

TUTORIAL VI:  
Using MATCONT to find and continue orbits homoclinic to  
hyperbolic equilibria

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June 13, 2021

This session was tested on MATCONT7.3 with MATLAB2020b, R9.9. MATCONT provides several tools for the initialization and continuation of connecting orbits. Orbits homoclinic to saddle equilibria can be initialized from Bogdanov-Takens points. Orbits homoclinic to hyperbolic and to saddle-node equilibria can be initialized from periodic orbits with a large period, typically in cases where the period tends to infinity when a system parameter converges to a homoclinic value.

Orbits homoclinic to saddles can be initialized by homotopy, starting from a saddle equilibrium. This method is summarized in Appendix. Orbits homoclinic to saddle-nodes can also be initialized by homotopy, starting from a limit point (LP) equilibrium. Heteroclinic connections can be initialized from pairs of hyperbolic equilibria (under conditions that involve the dimensions of the stable and unstable manifolds of these equilibria)<sup>1</sup>.

In the present tutorial we consider in detail the homotopy method to initialize and continue homoclinic connections to hyperbolic equilibria.

## 1 Traveling pulses in the FitzHugh-Nagumo model

The following system of partial differential equations is the FitzHugh-Nagumo model of the nerve impulse propagation along an axon:

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - f_a(u) - v, \\ \frac{\partial v}{\partial t} = bu, \end{cases} \quad (1)$$

where  $u = u(x, t)$  represents the membrane potential;  $v = v(x, t)$  is a phenomenological “recovery” variable;  $f_a(u) = u(u - a)(u - 1)$ ,  $1 > a > 0$ ,  $b > 0$ ,  $-\infty < x < +\infty$ ,  $t > 0$ .

*Traveling waves* are solutions to these equations of the form

$$u(x, t) = U(\xi), \quad v(x, t) = V(\xi), \quad \xi = x + ct,$$

where  $c$  is an a priori unknown wave propagation speed. The functions  $U(\xi)$  and  $V(\xi)$  satisfy the system of three ordinary differential equations

$$\begin{cases} U' = W, \\ W' = cW + f_a(U) + V, \\ V' = \frac{b}{c}U. \end{cases} \quad (2)$$

System (2) is called a *wave system*. It depends on three positive parameters  $(a, b, c)$ . Any bounded orbit of (2) corresponds to a traveling wave solution of the FitzHugh-Nagumo system (1) at parameter values  $(a, b)$  propagating with velocity  $c$ .

For all  $c > 0$  the wave system has a unique equilibrium  $0 = (0, 0, 0)$  with one positive eigenvalue  $\lambda_1$  and two eigenvalues  $\lambda_{2,3}$  with negative real parts. The equilibrium can be either a saddle or a saddle-focus and has in both cases a one-dimensional unstable and a two-dimensional stable invariant manifold  $W^{u,s}(0)$ . The transition between saddle and saddle-focus cases is caused by the presence of a double negative eigenvalue; for fixed  $b > 0$  this happens on the curve

$$D_b = \{(a, c) : c^4(4b - a^2) + 2ac^2(9b - 2a^2) + 27b^2 = 0\}.$$

A branch  $W_1^u(0)$  of the unstable manifold leaving the origin into the positive octant can return back to the equilibrium, forming a homoclinic orbit  $\Gamma_0$  at some parameter values.

For  $b > 0$ , these parameter values form a curve  $P_b^{(1)}$  in the  $(a, c)$ -plane that can only be found numerically. As we shall see, this curve passes through the saddle-focus region delimited by  $D_b$ .

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<sup>1</sup>The homotopy method for connections homoclinic to saddle-node starts by selecting **Type|Initial point|Limit point** and **Type|Curve|ConnectionSaddleNode** in the MATCONT main panel. The homotopy method for heteroclinic connections is another variant and starts by selecting **Type|Initial point|Equilibrium** and **Type|Curve|ConnectionHet** in the MATCONT main panel.

Any homoclinic orbit defines a traveling *impulse*. The shape of the impulse depends on the type of the corresponding equilibrium: It has a monotone “tail” in the saddle case and an oscillating “tail” in the saddle-focus case.

