Lecture 2

Equilibrium bifurcations of ODEs and their numerical analysis

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Contents

1. Equilibria of ODEs and their simplest (codim 1) bifurcations

- Consider a smooth ODE system

$$\dot{u} = f(u, \alpha), \quad u \in \mathbb{R}^n, \alpha \in \mathbb{R}^m.$$

- An equilibrium u_0 satisfies

$$f(u_0, \alpha_0) = 0$$

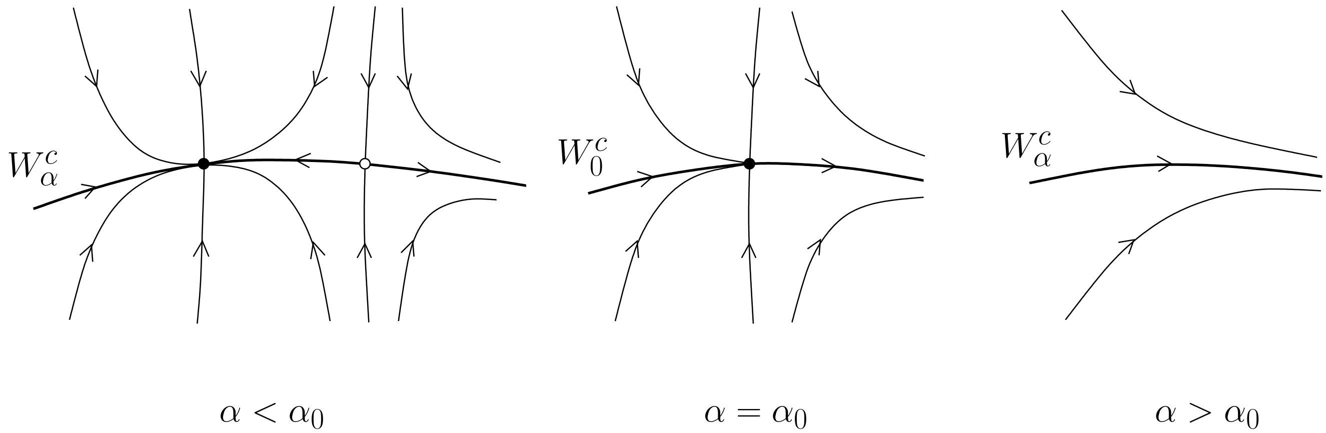
and its Jacobian matrix $A = f_u(u_0, \alpha_0)$ has eigenvalues $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$.

- Critical cases:

  - **Fold (LP)**: $\lambda_1 = 0$;

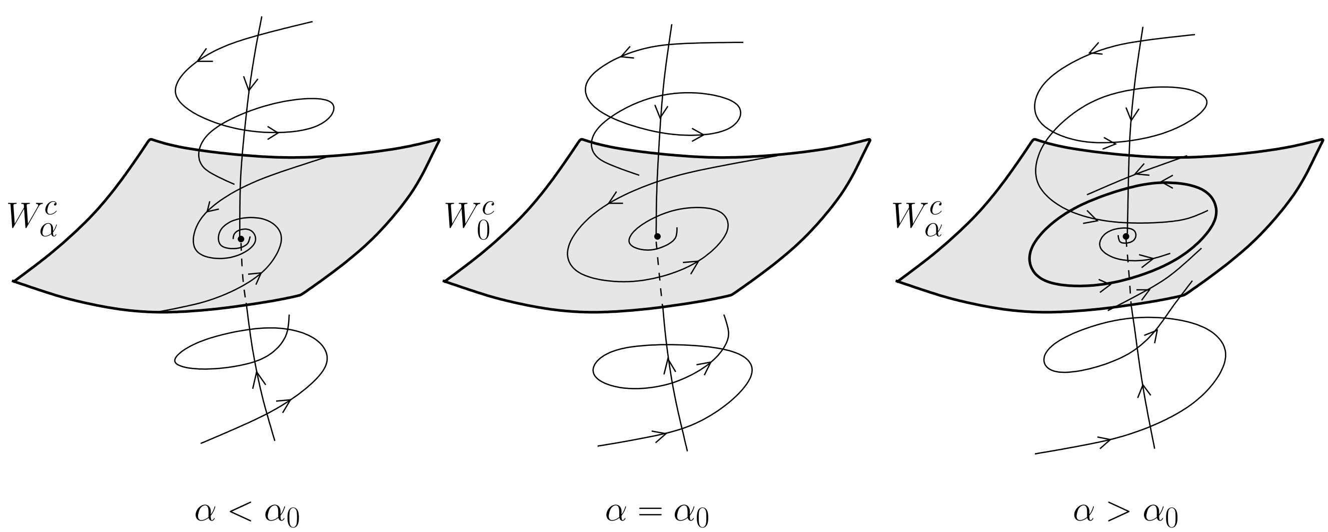
  - **Andronov-Hopf (H)**: $\lambda_{1,2} = \pm i\omega_0$, $\omega_0 > 0$.

