

TUTORIAL V:  
Continuation of homoclinic orbits with MATCONT

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This session is devoted to location and continuation of orbits homoclinic to hyperbolic equilibria in autonomous systems of ODEs depending on two parameters

$$\dot{u} = f(u, \alpha), \quad u \in \mathbb{R}^n, \alpha \in \mathbb{R}^2,$$

and to detection of their codim 2 bifurcations.

## 1 Traveling pulses in the FitzHugh-Nagumo model

The following system of partial differential equations is the FitzHugh-Nagumo model of the nerve impulse propagation along an axon:

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - f_a(u) - v, \\ \frac{\partial v}{\partial t} = bu, \end{cases} \quad (1)$$

where  $u = u(x, t)$  represents the membrane potential;  $v = v(x, t)$  is a phenomenological “recovery” variable;  $f_a(u) = u(u - a)(u - 1)$ ,  $1 > a > 0$ ,  $b > 0$ ,  $-\infty < x < +\infty$ ,  $t > 0$ .

*Traveling waves* are solutions to these equations of the form

$$u(x, t) = U(\xi), \quad v(x, t) = V(\xi), \quad \xi = x + ct,$$

where  $c$  is an a priori unknown wave propagation speed. The functions  $U(\xi)$  and  $V(\xi)$  satisfy the system of three ordinary differential equations

$$\begin{cases} \dot{U} = W, \\ \dot{W} = cW + f_a(U) + V, \\ \dot{V} = \frac{b}{c}U, \end{cases} \quad (2)$$

where the dot means differentiation with respect to “time”  $\xi$ . System (2) is called a *wave system*. It depends on three positive parameters  $(a, b, c)$ . Any bounded orbit of (2) corresponds to a traveling wave solution of the FitzHugh-Nagumo system (1) at parameter values  $(a, b)$  propagating with velocity  $c$ .

For all  $c > 0$  the wave system has a unique equilibrium  $0 = (0, 0, 0)$  with one positive eigenvalue  $\lambda_1$  and two eigenvalues  $\lambda_{2,3}$  with negative real parts. The equilibrium can be either a saddle or a saddle-focus and has in both cases a one-dimensional unstable and a two-dimensional stable invariant manifolds  $W^{u,s}(0)$ . The transition between saddle and saddle-focus cases is caused by the presence of a double negative eigenvalue; for fixed  $b > 0$  this happens on the curve

$$D_b = \{(a, c) : c^4(4b - a^2) + 2ac^2(9b - 2a^2) + 27b^2 = 0\}.$$

A branch  $W_1^u(0)$  of the unstable manifold leaving the origin into the positive octant can return back to the equilibrium, forming a homoclinic orbit  $\Gamma_0$  at some parameter values.

For  $b > 0$ , these parameter values form a curve  $P_b^{(1)}$  in the  $(a, c)$ -plane that can only be found numerically. As we shall see, this curve passes through the saddle-focus region delimited by  $D_b$ . Any homoclinic orbit defines a traveling *impulse*. The shape of the impulse depends on the type of the corresponding equilibrium: It has a monotone “tail” in the saddle case and an oscillating “tail” in the saddle-focus case.

The saddle quantity  $\sigma_0 = \lambda_1 + \text{Re } \lambda_{2,3}$  is always positive for  $c > 0$ . Therefore, the phase portraits of (2) near the homoclinic curve  $P_b^{(1)}$  are described by Shilnikov’s Theorems. In particular, near the homoclinic bifurcation curve  $P_b^{(1)}$  in the saddle-focus region, system (2) has an infinite number of saddle cycles. These cycles correspond to *periodic wave trains* in the FitzHugh-Nagumo model (1). Secondary homoclinic orbits existing in (2) near the primary saddle-focus homoclinic

bifurcation correspond to *double traveling impulses* in (1). An infinite number of the corresponding secondary homoclinic bifurcation curves  $P_{b,j}^{(2)}$  in (2) originate at each point  $A_{1,2}$ , where  $P_b^{(1)}$  intersects  $D_b$ .