The saddle quantity  $\sigma_0 = \lambda_1 + \operatorname{Re} \lambda_{2,3}$  is always positive for  $c > 0$ . Therefore, the phase portraits of (2) near the homoclinic curve  $P_b^{(1)}$  are described by Shilnikov’s Theorems. In particular, near the homoclinic bifurcation curve  $P_b^{(1)}$  in the saddle-focus region, system (2) has an infinite number of saddle cycles. These cycles correspond to *periodic wave trains* in the FitzHugh-Nagumo model (1). Secondary homoclinic orbits existing in (2) near the primary saddle-focus homoclinic bifurcation correspond to *double traveling impulses* in (1). An infinite number of the corresponding secondary homoclinic bifurcation curves  $P_{b,j}^{(2)}$  in (2) originate at each point  $A_{1,2}$ , where  $P_b^{(1)}$  intersects  $D_b$ .

We will locate a critical value of  $c$  for  $a = 0.15$  and  $b = 0.0025$ , at which (2) has a homoclinic orbit, continue this homoclinic orbit with respect to the parameters  $(a, c)$ , and detect the codim 2 bifurcations points  $A_{1,2}$  in  $P_b^{(1)}$ .

## 1.1 System specification

Specify a new ODE system with the coordinates  $(U, W, V)$ , parameters  $(a, b, c)$ , and time  $xi$ :

```
U' = W
W' = c*W + U*(U - a)*(U - 1.0) + V
V' = b*U/c
```

Generate the derivatives of order 1, 2, and 3 symbolically.

## 1.2 Location of a homoclinic orbit by homotopy

This consists of several steps, each presented in a separate subsection.

### 1.2.1 Approximating the unstable manifold by integration

Select **Type|Initial point|Equilibrium** and **Type|Curve|ConnectionSaddle**.

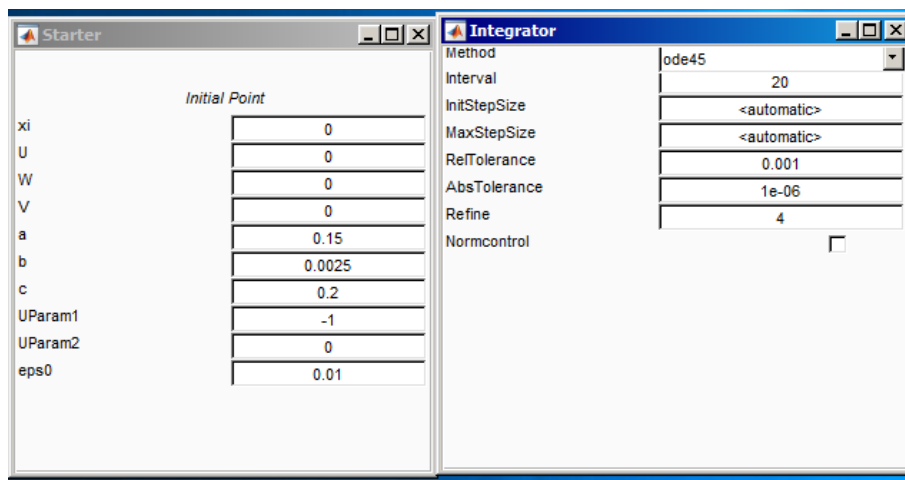


Figure 1: **Starter** and **Integrator** windows for the integration of the unstable manifold.

In the appearing **Integrator** window, increase the integration **Interval** to 20 (see the right panel of Figure 1).

Via the **Starter** window, input the initial values of the system parameters

a            0.15  
 b            0.0025  
 c            0.2

as well as

Uparam1     -1  
 eps0         0.01

that specify direction and distance of the displacement from the saddle

U            0  
 W            0  
 V            0

along the unstable eigenvector<sup>2</sup>. The **Starter** window should look like in the left panel of Figure 1.

Open a **2Dplot** window with **Window/Output|Graphic|2Dplot**. Select U and V as variables along the corresponding axes and input the following plotting ranges

Abcissa:         -0.2            0.5  
 Ordinate:       -0.05           0.1

Start **Compute|Forward**. You will get an orbit approximating the unstable manifold that departs from the saddle in a non-monotone way, see Figure 2. This orbit does not resemble a homoclinic orbit.

