

TUTORIAL V:  
Codimension 2 bifurcations of limit cycles in MATCONT

Yu.A. Kuznetsov  
Department of Mathematics  
Utrecht University  
Budapestlaan 6  
3508 TA, Utrecht  
The Netherlands

F. Della Rossa  
DEI, Politecnico di Milano  
Via Ponzio 35/5  
20133 Milano  
Italy

V. De Witte & W. Govaerts  
Department of Applied Mathematics and Computer Science  
Ghent University  
Krijgslaan 281-S9  
B-9000, Gent  
Belgium

June 10, 2021

This session was tested on MATCONT7.3 with MATLAB2020b, R9.9. It is devoted to the numerical continuation of codim 1 bifurcations of limit cycles in systems of autonomous ODEs depending on two parameters

$$\dot{x} = f(x, \alpha), \quad x \in \mathbb{R}^n, \alpha \in \mathbb{R}^2,$$

and detection of their codim 2 bifurcations. We will also switch at some codim 2 equilibrium bifurcations to the continuation of codim 1 bifurcation curves rooted there. Simultaneously, we will illustrate how MATCONT can be used to study bifurcations of periodic solutions of periodically-forced ODEs.

## 1 Introduction

In this tutorial, we will reconstruct the bifurcation diagram<sup>1</sup> presented in Figure 1. This is a partial bifurcation diagram for periodic solutions of the following periodically-forced system:

$$\begin{cases} \dot{x} &= r \left(1 - \frac{x}{K}\right) x - p(x, t)y, \\ \dot{y} &= ep(x, t)y - dy, \end{cases} \quad (1)$$

where  $x$  and  $y$  are densities of individuals, respectively of prey and predator populations. The parameters present in system (1) are the intrinsic growth rate  $r$ , the carrying capacity  $K$ , the

<sup>1</sup>Yu.A. Kuznetsov, S. Muratori, and S. Rinaldi. *Bifurcations and chaos in a periodic predator-prey model*, Internat. J. Bifur. Chaos Appl. Sci. Engr., 2 (1992), 117-128.

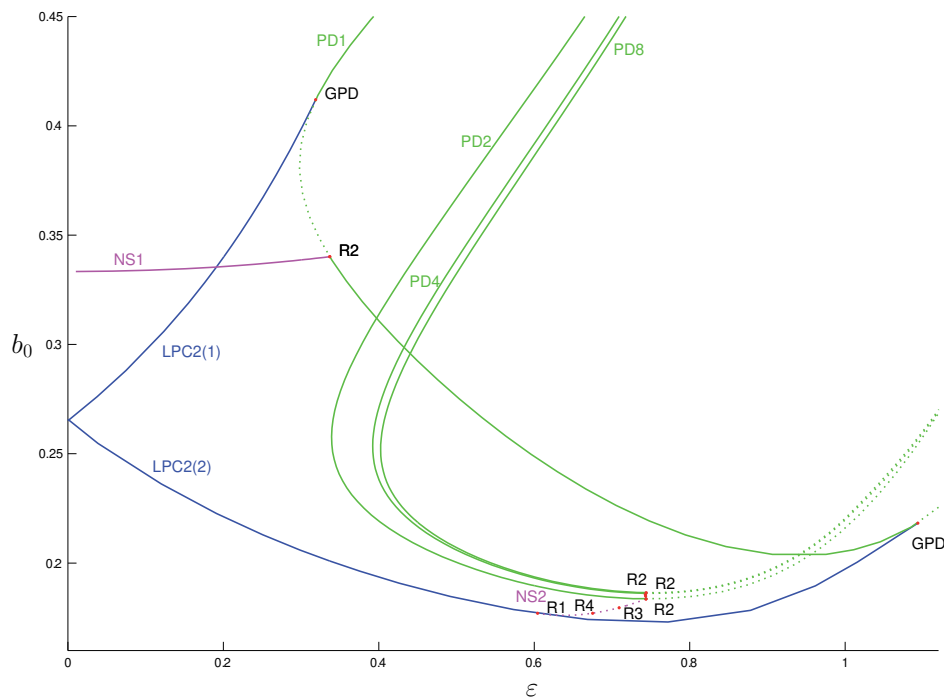


Figure 1: Bifurcation diagram of periodic solutions of model (1). Blue lines correspond to fold bifurcations of cycles, green - to period-doubling bifurcations, and magenta - to Neimark-Sacker bifurcations. Continuous/dotted curves correspond to super-/sub-critical bifurcations, respectively.

efficiency  $e$  and the death rate  $d$  of the predator. The function  $p(x, t)$  is a functional response, for which the Holling type II is chosen, with constant attack rate  $a$  and half saturation  $b(t)$  that varies periodically with period one (year), i.e.

$$p(x, t) = \frac{ax}{b(t) + x}, \quad b(t) = b_0(1 + \varepsilon \cos(2\pi t)).$$

Instead of time-periodic system (1), we consider the extended autonomous system

$$\begin{cases} \dot{x} &= r \left(1 - \frac{x}{K}\right) x - \frac{axy}{b_0(1 + \varepsilon u) + x}, \\ \dot{y} &= e \frac{axy}{b_0(1 + \varepsilon u) + x} - dy, \\ \dot{u} &= u - 2\pi v - (u^2 + v^2)u, \\ \dot{v} &= 2\pi u + v - (u^2 + v^2)v, \end{cases} \quad (2)$$

where the last two equations are decoupled from the first two and have a stable limit cycle with

$$\begin{cases} u(t) &= \cos(2\pi t + \varphi), \\ v(t) &= \sin(2\pi t + \varphi), \end{cases}$$

and a phase shift  $\varphi$  depending on the initial conditions.

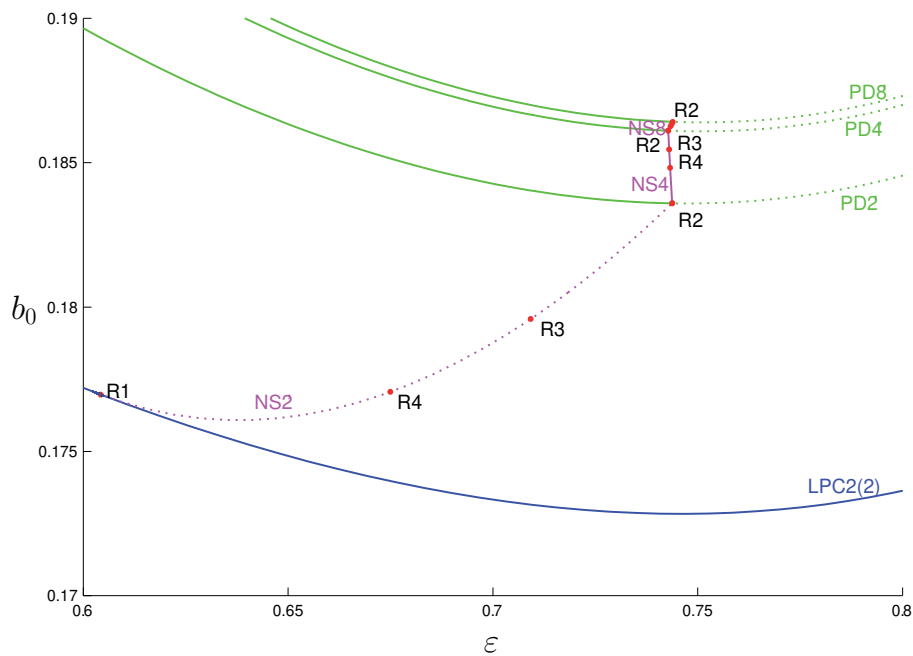


Figure 2: Cascade of 1:2 resonances.

A zoomed-in part of Figure 1 is shown in Figure 2. System (2) was used as an example in a recent paper<sup>2</sup> devoted to the computation of critical normal form coefficients at codim 2 bifurcations of limit cycles. Test functions to detect codim 2 bifurcations of limit cycles are described in a previous publication<sup>3</sup>.

<sup>2</sup>V. De Witte, F. Della Rossa, W. Govaerts, and Yu.A. Kuznetsov. *Numerical periodic normalization for codim 2 bifurcations of limit cycles: Computational formulas, numerical implementation, and examples*. SIAM J. Appl. Dyn. Syst. 12 (2013), 722-788.

<sup>3</sup>Yu.A. Kuznetsov, W. Govaerts, E.J. Doedel, and A. Dhooge. *Numerical periodic normalization for codim 1 bifurcations of limit cycles*. SIAM J. Numer. Anal. 43 (2005), 1407-1435.

## 2 Specify the model in MATCONT

We assume that MATCONT is installed and is started. To specify system (2) in MATCONT, click **Select|System|New** in the main window, and then define the new system as in Figure 3.