We will locate a critical value of  $c$  for  $a = 0.15$  and  $b = 0.0025$ , at which (2) has a homoclinic orbit, continue this homoclinic orbit with respect to the parameters  $(a, c)$ , and detect the codim 2 bifurcations points  $A_{1,2}$  in  $P_b^{(1)}$ .

## 2 System specification

Start a version of MATCONT that supports homoclinic continuation, and specify a new ODE system with the coordinates  $(U, W, V)$  and time  $t$ <sup>1</sup>:

$$\begin{aligned} U' &= W \\ W' &= cc*W + U*(U - aa)*(U - 1.0) + V \\ V' &= bb*U/cc \end{aligned}$$

The parameters  $a, b, c$  are denoted by **aa, bb, cc**, respectively. Generate the derivatives of order 1, 2, and 3 symbolically.

## 3 Location of a homoclinic orbit by homotopy

This consists of several steps, each presented in a separate subsection.

### 3.1 Approximating the unstable manifold by integration

Select **Type|Initial point|Equilibrium** and **Type|Curve|ConnectionSaddle**.

In the appearing **Integrator** window, increase the integration **Interval** to 20 (see the right panel of Figure 1).

Via the **Starter** window, input the initial values of the system parameters

<sup>1</sup>Due to MATLAB restrictions, the name **xi** cannot be used here !

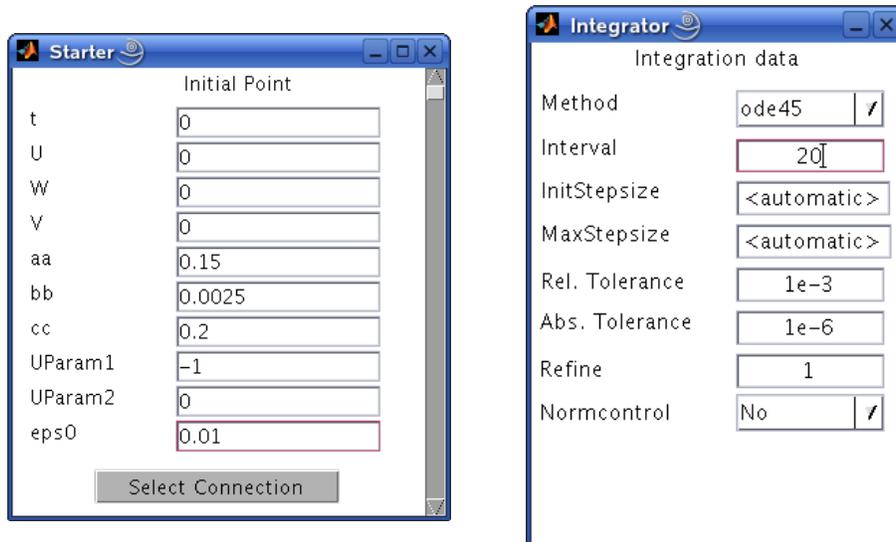


Figure 1: **Starter** and **Integrator** windows for the integration of the unstable manifold.

```
aa      0.15
bb      0.0025
cc      0.2
```

as well as

```
Uparam1 -1
eps0     0.01
```

that specify direction and distance of the displacement from the saddle

```
x0_U     0
x0_W     0
x0_V     0
```

along the unstable eigenvector<sup>2</sup>. The **Starter** window should look like in left panel of Figure 1.

Open a **2Dplot** window with **Window|Graphic|2Dplot**. Select **U** and **V** as variables along the corresponding axes and input the following plotting region

```
Abcissa:   -0.2      0.5
Ordinate:  -0.05     0.1
```

Start **Compute|Forward**. You will get an orbit approximating the unstable manifold that departs from the saddle in a nonmonotone way, see Figure 2. This orbit does not resemble a

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<sup>2</sup>**Uparam2** is only used when  $\dim W^u = 2$ . On some platforms, **Uparam1=1** should be used to select the correct direction.