- **Generic LP bifurcation:** $\lambda_1 = 0$



Collision of two equilibria.

- **Generic H bifurcation**: $\lambda_{1,2} = \pm i\omega_0$



Birth of a limit cycle.

2. Detection of LP and H bifurcations

- Monitor eigenvalues of $A(u,\alpha) = f_u(u,\alpha)$ along the **equilibrium curve**

$$f(u,\alpha) = 0, \qquad u \in \mathbb{R}^n,\ \alpha \in \mathbb{R}.$$

- Test function for LP: $\psi_{LP} = V_{n+1}$, the α-component of the normalized tangent vector to the equilibrium curve in the (u,α)-space.

- Test function for H:

$$\psi_H = \det(2A(u,\alpha) \odot I_n),$$

where $\odot$ denotes the **bialternate matrix product** with elements

$$(A \odot B)_{(i,j),(k,l)} = \frac{1}{2}\left\{ \begin{vmatrix} a_{ik} & a_{il} \\ b_{jk} & b_{jl} \end{vmatrix} + \begin{vmatrix} b_{ik} & b_{il} \\ a_{jk} & a_{jl} \end{vmatrix} \right\},$$

where $i > j,\ k > l$.

Wedge product of vectors

- Two index pairs $(i,j),(k,l)$ are listed in the **lexicographic order** if either $i<k$ or ($i=k$ and $j<l$).

- The **wedge product** of two vectors $v,w \in \mathbb{C}^n$ is a vector $v \wedge w \in \mathbb{C}^m$, $m = \frac{n(n-1)}{2}$, with the components:

$$(v \wedge w)_{(i,j)} = v_i w_j - v_j w_i, \quad n \geq i > j \geq 1,$$

  listed in the lexicographic order of their index pairs.

- For any $v, w, w^{1,2} \in \mathbb{C}^n$, $\lambda \in \mathbb{C}$: $v \wedge w = -w \wedge v$ and

$$v \wedge (\lambda w) = \lambda (v \wedge w), \;\; v \wedge (w^1 + w^2) = v \wedge w^1 + v \wedge w^2.$$

- If $e^i \in \mathbb{C}^n$, $n \geq i \geq 1$, form a basis in $\mathbb{C}^n$, then $e^i \wedge e^j \in \mathbb{C}^m$, $n \geq i > j \geq 1$, form a basis in $\mathbb{C}^m$.

Bialternate matrix product

- The matrix of the linear transformation of $\mathbb{C}^m$ defined by

  $$(v \wedge w) \mapsto (A \odot B)(v \wedge w) = \frac{1}{2}(Av \wedge Bw - Aw \wedge Bv)$$

  in the standard basis $\{e^i \wedge e^j\}$ is called the **bialternate product** of two matrices $A, B \in \mathbb{C}^{n \times n}$.

- **Stéphanos Theorem** *If $A \in \mathbb{C}^{n \times n}$ has eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, then*

  $$\begin{array}{rl} (i) & A \odot A \text{ has eigenvalues } \lambda_i \lambda_j, \\ (ii) & 2A \odot I_n \text{ has eigenvalues } \lambda_i + \lambda_j, \end{array}$$

  where $n \geq i > j \geq 1$.

  Indeed, if $\{v^i\}$ are linearly-independent eigenvectors of A, then $v^i \wedge v^j$ is an eigenvector of both $A \odot A$ and $2A \odot I_n$.