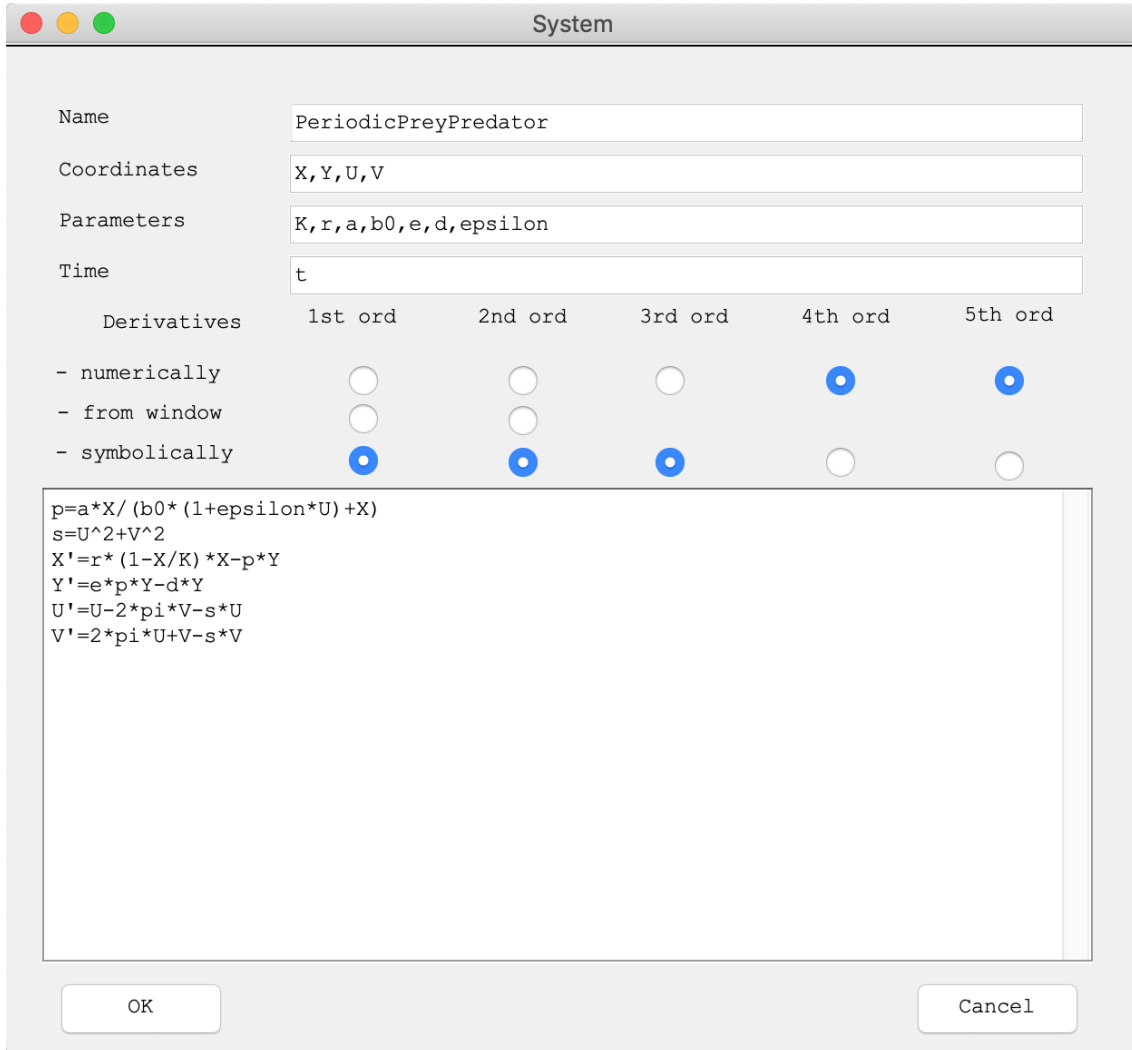


Figure 3: The **System** window. Click **OK** to process the system.

## 3 Find an initial solution

First, we need a periodic solution to start the continuation analysis. To get such a solution, we start numerical integration at some initial point, hoping that the orbit will converge to a stable limit cycle. When clicking the menu **Type|Initial Point|Point**, two windows are opened, the **Starter** and the **Integrator** windows. In the **Starter** window, set  $t = 0$  (actually, it is default) and input the following initial phase and parameter values

$$X = Y = 0.1, \quad U = 1, \quad V = 0, \quad K = 1, \quad r = 6.28, \quad a = 12.56, \quad b_0 = 0.25, \quad e = 1, \quad d = 6.28, \quad \varepsilon = 0.2,$$

and set **Interval** to 100 in the **Integrator** window, see Figure 4.

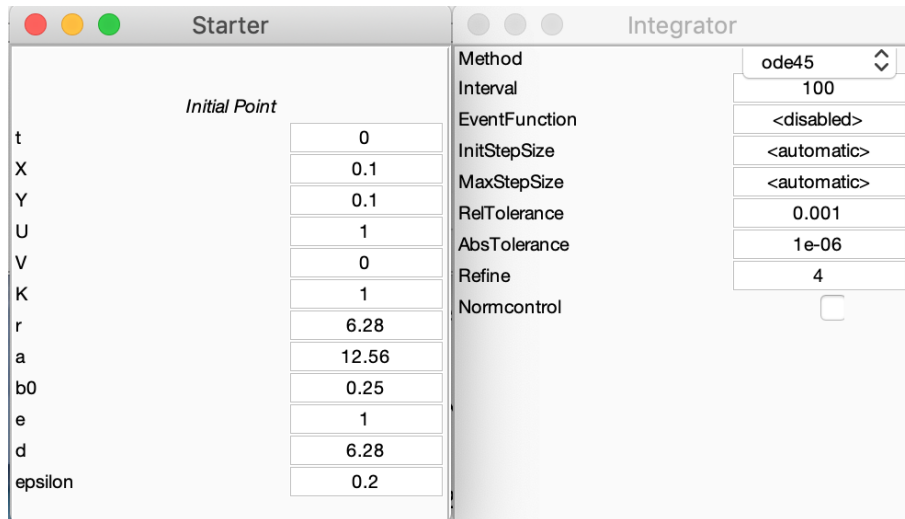


Figure 4: Initial data for integration in the **Starter** and **Integrator** windows.

Open a plot window, through the menu **Window/Output|Graphic|2D plot**, which (by default) visualizes the plane of the first two phase variables within the range  $[0, 1]$  for both variables. By computing the orbit with the time-integration, through the menu **Compute|Forward**, we obtain the result shown in Figure 5. **Close** the **Control** window. We now want to select the

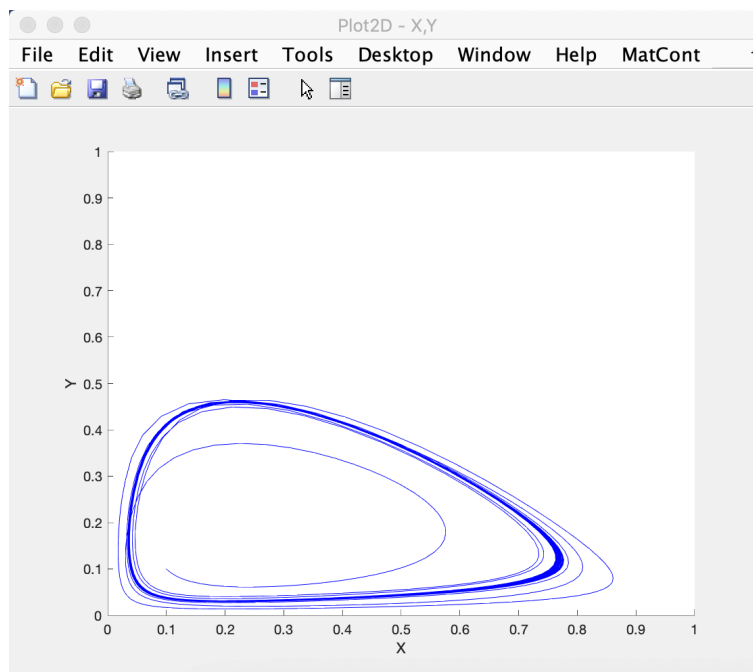


Figure 5: The **Plot2D** window showing the orbit converging to a stable cycle with period 2.

asymptotic cycle in order to start its continuation with respect to a parameter. First we select the **Last Point** of the computed orbit, through the MATCONT menu **Select|Initial Point** that opens a **Data Browser** window, see Figure 6. Note this window can also be opened directly from the