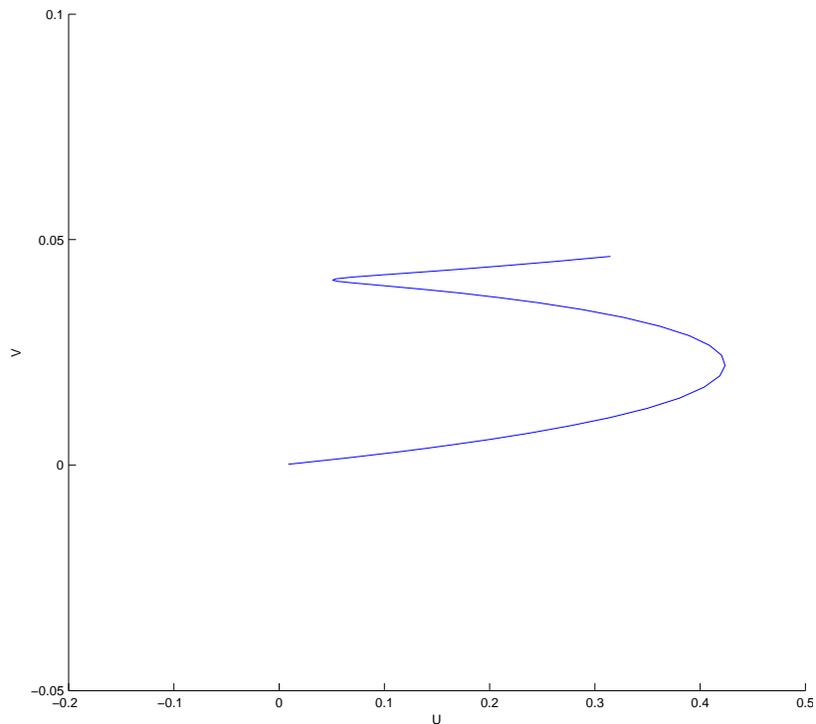


Figure 2: A segment of the unstable manifold of the saddle at the initial parameter values.

homoclinic orbit.

Press **Select Connection** button in the **Starter** window. MATCONT will search for a point in the computed orbit where the distance to the *stable* eigenspace of the Jacobian matrix of the saddle is stopped decreasing for the last time. This point is selected as the end-point of the initial connecting orbit (as we shall see, it corresponds to the time-interval  $T=8.40218$ ). The program will ask to choose the BVP-discretization parameters **ntst** and **ncol** that will be used in all further continuations. Set **ntst** equal to 50 and keep **ncol** equal to 4 (Figure 3). Press **OK**.

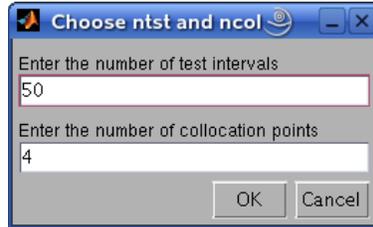


Figure 3: The discretization parameters for homotopy BVPs.

### 3.2 Homotopy towards the stable eigenspace

In the new **Starter** window, activate the parameters **cc**, **SParam1**, and **eps1** (see Figure 4), and **Compute|Backward**. A family of curves will be produced by continuation (see Figure 5) and the message

**SParam equal to zero**

will indicate that the end-point has arrived at the stable eigenspace of the saddle (i.e. reached the plane tangent to the stable invariant manifold at the saddle and given by the condition  $SParam1=0$ ). The corresponding orbit segment is labeled **HTHom**. Stop the continuation there.