Now press the **View Result** button in the **Control** window. This opens a **Data Browser** window as shown in Figure 3.

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<sup>2</sup>Uparam2 is only used when  $\dim W^u \geq 2$ . On some platforms, Uparam1=1 should be used to select the correct direction.

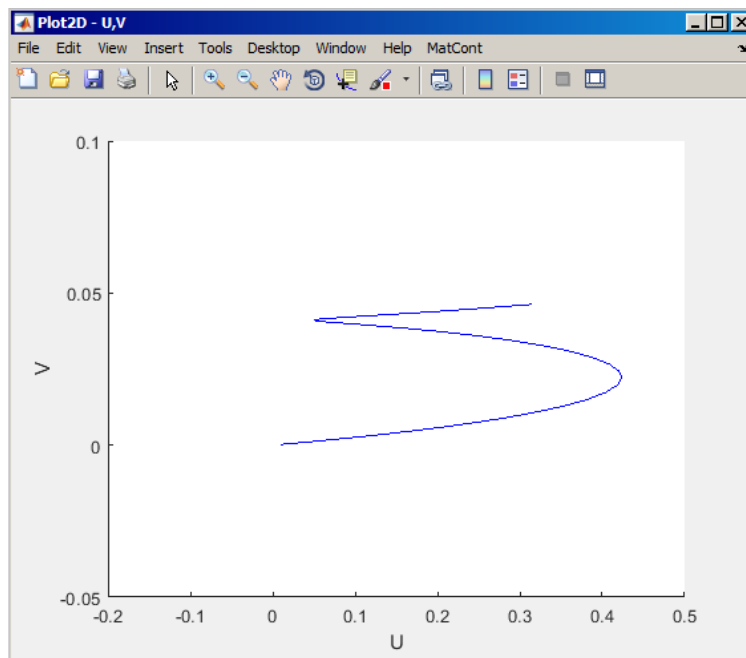


Figure 2: A segment of the unstable manifold of the saddle at the initial parameter values.

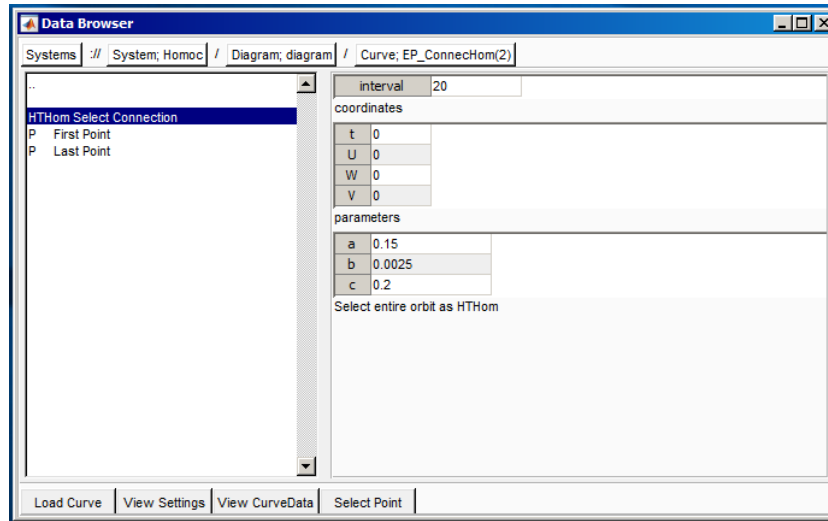


Figure 3: **Data Browser** window with highlighted **HTHom Select Connection**.

Click the **HTHom Select Connection** and then select it as initial point by clicking **Select Point** at the bottom of the **Data Browser** window.

MATCONT will search for a point in the computed orbit where the distance to the *stable* eigenspace of the Jacobian matrix of the saddle is stopped decreasing for the last time. This point is selected as the end-point of the initial connecting orbit (as we shall see, it corresponds to the time-interval  $T=8.40218$ ). The program will ask to choose the BVP-discretization parameters **ntst** and **ncol** that will be used in all further continuations. Set **ntst** equal to 50 and keep **ncol** equal to 4 (Figure 4). Press OK.