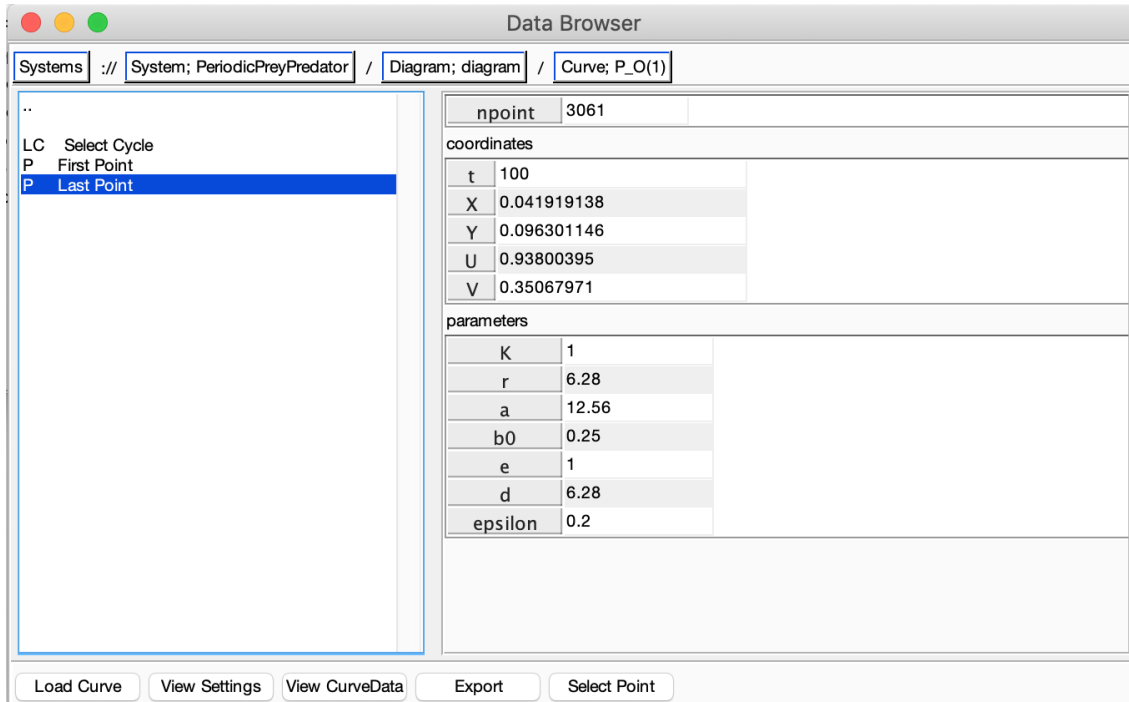


Figure 6: The **Data Browser** window with several points in the computed orbit  $0.P(1)$ ; highlight **Last Point** and click **Select Point**.

**Control** window by pressing **View result** when the computation is finished. Then clear the plot with **MatCont|Clear** menu in the **Plot2D** window. Change the **Interval** in the **Integrator** window to 1, and compute the new orbit through **Compute|Forward**. In order to obtain a closed orbit, the integration should be extended once, by clicking the **Compute|Extend** in the main window. Now we can select the obtained limit cycle through the **Data Browser** window that opens, e.g. with **Select|Initial Point** menu. Choose **Select Cycle**, and then press **Select Point** button. A small window shown in Figure 7 will appear. Click **OK** to use the default tolerance

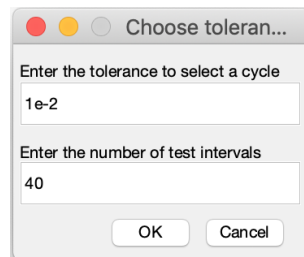


Figure 7: The default discretisation parameters are **OK**.

and the number of test intervals ( $ntst$ ). After that, two windows will open automatically to prepare for the continuation of the limit cycle: new **Starter** and **Continuer** windows.

## 4 Continuation of periodic solutions

Check the parameter  $b_0$  and the period of the limit cycle as free (active) parameters in the **Starter** window (note that **Period** is approximately equal to 2). Turn on the detection of all singularities except **Branch Point of Cycle**, and change **MaxStepsize** to 0.5 and **MaxNumPoints** to 30 in the **Continuer** window, see Figure 8.

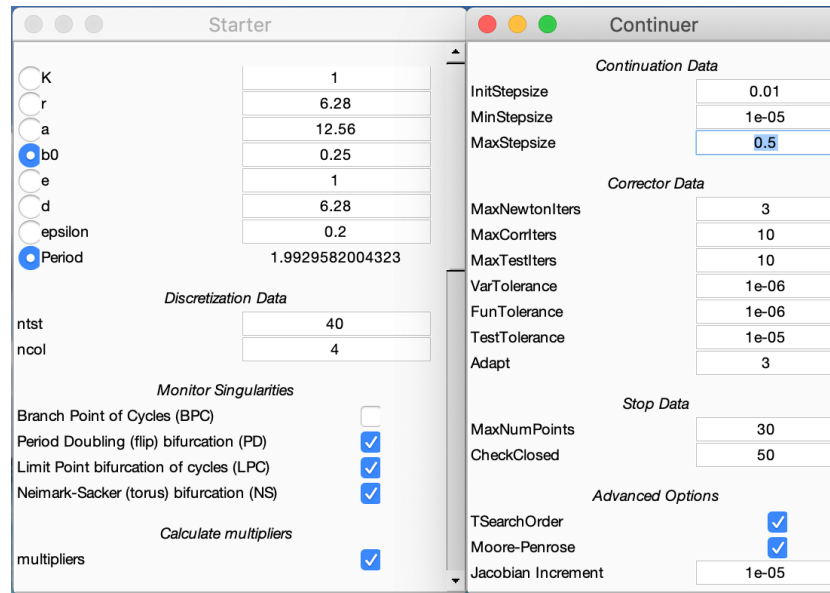


Figure 8: The **Starter** and **Continuer** windows to initialize the cycle continuation in  $b_0$ .

Start continuation as usual with **Compute|Forward** in the main window, and observe a family of limit cycles in the **Plot2D** window. There will be a special cycle detected, and in the MATLAB Command window the following message will appear

```
Limit point cycle (period = 2.000000e+00, parameter = 3.399917e-01)
Normal form coefficient = 1.341555e+09
```

that indicates a fold (limit point) bifurcation of cycles (LPC). This will be the first point to compute the bifurcation diagram. Note that (in general) the normal form coefficients depend very much on the details of the computations and can be reliably used only if the bifurcation points are computed to high accuracy. **Resume** the computation and extend it once to detect another LPC bifurcation:

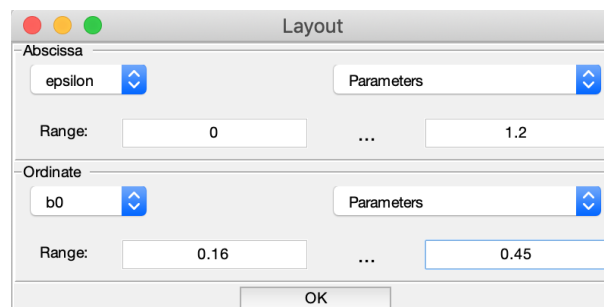


Figure 9: **Layout** window.

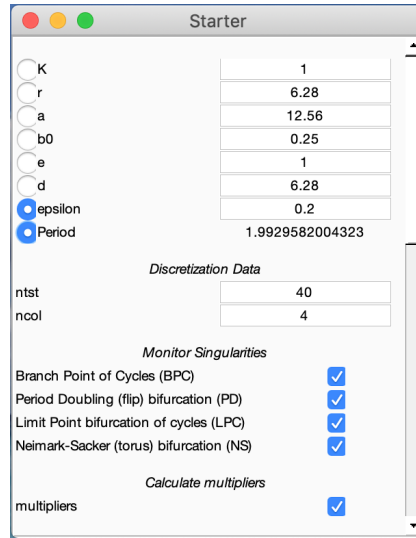


Figure 10: The **Starter** window to initialize the cycle continuation w.r.t.  $\varepsilon$ .

Limit point cycle (period = 2.000000e+00, parameter = 2.213011e-01)  
 Normal form coefficient = -1.621661e+10

**Stop** the computation, close the **Control** window, and save the computed curve LC<sub>LC</sub>(1) as LC<sub>1with2LPC</sub> via **Select|Curve** and using the **Rename** button in the **Data Browser** window that can be closed afterwards. **Change** the variables on the axes of the **Plot2D** window using **MatCont|Layout** and selecting the two system parameters  $\varepsilon$  and  $b_0$  as **Abscissa** and **Ordinate**, see Figure 9. Change the ranges into  $\varepsilon \in [0, 1.2]$  and  $b_0 \in [0.16, 0.45]$  and clear the **Plot2D** window through **MatCont|Clear**.