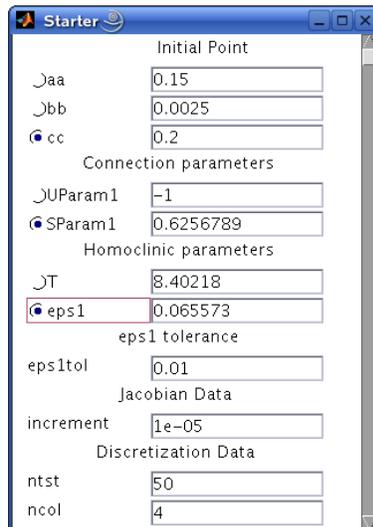


Figure 4: **Starter** window for the homotopy towards the stable eigenspace.

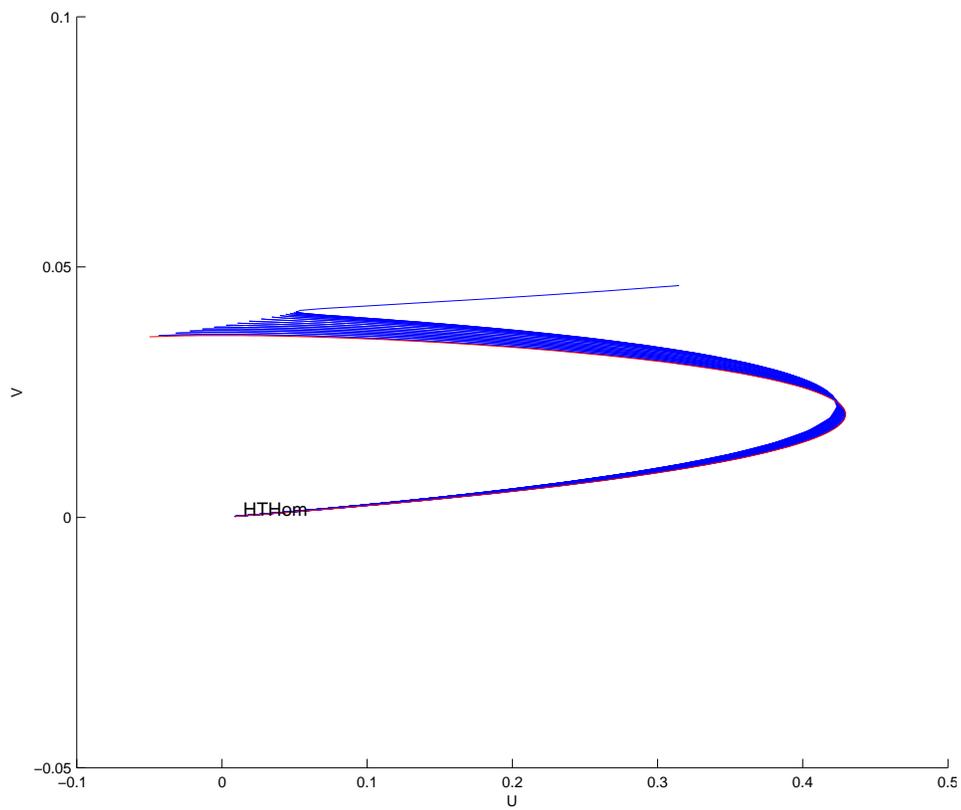


Figure 5: The unstable manifold with the end-point in the stable eigenspace of the saddle.

### 3.3 Homotopy of the end-point towards the saddle

The obtained segment is still far from the homoclinic orbit but can be selected as the initial point for a homotopy of the end-point towards the saddle. Select

2) HTHom: SParam equal to zero

via **Select|Initial point** menu.

In the **Continuer** window, set **MaxStepsize** to 0.5, see in the right panel of Figure 6.