- For two nonsingular matrices A and B:

  $$\begin{aligned} (AB) \odot (AB) &= (A \odot A)(B \odot B), \\ (A \odot A)^{-1} &= A^{-1} \odot A^{-1}. \end{aligned}$$

3. Continuation of LP and Hopf bifurcations

3.1. Bordering technique

3.2. Continuation of LP bifurcation

3.3. Continuation of Hopf bifurcation

3.1. Bordering technique

Let $M \in \mathbb{R}^{n\times n}, \quad v_j, b_j, c_j \in \mathbb{R}^n, \; g_{ij}, d_{ij} \in \mathbb{R}$

- Suppose the following system has invertible matrix:

$$\begin{pmatrix} M & b_1 \\ c_1^\top & d_{11} \end{pmatrix} \begin{pmatrix} v_1 \\ g_{11} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

  Then M has rank defect 1 if and only if $g_{11} = 0$. Indeed, by Cramer's rule

$$g_{11} = \frac{\det M}{\det \begin{pmatrix} M & b_1 \\ c_1^\top & d_{11} \end{pmatrix}}.$$

- Suppose the following system has invertible matrix:

$$\begin{pmatrix} M & b_1 & b_2 \\ c_1^\top & d_{11} & d_{12} \\ c_2^\top & d_{21} & d_{22} \end{pmatrix} \begin{pmatrix} v_1 & v_2 \\ g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

  Then M has rank defect 2 if and only if

$$g_{11} = g_{12} = g_{21} = g_{22} = 0.$$

3.2. Continuation of LP bifurcation

- At a generic LP bifurcation $A(u,\alpha) = f_u(u,\alpha)$ has rank defect 1.

- Defining system: $x = (u,\alpha) \in \mathbb{R}^{n+2}$

$$\begin{cases} f(u,\alpha) &= 0, \\ G(u,\alpha) &= 0, \end{cases}$$

  where G is computed by solving the *bordered system*

$$\begin{pmatrix} A(u,\alpha) & p_1 \\ q_1^{\top} & 0 \end{pmatrix} \begin{pmatrix} q \\ G \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

- Vectors $q_1, p_1 \in \mathbb{R}^n$ are adapted along the LP-curve to make the matrix of the linear system nonsingular.

- (G_u, G_α) can be computed efficiently using the adjoint linear system.

Derivatives of G

The α-derivative of the bordered system

$$\begin{pmatrix} A(u,\alpha) & p_1 \\ q_1^\top & 0 \end{pmatrix} \begin{pmatrix} q_\alpha \\ G_\alpha \end{pmatrix} + \begin{pmatrix} A_\alpha(u,\alpha) & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} q \\ G \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

implies

$$\begin{pmatrix} A(u,\alpha) & w_1 \\ q_1^\top & 0 \end{pmatrix} \begin{pmatrix} q_\alpha \\ G_\alpha \end{pmatrix} = - \begin{pmatrix} A_\alpha(u,\alpha) & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} q \\ G \end{pmatrix}$$

Multiplication from the left by $(p^\top \;\; h)$ satisfying

$$\begin{pmatrix} A^\top(u,\alpha) & q_1 \\ p_1^\top & 0 \end{pmatrix} \begin{pmatrix} p \\ h \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

gives

$$G_\alpha = -p^\top A_\alpha(u,\alpha) q = -\langle p, A_\alpha(u,\alpha) q \rangle .$$

3.3. Continuation of Hopf bifurcation

- At a generic Hopf bifurcation $A^2(u,\alpha)+\omega_0^2 I_n$ has rank defect 2.

- Defining system: $x = (u, \alpha, \kappa) \in \mathbb{R}^{n+3}$

$$\begin{cases} f(u,\alpha) &= 0, \\ G_{11}(u,\alpha,\kappa) &= 0, \\ G_{22}(u,\alpha,\kappa) &= 0, \end{cases}$$

where $\kappa = \omega_0^2$ and G_{ij} are computed by solving

$$\begin{pmatrix} A^2(u,\alpha)+\kappa I_n & p_1 & p_2 \\ q_1^\top & 0 & 0 \\ q_2^\top & 0 & 0 \end{pmatrix} \begin{pmatrix} r & s \\ G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

- Vectors $q_{1,2}, p_{1,2} \in \mathbb{R}^n$ are adapted to ensure unique solvability.

- Efficient computation of derivatives of G_{ij} is possible.

Remarks on continuation of bifurcations

- For each defining system holds: *Simplicity of the bifurcation + Transversality ⇒ Regularity of the defining system*.

- Border adaptation using solutions of the adjoint linear system.