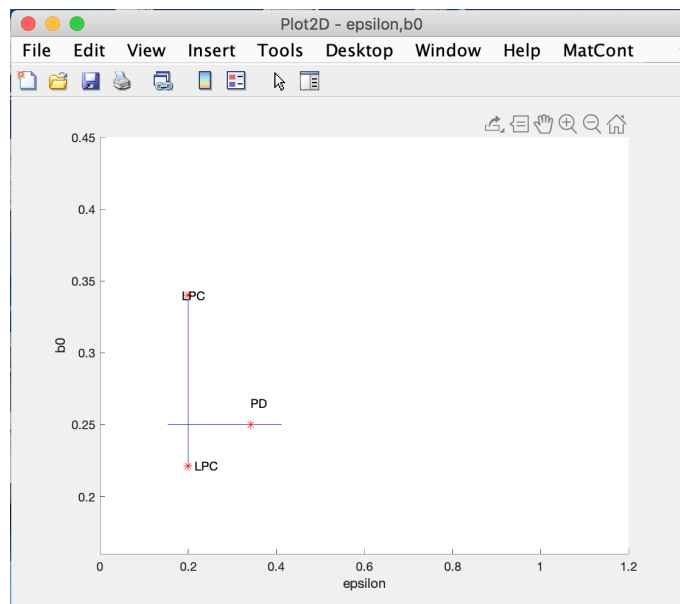


Figure 11: **Plot2D** window showing the cycle continuations and the detected bifurcations.



Activate now parameter  $\varepsilon$  in the **Starter** window, while keeping the **Period** active. Turn on the detection of all singularities as shown in Figure 10, and start the continuation with **Compute|Backward**.

After some continuation steps, a period-doubling bifurcation will be detected with the message

```
Period Doubling (period = 2.000000e+00, parameter = 3.417832e-01)
Normal form coefficient = -2.749889e-01
```

Depending on details of the computations, it can be necessary to compute **Forward** to find this point. Stop computation, and save this curve as usual under the name **LC2withPD** through **Select|Curve** and the **Data Browser**. If we redraw the bifurcation diagram obtained so far (**MatCont|Redraw Diagram** in the **Plot2D** window), we obtain the lines shown in Figure 11.

## 5 Continuation of LPC curves

Load the **LC2with2LPC** via **Select|Curve** menu in the **Data Browser** window. Select the first fold (LPC) bifurcation point in the limit cycle curve **LC2with2LPC** as a starting point for the two-parameter continuation using **Select|Initial Point** menu that opens the **Data Browser** window, see Figure 12. The **Curve Type** field in the main window automatically changes to **LPC** (with the

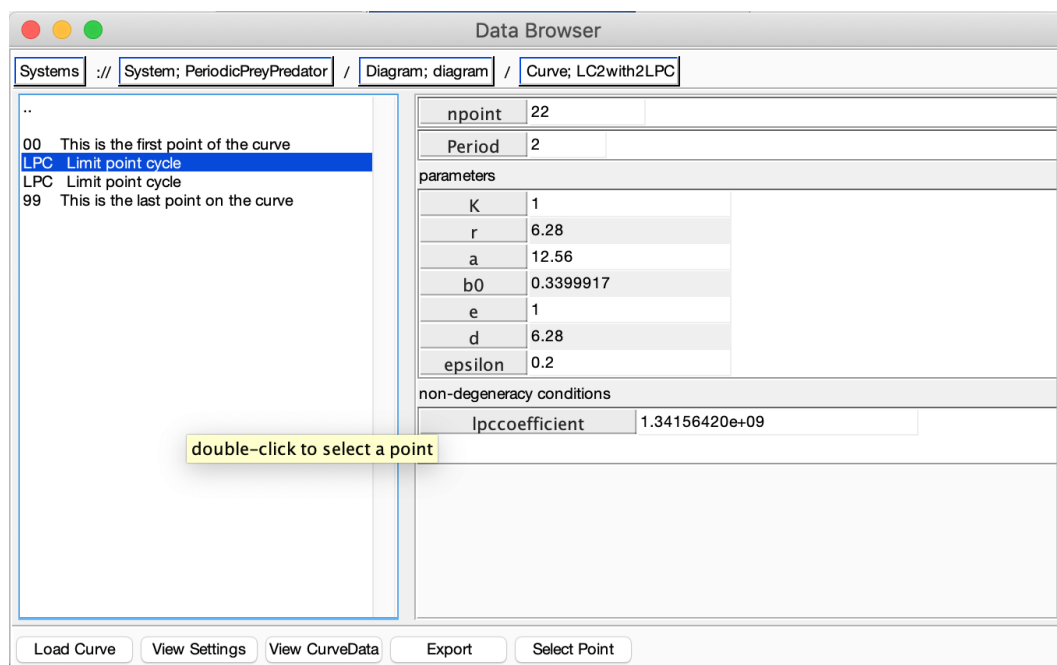


Figure 12: Select first LPC bifurcation.

**Initializer** set to `init LPC_LPC`). Activate the parameters  $b_0$  and  $\varepsilon$  in the **Starter** window, and check that monitoring of all singularities is on. Set **MaxStepsize** equal to 0.1 in the **Continuer**, and compute the LPC curve both forward and backward until it leaves the **Plot2D** window, or the computations stop. Extend computations, if necessary. One of these computations may lead to the **Current step size too small error** message. Save the two obtained **LPC\_LPC** curves as **LPC1for** and **LPC1back**, respectively. Select now the second LPC point in the curve **LC2with2LPC** as the new starting point and perform the  $(\varepsilon, b_0)$ -continuation of LPC curve as above (with **Maxstepsize** equal to 0.1). Extend each computation twice. Save the two obtained **LPC\_LPC** curves as **LPC2for** and **LPC2back**, respectively. Figure 13 shows these LPC-curves.

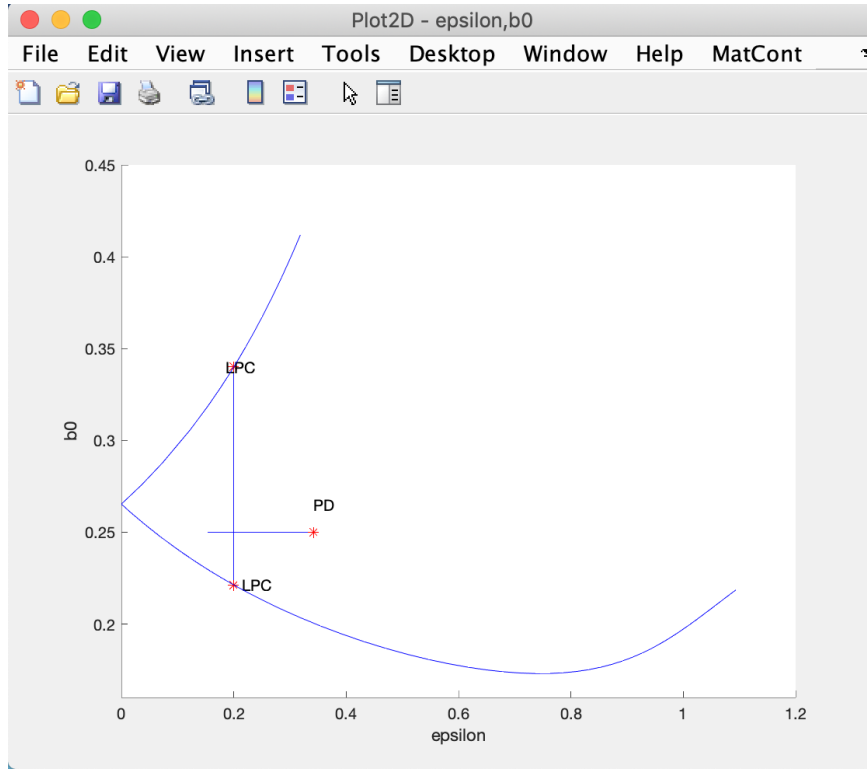


Figure 13: Two LPC bifurcation curves added.

Note that

1. The error message `Current step size too small` pops up during an LPC continuation, which in this case indicates that a codim 2 point is encountered. We will see later that there are Generalized Period Doubling (GPD) points (for a cycle with period one).
2. The two LPC curves meet at  $\varepsilon = 0$ . This is typical for periodically forced systems: At this point the unforced ODE has a limit cycle with period that equals 2.
3. By monitoring the test function for the resonance 1:1 along the lower LPC branch, a change of sign can be observed around  $\varepsilon = 0.6$ . We will return to this later.

## 6 Continuation of the primary PD and NS curves

A robust way to find the first period doubling bifurcation curve is to start with a period-1 limit cycle, that can be found by integration as in Section 3. Note that  $\varepsilon = 0.2$  and change  $b_0$  to 0.4. Using  $X = Y = 0.1, U = 1, V = 0$  and `Interval` equal to 100, we see (in a separate **Plot2D** window) that the corresponding orbit rapidly converges to a stable limit cycle with period 1. Selecting this cycle as in Section 3 and continuing it backward in  $\varepsilon$  (keeping `Period` active, but not monitoring for BPC singularity), we find a PD bifurcation of the period-1 cycle at  $\varepsilon = 0.30509\dots$ :

```
Period Doubling (period = 1.000000e+00, parameter = 3.050914e-01)
Normal form coefficient = 6.270372e-01
```

After that the continuation can be stopped.