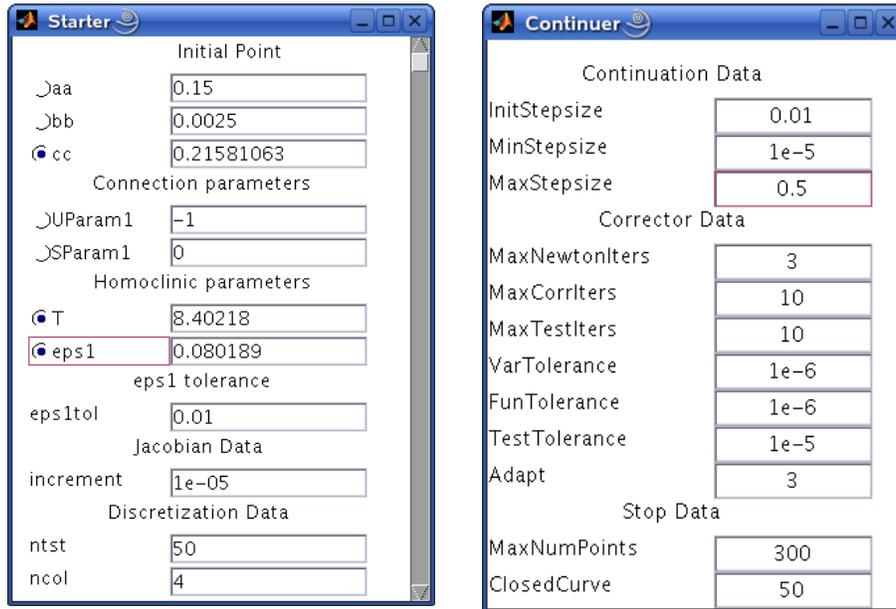


Figure 6: **Starter** and **Integrator** windows for the homotopy towards the saddle.

In the **Starter** window, **SParam1** now equals to zero, while the parameter **cc** is adjusted. Activate parameters **cc**, **T**, and **eps1** there. Set **eps1tol** equal to 0.01; this will be used as the target distance **eps1** from the end-point to the saddle.

Open a **Numeric** window to monitor the values of the active parameters. Clean the **2DPlot** window and **Compute | Forward**. You should get Figure 7, where the last computed segment is again labeled by HTHom. The message

`eps1 small enough`

appears in the main window and indicates that a good approximation of the homoclinic orbit is found. The begin- and the end-points are now both located near the saddle (at distance 0.01). The **Numeric** window at the last computed point is presented in Figure 8. It can be seen that the **eps1** became 0.01, while the time-interval **T** increased to 36.6206. Stop the continuation.

### 3.4 Continuation of the homoclinic orbit

Select just computed

2) HTHom: `eps1 small enough`

via **Select|Initial point** menu as the initial data. Select **Type|Curve|Homoclinic to Saddle** and check that the curve type is Hom, while the initial point is of type HTHom.

In the new **Starter** window, activate two system parameters: **aa** and **cc** as well as the homoclinic parameter **T** (see Figure 9). These parameters will vary along the homoclinic curve, while

