &= \omega\xi + Q(\xi, \eta), \end{cases}$$

where  $P, Q$  contain terms of order two and higher in  $(\xi, \eta)$ .

- (b) Introduce complex variables  $z = \xi + i\eta$  and  $\bar{z} = \xi - i\eta$ , and derive the equation

$$\dot{z} = i\omega z + \sum_{2 \leq j+k \leq 3} \frac{1}{j!k!} g_{jk} z^j \bar{z}^k + O(|z|^4).$$

- (c) Compute

$$l_1 = \frac{1}{2\omega^2} \text{Re}(ig_{20}g_{11} + \omega g_{21}).$$