Select the PD bifurcation point in the computed limit cycle curve as a starting point for the two-parameter continuation using **Select|Initial point** menu that opens the **Data Browser**

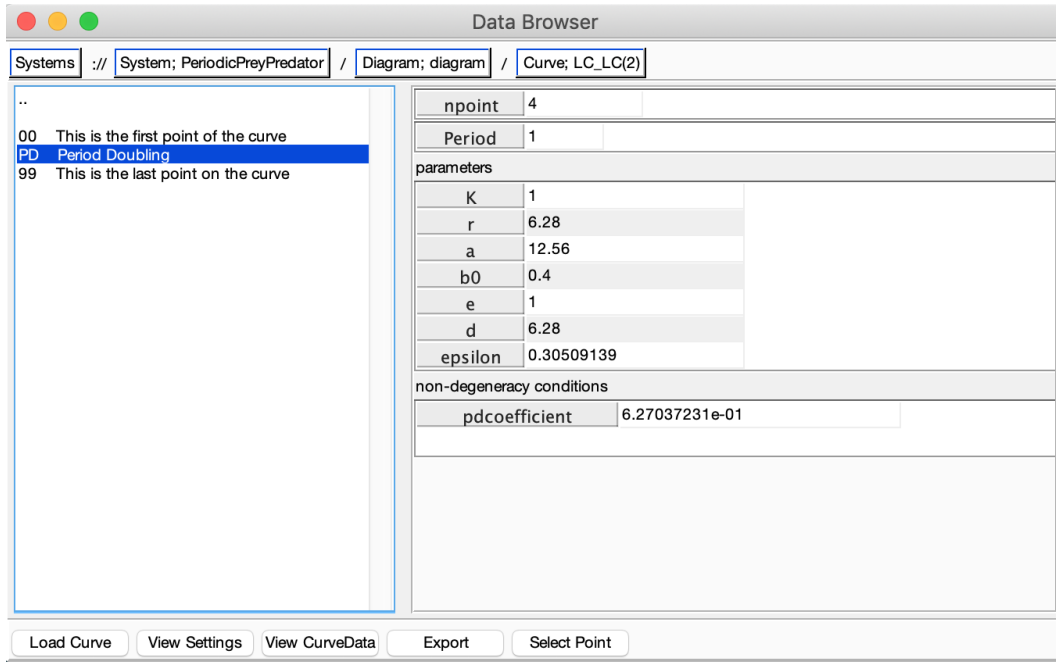


Figure 14: Select the Period Doubling bifurcation point.

window, see Figure 14. In order to start the two-parameter continuation of the PD curve, click **Type|Curve|Period-doubling** in the main MATCONT window. Make necessary changes in the appearing **Starter** and **Continuer** windows as indicated in Figure 15. Compute the period-

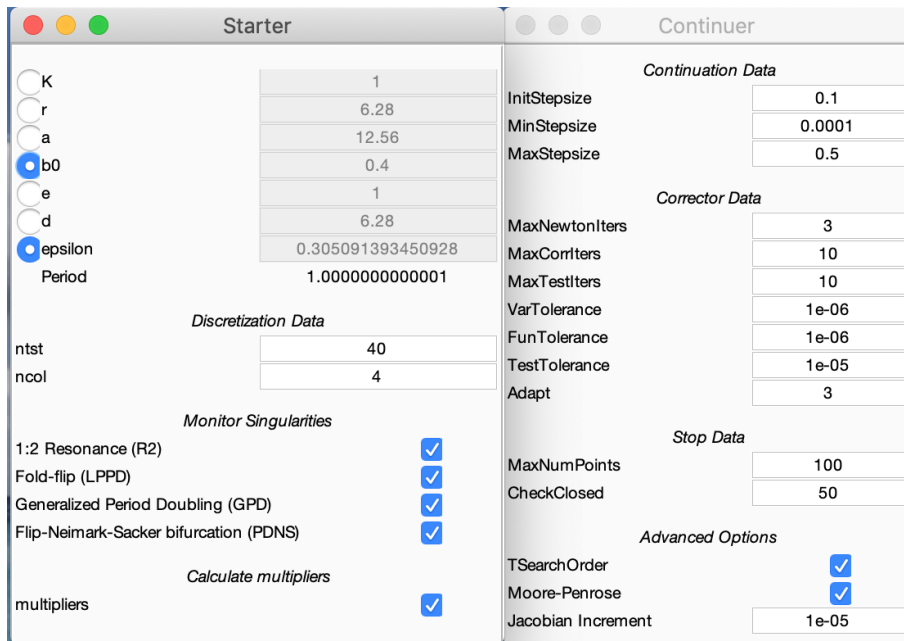


Figure 15: The **Starter** and **Continuer** windows to initialize the continuation of the PD curve for the period-1 cycle.

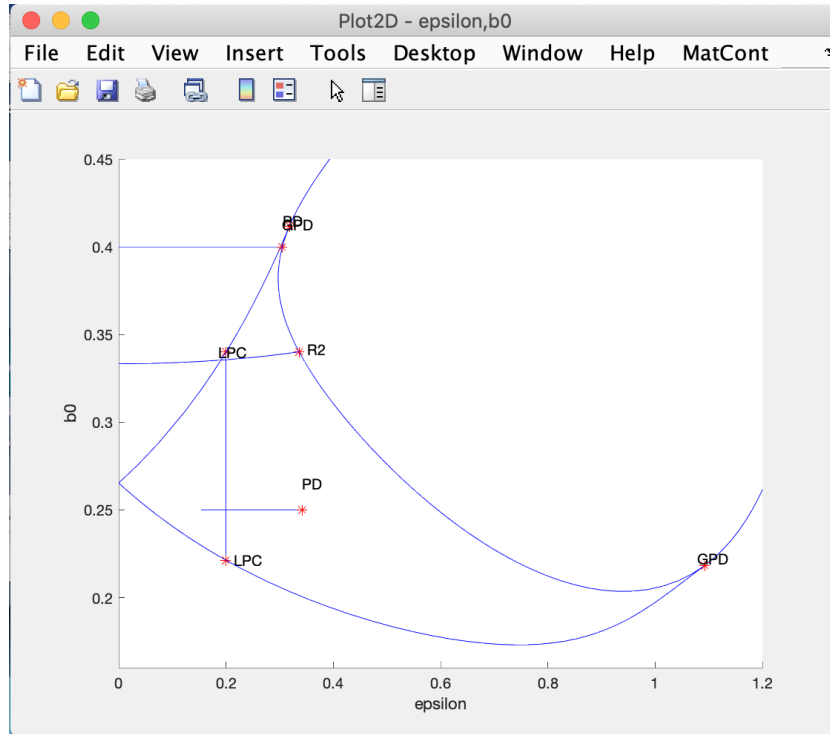


Figure 16: The bifurcation curves of the period-1 cycle added: PD (with two GPD and R2 points) and NS (rooted at R2).

doubling curve forward and backward. During the computations we obtain the following messages in the MATLAB Command Window:

```
Generalized period doubling(period = 1.000000e+00, parameters = 4.117418e-01, 3.182303e-01)
e = -5.859322e+01
```

```
Resonance 1:2 (period = 1.000000e+00, parameters = 3.401303e-01, 3.362213e-01)
(a,b)=(3.400441e+00, -1.292211e+01)
```

```
Generalized period doubling(period = 1.000000e+00, parameters = 2.183705e-01, 1.092944e+00)
e = 4.155484e+01
```

The result of the redrawing is presented in Figure 16. Notice that we have found two Generalized Period Doubling (GPD) and one Resonance 1:2 (R2) codimension 2 bifurcation points. Save the two obtained bifurcation branches as PD1with1GPD and PD1withR2GPD. Notice that these two branches are parts of a closed PD-curve that contains another R2 point far away.

The normal form analysis tells us that the R2 point is always a starting point for a Neimark-Sacker (NS) bifurcation curve of cycles with the same period. To start such an NS curve, select R2 Resonance 2:1 point in the PD1withR2GPD curve using **Select|Initial Point** and the **Data Browser** window, see Figure 17. The **Curve Type** automatically switches to the NS continuation (`init_R2_NS`). Ensure that  $b_0$  and  $\varepsilon$  are selected as active parameters and turn on detection of all singularities, see Figure 18. Do not forget to set **MaxStepsize** equal to 0.05. Start computations with **Compute|Forward** and stop them when the NS curve leaves the window at  $\varepsilon = 0$  (see Figure 16). At the corresponding value  $b_0$ , the unperturbed system has a Hopf bifurcation. Save the obtained Neimark-Sacker curve as NS1.