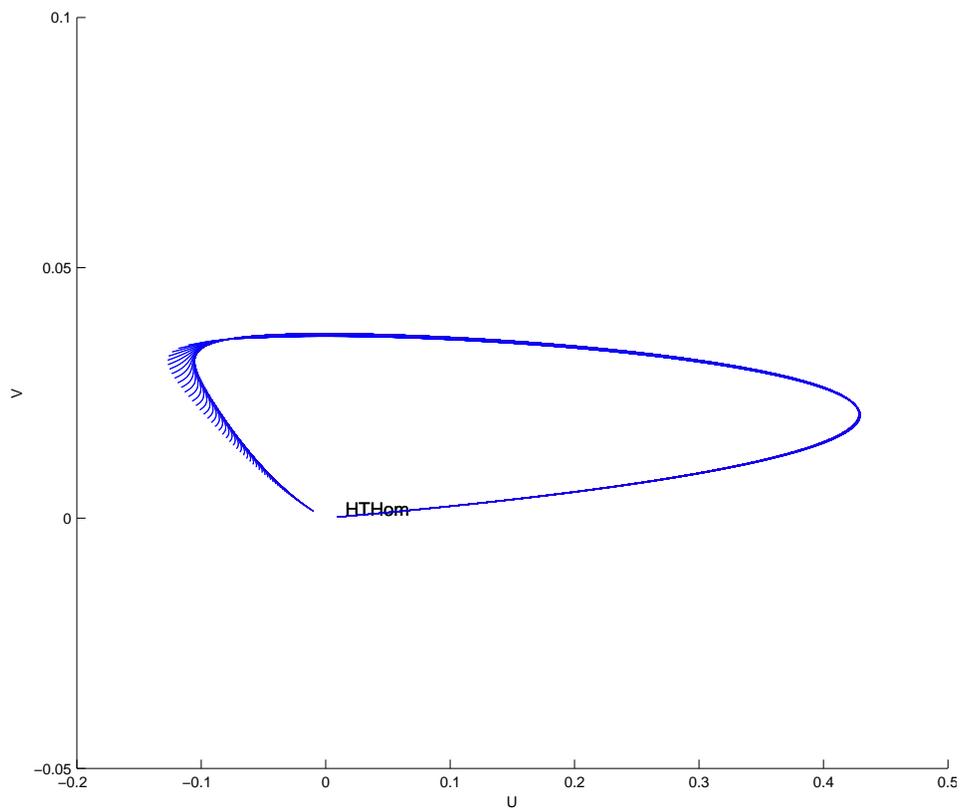


Figure 7: The homotopy results in the manifold segment with both the begin- and the end-points near the saddle.

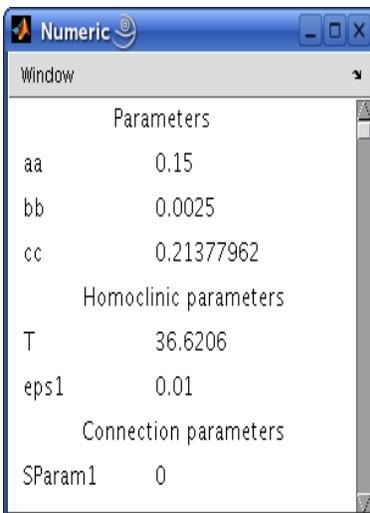


Figure 8: **Numeric** window at the last point of the homotopy towards the saddle.

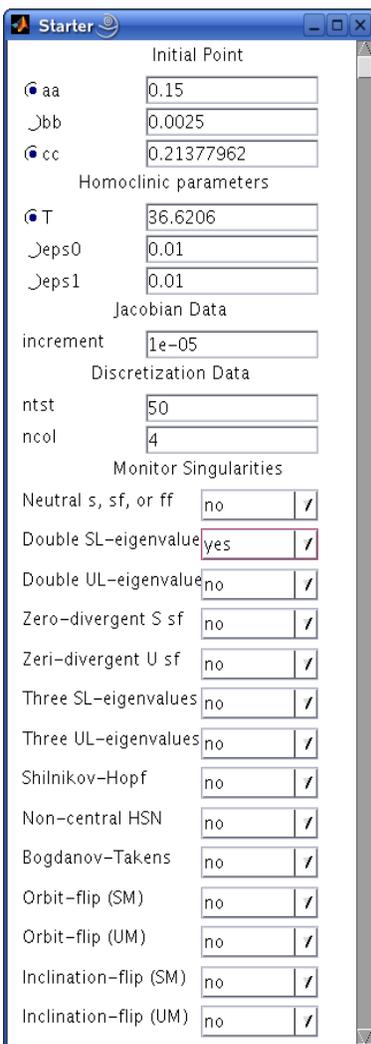


Figure 9: **Starter** window for the two-parameter homoclinic continuation.

both `eps0` and `eps1` (the begin- and end-distances to the saddle) will be fixed, see Figure 9. Also, choose **Yes** to detect the singularity **Double SL-eigenvalue** (*double stable leading eigenvalue*) along the homoclinic curve.

In the **Continuer** window, increase the `MaxStepsize` to 1.