- Alternatives to bordering for LP:

$$\begin{cases} f(u,\alpha) &= 0, \\ f_u(u,\alpha)q &= 0, \\ \langle q, q_0\rangle - 1 &= 0 \end{cases} \quad \text{or} \quad \begin{cases} f(u,\alpha) &= 0, \\ \det(f_u(u,\alpha)) &= 0. \end{cases}$$

- Alternatives to bordering for H:

$$\begin{cases} f(u,\alpha) &= 0, \\ f_u(u,\alpha)q + \omega p &= 0, \\ f_u(u,\alpha)p - \omega q &= 0, \\ \langle q, q_0\rangle + \langle p, p_0\rangle - 1 &= 0, \\ \langle q, p_0\rangle - \langle q_0, p\rangle &= 0 \end{cases}$$

or

$$\begin{cases} f(u,\alpha) &= 0, \\ \det(2f_u(u,\alpha) \odot I_n) &= 0. \end{cases}$$

4. Computation of normal forms for LP and Hopf bifurcations

4.1. Normal forms on center manifolds

4.2. Fredholm's Alternative

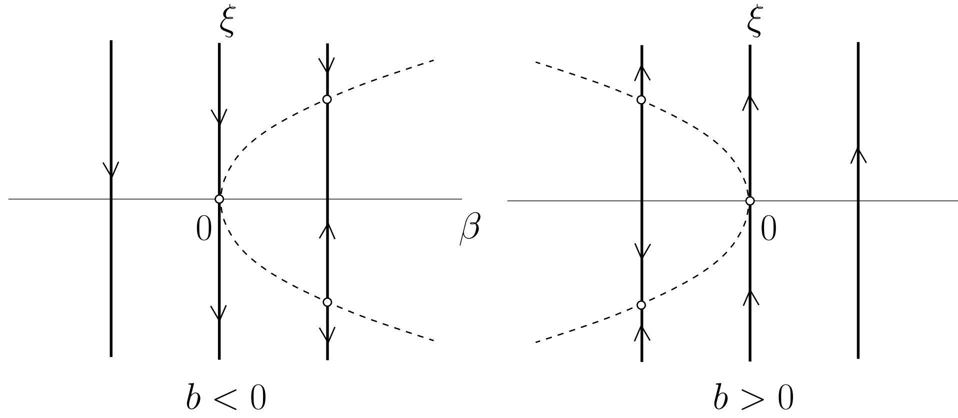
4.3. Critical LP-coefficient

4.4. Critical H-coefficient

4.5. Approximation of multilinear forms by finite differences

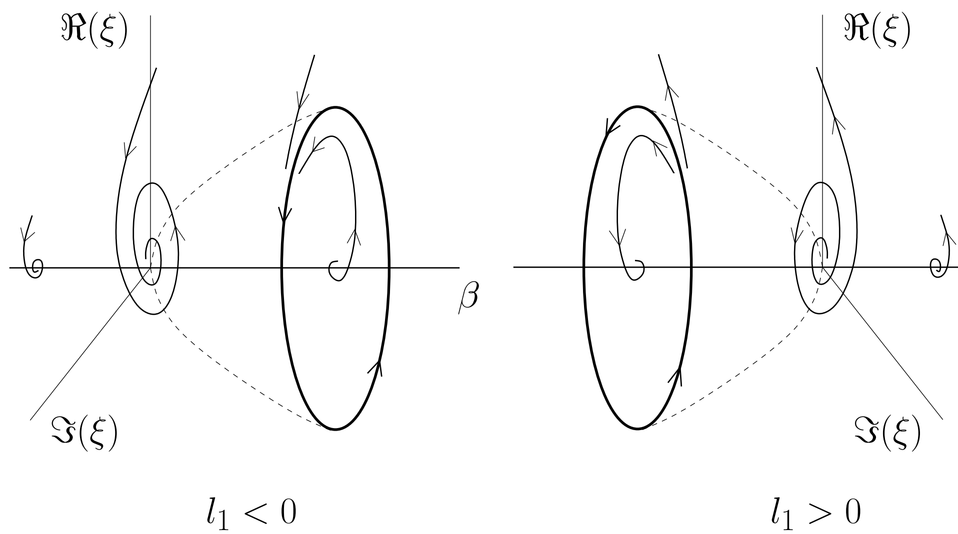
4.1. Normal forms on center manifolds

- LP: $\dot{\xi} = \beta + b\xi^2,\ b \neq 0$



Equilibria: $\beta + b\xi^2 = 0 \Rightarrow \xi_{1,2} = \pm\sqrt{-\frac{\beta}{b}}$