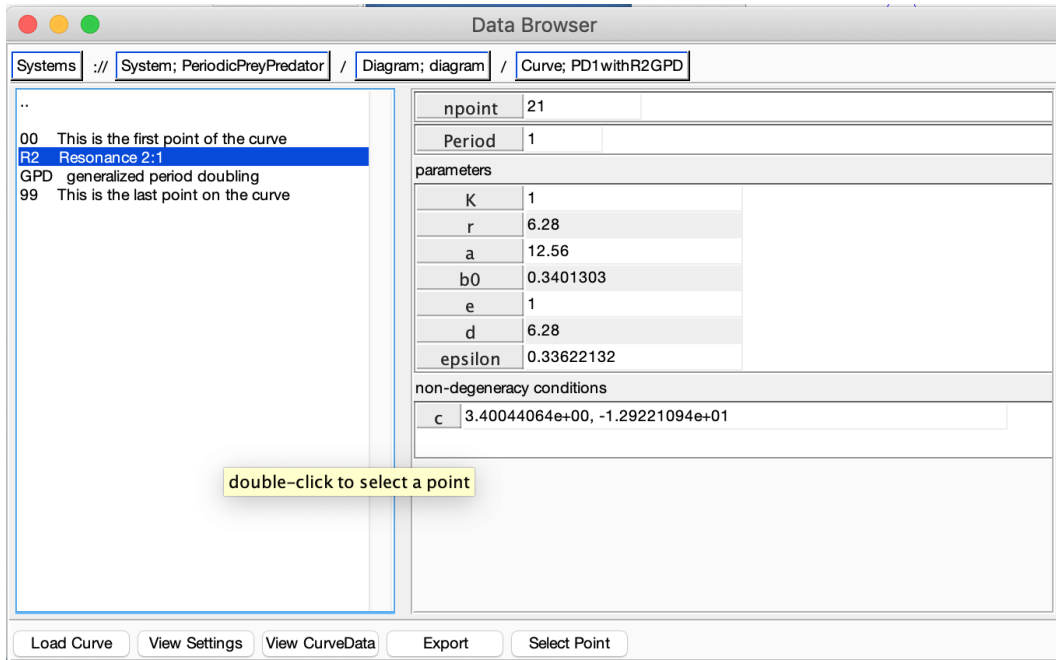


Figure 17: Select R2.

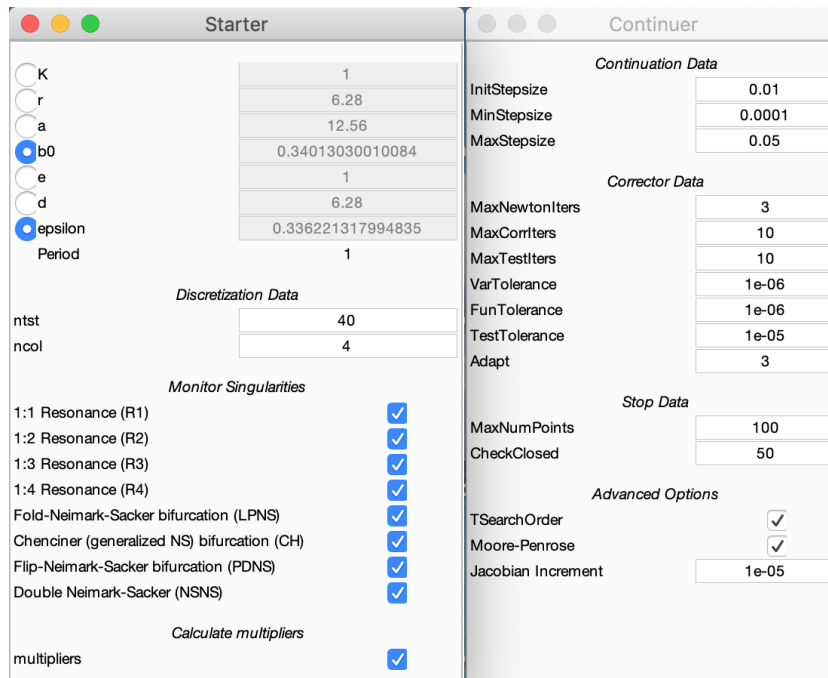


Figure 18: The **Starter** and **Continuer** windows to initialize the continuation of the Neimark-Sacker bifurcation curve from the resonance 2:1 point R2.

## 7 Continuation of the secondary PD and NS curves

The period-doubling bifurcation curve for cycles with period 2 can be started at the PD bifurcation point in the limit cycle curve LC2withPD computed in Section 4. Select this point as in Section 4 and

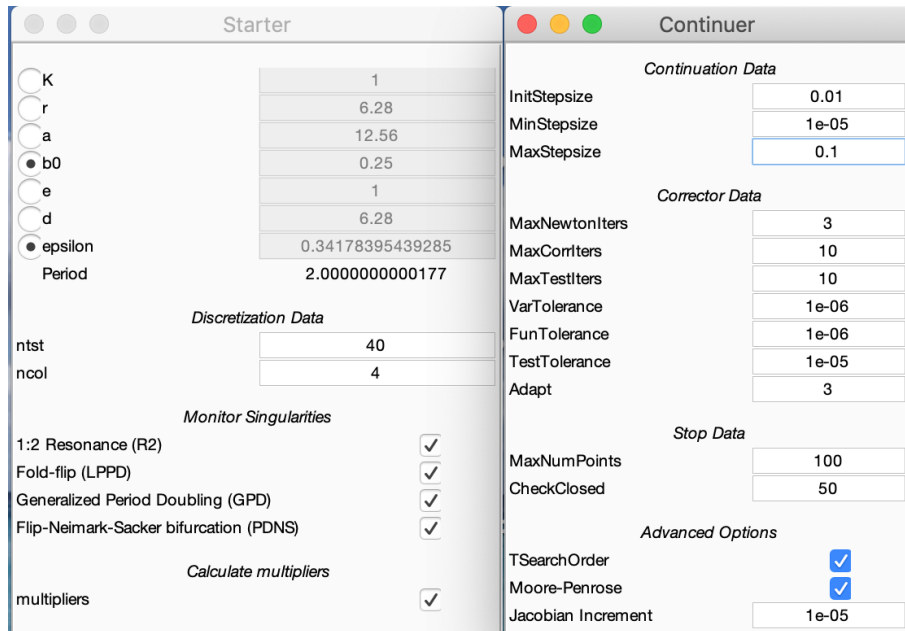


Figure 19: The **Starter** and **Continuer** windows to initialize the continuation of the PD curve for the period-2 cycle.

click **Type|Curve|Period-doubling** in the main MATCONT window. Activate the parameters  $b_0$  and  $\varepsilon$  and make necessary changes in the appearing **Starter** and **Continuer** windows as indicated in Figure 19. Notice that **Period** is equal to 2 now.

Compute the period-doubling curve forward and backward (see Figure 20). Save the computed curves as PD2for and PD2back, respectively. During the continuation, another codim 2 point is found:

Resonance 1:2 (period = 2.000000e+00, parameters = 1.838264e-01, 7.431615e-01)  
 (a,b)=(-6.554884e+01, 1.623625e+01)

As we have already mentioned, this point can be used as a starting point of a Neimark-Sacker bifurcation curve, in this case for cycles with period 2. Now proceed as in Section 6, i.e. select **R2 Resonance 2:1** point in the PD2for curve using **Select|Initial Point** and the **Data Browser** window. The **Curve Type** automatically switches to the NS continuation (`init_R2_NS`). Activate parameters  $b_0$  and  $\varepsilon$ , and turn on detection of all singularities. **Compute|Forward** will produce an NS curve connecting the lower R2 point with the lower branch of the LPC curve, see Figure 20. Save this curve as NS2.

While continuing the NS2, several messages appear in the MATLAB Command window, indicating the detection of codim 2 bifurcations:

Resonance 1:3 (period = 2.000000e+00, parameters = 1.798043e-01, 7.084323e-01)  
 (b,Re(c))=(6.213391e+00 + (1.363706e+00) i,9.133716e+00)

Resonance 1:4 (period = 2.000000e+00, parameters = 1.772744e-01, 6.740362e-01)  
 (A,d)=(1.032144e-01+ (-7.519275e-01) i, 4.459229e+01+ (-1.030700e+02) i)

Resonance 1:1 (period = 2.000000e+00, parameters = 1.771951e-01, 6.029715e-01)  
 ab=4.925658e+02

These messages indicate strong resonances in the NS2 curve.