Change the attributes of the **2Dplot** window: Select `aa` and `cc` as the abscissa and ordinate with the visibility limits

```
Abscissa:      0          0.3
Ordinate:     0          0.8
```

Now you are ready to start the continuation. **Compute|Forward** and **Backward**, resume computations at special points, and terminate them when the computed points leave the positive quadrant of the  $(a, c)$ -plane. Two special points will be detected, where the equilibrium undergoes the saddle-to-saddle-focus transition. These are codim 2 bifurcation points  $A_{1,2}$  introduced in Section 1.

Delete all previously computed curves except the last two, namely

```
HTHom_Hom(1)
HTHom_Hom(1)
```

and **Plot|Redraw diagram**. This should produce Figure 10.

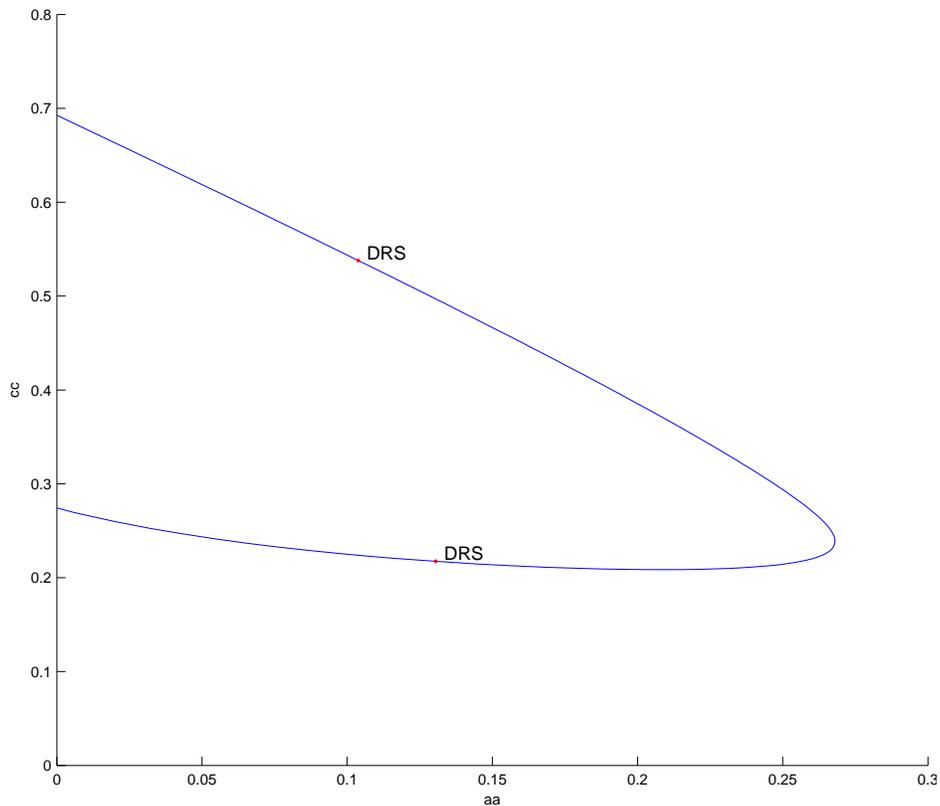


Figure 10: The homoclinic bifurcation curve in the  $(a, c)$ -plane. The saddle to saddle-focus transitions  $A_{1,2}$  are labeled by **DRS**.