- H: $\dot{\xi} = (\beta + i\omega)\xi + c\xi|\xi|^2,\ l_1 = \frac{1}{\omega}\Re(c) \neq 0$



Limit cycle:

$$\begin{cases} \dot{\rho} &=& \rho(\beta + \Re(c)\rho^2), \\ \dot{\varphi} &=& \omega + \Im(c)\rho^2, \end{cases} \Rightarrow \rho_0 = \sqrt{-\frac{\beta}{\Re(c)}}$$

4.2. Fredholm's Alternative

- **Lemma 1** *The linear system* $Ax = b$ *with* $b \in \mathbb{R}^n$ *and a singular* $n \times n$ *real matrix* A *is solvable if and only if* $\langle p, b \rangle = 0$ *for all* p *satisfying* $A^\mathsf{T} p = 0$.

  Indeed, $\mathbb{R}^n = L \oplus R$ with $L \perp R$, where

  $$L = \mathcal{N}(A^\mathsf{T}) = \{p \in \mathbb{R}^n : A^T p = 0\}$$

  and

  $$R = \{x \in \mathbb{R}^n : x = Ay \quad \text{for some} \quad y \in \mathbb{R}^n\}.$$

  The proof is completed by showing that the orthogonal complement $L^\perp$ to L coincides with R.

- In the complex case:

$$\begin{array}{rcl} \mathbb{R}^n & \Rightarrow & \mathbb{C}^n \\ \langle p, b \rangle & = & \bar{p}^\mathsf{T} b \\ A^\mathsf{T} & \Rightarrow & A^* = \bar{A}^\mathsf{T} \end{array}$$

4.3. Critical LP-coefficient b

- Let $Aq = A^T p = 0$ with $\langle q, q\rangle = \langle p, q\rangle = 1$.

- Write the RHS at the bifurcation as
$$F(u) = Au + \frac{1}{2}B(u,u) + O(\|u\|^3),$$
and locally represent the center manifold W_0^c as the graph of a function $H : \mathbb{R} \to \mathbb{R}^n$,
$$u = H(\xi) = \xi q + \frac{1}{2}h_2\xi^2 + O(\xi^3),\ \ \xi \in \mathbb{R},\ h_2 \in \mathbb{R}^n.$$
The restriction of $\dot{u} = F(u)$ to W_0^c is
$$\dot{\xi} = G(\xi) = b\xi^2 + O(\xi^3).$$

- The invariance of the center manifold $H_\xi(\xi)\dot{\xi} = F(H(\xi))$ implies
$$H_\xi(\xi)G(\xi) = F(H(\xi))$$
Substitute all expansions into this **homological equation** and collect the coefficients of the ξ^j-terms.

We have

$$A(\xi q + \frac{1}{2}h_2\xi^2) \;\; + \;\; \frac{1}{2}B(\xi q, \xi q) + O(|\xi|^3)$$
$$= b\xi^2 q \;\; + \;\; b\xi^3 h_2 \;\; + \;\; O(|\xi|^4)$$

- The ξ-terms give the identity: $Aq = 0$.

- The ξ^2-terms give the equation for h_2:

  $$Ah_2 = -B(q,q) + 2bq.$$

  It is singular and its **Fredholm solvability**

  $$\langle p, -B(q,q) + 2bq \rangle = 0$$

  implies

  $$b = \frac{1}{2}\langle p, B(q,q)\rangle$$

4.4. Critical H-coefficient c

- $Aq = i\omega_0 q, A^{\mathsf{T}} p = -i\omega_0 p, \langle q, q\rangle = \langle p, q\rangle = 1.$

- Write

  $$F(u) = Au + \frac{1}{2}B(u,u) + \frac{1}{3!}C(u,u,u) + O(\|u\|^4)$$

  and locally represent the center manifold W_0^c as the graph of a function $H : \mathbb{C} \to \mathbb{R}^n$,

  $$u = H(\xi, \bar{\xi}) = \xi q + \bar{\xi}\,\bar{q} + \sum_{2\le j+k\le 3} \frac{1}{j!k!} h_{jk}\xi^j \bar{\xi}^k + O(|\xi|^4).$$