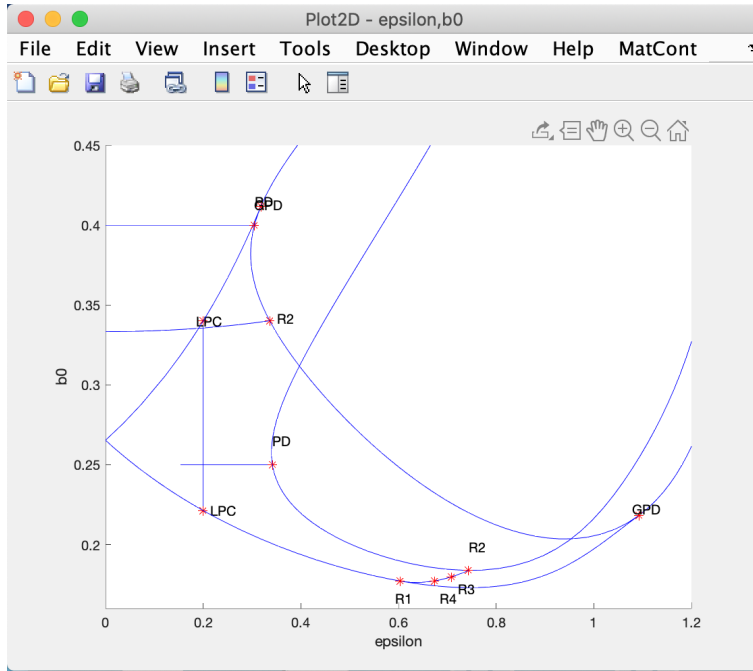


Figure 20: The secondary period-doubling curve with the resonance 1:2 point R2 and the Neimark-Sacker bifurcation curve rooted there are added.

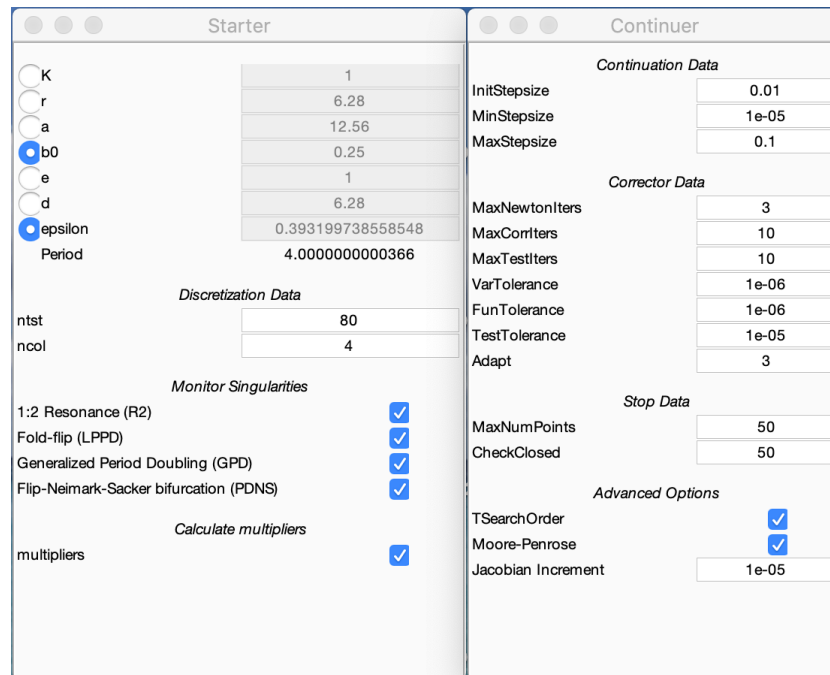


Figure 21: The **Starter** and **Continuer** windows to initialize the continuation of the PD curve for the period-4 cycle.

## 8 Cascade of 1:2 resonances

Actually, the computed normal form coefficients  $(a, b)$  at the resonance 1:2 in Section 7 indicate that this R2 point is also a starting point for the Neimark-Sacker bifurcation curve of the period-4

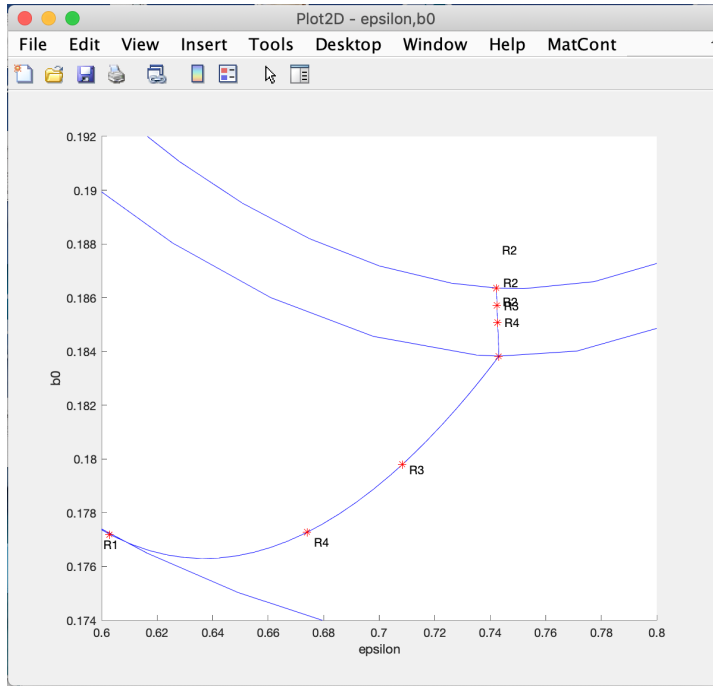


Figure 22: The Neimark-Sacker bifurcation curves rooted at the resonance 1:2 of the period-2 limit cycles; compare with Figure 2.

limit cycles. Indeed, we have  $a < 0$ . To compute this NS curve, the easiest way is to continue first the period-doubling curve for period-4 cycles, detect another R2 point in it, and compute the NS curve for the period-4 cycles that is rooted there with the standard procedure.

Thus, select the PD bifurcation point in the limit cycle curve `LC2withPD` computed in Section 4, and start the (default) continuation of the doubled limit cycle from this point with free parameter  $b_0$  and `ntst` equal to 80. Another PD bifurcation will be detected, from which the two-parameter continuation of the corresponding period-doubling curve can be started forward and backward, see the **Starter** and **Continuer** windows in Figure 21. **ComputeForward** and **ComputeBackward** and save the obtained branches of the period-doubling curve as `PD4for` and `PD4back`, respectively. The continuation revealed a resonance 1:2 point R2:

```
Resonance 1:2 (period = 4.000000e+00, parameters = 1.863585e-01, 7.422188e-01)
(a,b)=(-2.684191e+02, -1.815325e+01)
```

This R2 point can now be used to start the continuation of the NS curve for period-4 cycles. The computations reveal the usual sequence of strong resonances along the NS-curve:

```
Resonance 1:2 (period = 4.000000e+00, parameters = 1.863585e-01, 7.422133e-01)
(a,b)=(-2.684286e+02, -1.810606e+01)
```

```
Resonance 1:3 (period = 4.000000e+00, parameters = 1.857078e-01, 7.424370e-01)
(b,Re(c))=(6.081223e+00 + (1.049292e+01) i,-8.800450e+00)
```

```
Resonance 1:4 (period = 4.000000e+00, parameters = 1.850691e-01, 7.426702e-01)
(A,d)=(-1.337785e-02+ (-6.454597e-01) i, 4.165935e+02+ (-4.857669e+02) i)
```

Figure 22 is a zoom in the bifurcation diagram that shows both period-2 and -4 NS curves rooted at the resonance 1:2 of the period-2 limit cycles.



Actually, there is an infinite accumulating sequence of 1:2 resonances in the period-doubling curves, which are connected by the Neimark-Sacker curves of increasing periods.

## 9 Final picture

Removing now all auxiliary curves and making **MatCont|Redraw Diagram** in the **Plot2D** window, one obtains Figure 23. This figure, that should be compared with Figure 1, also contains the period-8 period-doubling curve. It can be further edited in MATLAB to (re)move labels and change colors of the curves, and finally produce eps or pdf files.