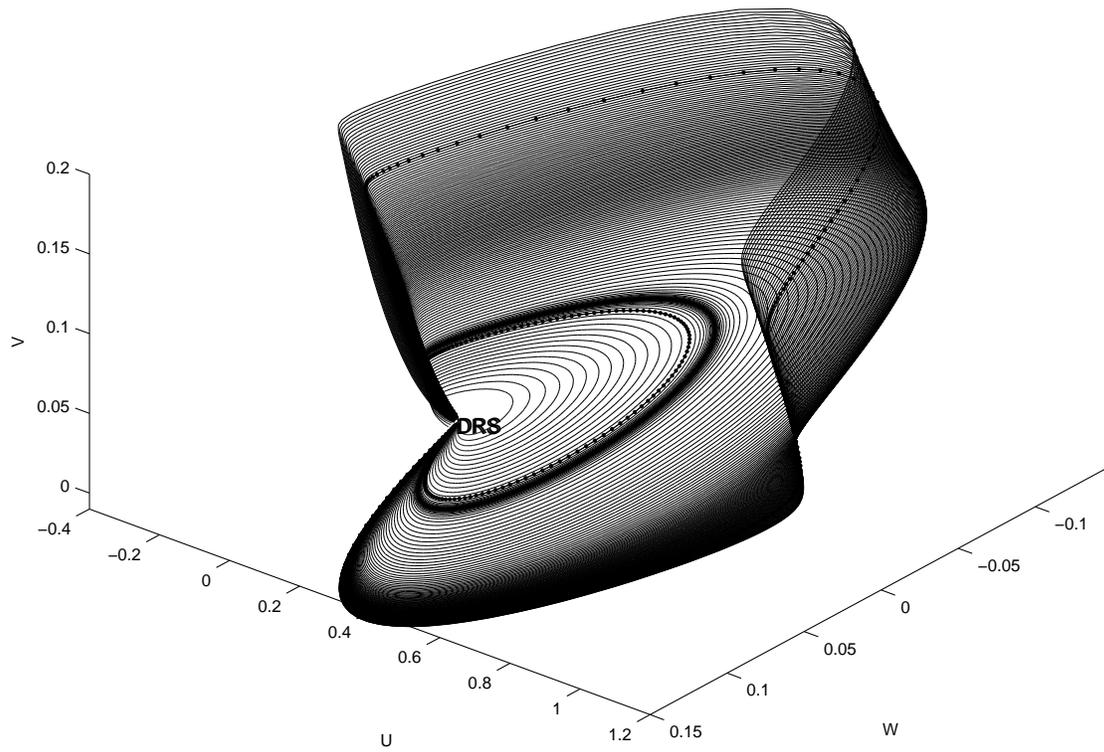


Figure 11: The family of homoclinic homoclinic orbits in the phase space of system (2) for  $b = 0.0025$ .

To verify that that all computed points indeed correspond to homoclinic orbits, open a **3Dplot** window and select U,W and V as variables along the coordinate axes with the visibility limits

Abcissa:                -0.4            1.2  
 Ordinate:              -0.15          0.15  
 Applicate:            -0.01          0.2

respectively. **Plot | Redraw diagram** in this new window should produce Figure 11 after an appropriate rotation.

## 4 Additional Problems

A. Consider the famous *Lorenz system*

$$\begin{cases} \dot{x} &= \sigma(-x + y), \\ \dot{y} &= rx - y - xz, \\ \dot{z} &= -bz + xy, \end{cases}$$

with the standard parameter value  $b = \frac{8}{3}$ . Use MATCONT to analyse its homoclinic bifurcations:

1. Locate at  $\sigma = 10$  the bifurcation parameter value  $r_{\text{Hom}}$  corresponding to the primary orbit homoclinic to the origin. *Hint:* Use homotopy starting from  $r = 15.5$ .
2. Compute the primary homoclinic bifurcation curve in the  $(r, \sigma)$ -plane for  $b = \frac{8}{3}$ . Try to reach  $r = 100$  and  $\sigma = 100$ .
3. Locate and continue in the same  $(r, \sigma)$ -plane several secondary homoclinic to the origin orbits in the Lorenz system. *Hint:* These orbits make turns around both nontrivial equilibria. The simplest one can be found starting from  $(\sigma, r) = (10, 55)$ .

B. Study with MATCONT homoclinic bifurcations in the *adaptive control system of Lur'e type*:

$$\begin{cases} \dot{x} &= y, \\ \dot{y} &= z, \\ \dot{z} &= -\alpha z - \beta y - x + x^2, \end{cases}$$

where  $\alpha$  and  $\beta$  are parameters.