  The restriction of $\dot{u} = F(u)$ to W_0^c is

  $$\dot{\xi} = G(\xi, \bar{\xi}) = i\omega_0 \xi + c\xi|\xi|^2 + O(|\xi|^4).$$

- The invariance of W_0^c

  $$H_\xi(\xi, \bar{\xi})\dot{\xi} + H_{\bar{\xi}}(\xi, \bar{\xi})\dot{\bar{\xi}} = F(H(\xi, \bar{\xi}))$$

  implies

  $$H_\xi(\xi, \bar{\xi})G(\xi, \bar{\xi}) + H_{\bar{\xi}}(\xi, \bar{\xi})\overline{G}(\xi, \bar{\xi}) = F(H(\xi, \bar{\xi})).$$

- Quadratic ξ^2- and $|\xi|^2$-terms give

$$\begin{aligned} h_{20} &= (2i\omega_0 I_n - A)^{-1} B(q,q), \\ h_{11} &= -A^{-1} B(q,\overline{q}). \end{aligned}$$

- Cubic $w^2\overline{w}$-terms give the singular system

$$\begin{aligned} (i\omega_0 I_n - A) h_{21} &= C(q,q,\overline{q}) \\ &+ B(\overline{q}, h_{20}) + 2B(q, h_{11}) \\ &- 2cq. \end{aligned}$$

  The solvability of this system implies

$$\begin{aligned} c &= \frac{1}{2}\langle p, C(q,q,\overline{q}) \\ &+ B(\overline{q}, (2i\omega_0 I_n - A)^{-1} B(q,q)) \\ &- 2B(q, A^{-1} B(q,\overline{q}))\rangle \end{aligned}$$

- The **first Lyapunov coefficient**

$$l_1 = \frac{1}{\omega_0} \Re(c).$$

4.5. Approximation of multilinear forms by finite differences

- Finite-difference approximation of directional derivatives:

$$
\begin{aligned}
B(q,q) &= \frac{1}{h^2}\left[f(u_0+hq,\alpha_0)+f(u_0-hq,\alpha_0)\right] \\
&+ O(h^2) \\
C(r,r,r) &= \frac{1}{8h^3}\left[f(u_0+3hr,\alpha_0)-3f(u_0+hr,\alpha_0)\right. \\
&+ \left.3f(u_0-hr,\alpha_0)-f(u_0-3hr,\alpha_0)\right] \\
&+ O(h^2).
\end{aligned}
$$

- Polarization identities:

$$
B(q,r)=\frac{1}{4}\left[B(q+r,q+r)-B(q-r,q-r)\right],
$$

$$
\begin{aligned}
C(q,q,r) &= \frac{1}{6}[C(q+r,q+r,q+r) \\
&- C(q-r,q-r,q-r)] \\
&- \frac{1}{3}C(r,r,r).
\end{aligned}
$$

5. Detection of codim 2 bifurcations

- codim 2 cases along the LP-curve:
  - **Bogdanov-Takens (BT)**: $\lambda_{1,2} = 0$
    ($\psi_{BT} = \langle p, q \rangle$ with $\langle q, q \rangle = \langle p, p \rangle = 1$)
  - **fold-Hopf (ZH)**: $\lambda_1 = 0, \lambda_{2,3} = \pm i\omega_0$
    ($\psi_{ZH} = \det(2A \odot I_n)$)
  - **cusp (CP)**: $\lambda_1 = 0, b = 0$ ($\psi_{CP} = b$)

- Critical cases along the H-curve:
  - **Bogdanov-Takens (BT)**: $\lambda_{1,2} = 0$
    ($\psi_{BT} = \kappa$)
  - **fold-Hopf (ZH)**: $\lambda_{1,2} = \pm i\omega_0, \lambda_3 = 0$
    ($\psi_{ZH} = \det A$)
  - **double Hopf (HH)**: $\lambda_{1,2} = \pm i\omega_0, \lambda_{3,4} = \pm i\omega_1$
    ($\psi_{HH} = \det(2A^{\perp} \odot I_{n-2})$)
  - **Bautin (GH)**: $\lambda_{1,2} = \pm i\omega_0, l_1 = 0$
    ($\psi_{GH} = l_1$)