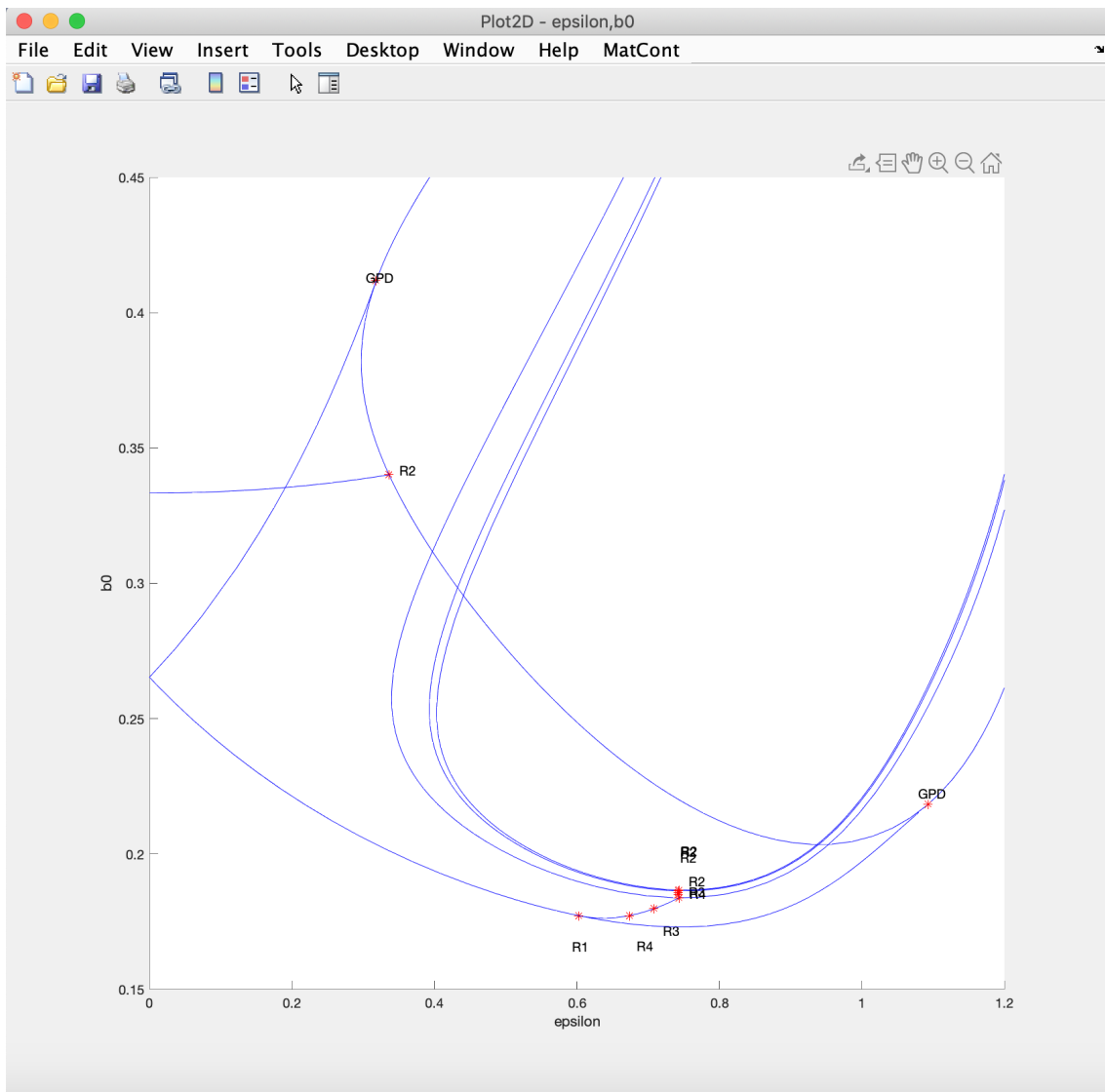


Figure 23: All computed bifurcation curves.

## 10 Additional Problems

A. Consider the following *seasonnaly forced epidemic model*

$$\begin{cases} \dot{S} &= \mu - \mu S - \beta(t)SI, \\ \dot{E} &= \beta(t)SI - (\mu + \alpha)E, \\ \dot{I} &= \alpha E - (\mu + \gamma)I, \end{cases} \quad (3)$$

that describes the spread of a non-lethal disease in a large population. Here the fractions of susceptible ( $S$ ), exposed ( $E$ ), and infective ( $I$ ) individuals are the state variables, while  $(\mu, \alpha, \gamma)$  are positive parameters. The value  $R = 1 - S - E - I$  gives the fraction of recovered (permanently immune) individuals, thus (3) is usually called the *SEIR-model*.

It is assumed that contact rate  $\beta(t)$  is periodic in time with period 1(year), namely

$$\beta(t) = \beta_0(1 + \delta \cos(2\pi t)),$$

where  $\beta_0 \geq 0$  is the mean contact rate and  $0 \leq \delta \leq 1$  is the degree of seasonality.

The aim is to study with MATCONT existence and stability of period-1,-2, and -3 cycles in (3) for fixed

$$\mu = 0.02, \quad \alpha = 35.842, \quad \gamma = 100,$$

corresponding to measles, when  $(\delta, \beta_0) \in [0, 0.6] \times [0, 6000]$ . This model was studied<sup>4</sup> some time ago using the interactive bifurcation software LOCBIF.

To analyse the model with MATCONT, you could use the following strategy:

1. Consider an equivalent autonomous system

$$\begin{cases} \dot{S} &= \mu - \mu S - \beta_0(1 + \delta u)SI, \\ \dot{E} &= \beta_0(1 + \delta u)SI - (\mu + \alpha)E, \\ \dot{I} &= \alpha E - (\mu + \gamma)I, \\ \dot{u} &= u - 2\pi v - (u^2 + v^2)u, \\ \dot{v} &= 2\pi u + v - (u^2 + v^2)v, \end{cases}$$

and introduce new variables

$$\begin{cases} s &= \ln S, \\ e &= \ln E, \\ i &= \ln I, \end{cases}$$

to better handle very small values of  $S, E$ , and  $I$ .

2. Find by simulations a period-1 cycle in the  $(s, e, i, u, v)$ -space at  $\delta = 0$  and  $\beta_0 = 5000$ . (*Hint*: Use a stiff integration Method, e.g. `ode23s`.)
3. Continue the found period-1 cycle w.r.t.  $\delta$  and find its period-doubling (PD) bifurcation.
4. Starting from the obtained PD-point, compute the period-doubling curve  $PD^{(1)}$  in the  $(\delta, \beta_0)$ -plane and locate two different generalized period-doubling (GPD) points. Report the parameter values corresponding to these codim 2 points.
5. Starting from the PD-point, continue the period-2 cycle and find two limit point of cycles (LPC) bifurcations.

Use the found LPC points to compute the  $LPC_{1,2}^{(2)}$  bifurcation curves in the  $(\delta, \beta_0)$ -plane and locate a cusp point of cycles (CPC) where they meet. Report the parameter values corresponding to this codim 2 bifurcation point. (*Hint*: MatCont will not detect CPC in this case, so use the numerical output to locate it approximately.)

What are the other end-points of these curves ?

---

<sup>4</sup>Kuznetsov, Yu.A. and Piccardi, C. *Bifurcation analysis of periodic SEIR and SIR epidemic models*. J. Math. Biol. 30 (1994), 109-121

6. Compute the bifurcation curve  $PD^{(2)}$  where the period-2 cycle exhibits a period-doubling bifurcation. (*Hint*: To locate a point on this curve, continue w.r.t.  $\beta_0$  a branch of the period-2 cycles, starting from the PD-point found in Step 3.)
  7. Compute a curve  $LPC^{(3)}$  where a stable and an unstable period-3 cycles are born via the LPC-bifurcation. (*Hint*: To locate a stable period-3 cycle, simulate the system at  $(\delta, \beta_0) = (0.1, 1200)$ .)  
Compute a curve  $PD^{(3)}$  corresponding to the period-doubling bifurcation of the stable period-3 cycle.
  8. Classify cycles existing in various domains in the  $(\delta, \beta_0)$ -plane to the left from the curves  $PD^{(2)}$  and  $PD^{(3)}$ .
- B. Consider a *two-patch predator-prey model*<sup>5</sup>, where predators can migrate between the two patches by diffusion. One of the patches experiences seasonal influences while the other can be seen as a wild-life refuge where human intervention minimises seasonal influences. The system is defined by

$$\begin{cases} \dot{x}_1 &= r_1 x_1 (1 - x_1) - \frac{c x_1 x_2}{x_1 + b_1 (1 + \varepsilon \cos t)}, \\ \dot{x}_2 &= -x_2 + \frac{c x_1 x_2}{x_1 + b_1 (1 + \varepsilon \cos t)} + \gamma (y_2 - x_2), \\ \dot{y}_1 &= r_2 y_1 (1 - y_1) - \frac{c y_1 y_2}{y_1 + b_2}, \\ \dot{y}_2 &= -y_2 + \frac{c y_1 y_2}{y_1 + b_2} + \gamma (x_2 - y_2). \end{cases} \quad (4)$$

The variables of  $x_1$  and  $x_2$  denote the numbers of individuals (or densities) respectively of prey and predator populations living outside the refuge and  $y_1$  and  $y_2$  are the corresponding numbers or densities inside. The intrinsic growth rates  $r_i$  and the constant attack rate  $c$  are parameters of the model. For the predator outside the refuge, the Holling type II is chosen as functional response with a half saturation which varies periodically with period  $2\pi$ . The terms with parameter  $\gamma$  describe the coupling of the two patches.

Fix the following parameter values

$$r_1 = 1, \quad r_2 = 1, \quad b_1 = 0.4, \quad \gamma = 0.1, \quad c = 2,$$

and use the half saturation  $b_2$  as a continuation parameter together with the amplitude of the seasonal forcing  $\varepsilon$ .

Study  $2\pi$ - and  $4\pi$ -cycles of (4) within the parameter region  $(b_2, \varepsilon) \in ]0, 0.4[ \times ]0, 1]$ . In particular, compute relevant period-doubling and Neimark-Sacker bifurcation curves in the  $(b_2, \varepsilon)$ -plane.

---

<sup>5</sup>Jansen, V.A.A.. *Regulation of predator-prey systems through spatial interactions: A possible solution to the paradox of enrichment*, *Oikos* 74 (1995), no. 3, 384-390.