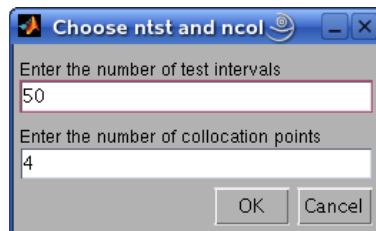


Figure 4: The discretization parameters for homotopy BVPs.

### 1.2.2 Homotopy towards the stable eigenspace

In the new **Starter** window, activate the parameters **c**, **SParam1**, and **eps1** (see Figure 5), and **ComputeBackward**. A family of curves will be produced by continuation (see Figure 6) and the message

**SParam equal to zero**

will indicate that the end-point has arrived at the stable eigenspace of the saddle (i.e. reached the plane tangent to the stable invariant manifold at the saddle and given by the condition  $SParam1=0$ ). The corresponding orbit segment is labeled **HTHom**.

Stop the continuation there.

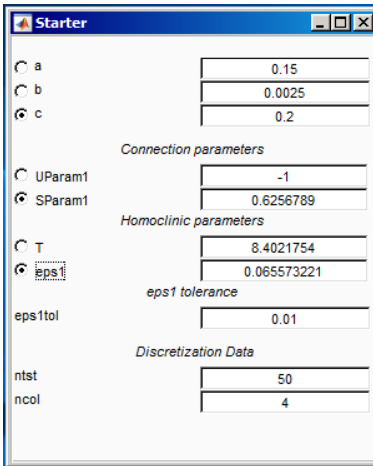


Figure 5: **Starter** window for the homotopy towards the stable eigenspace.

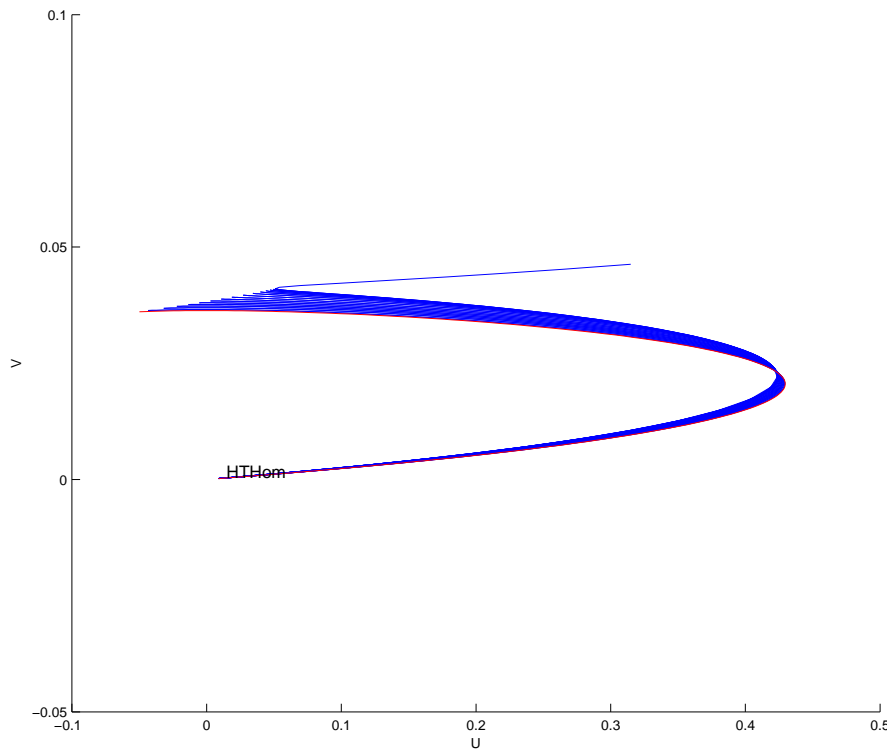


Figure 6: The unstable manifold with the end-point in the stable eigenspace of the saddle.

### 1.2.3 Homotopy of the end-point towards the saddle

The obtained segment is still far from the homoclinic orbit but can be selected as the initial point for a homotopy of the end-point towards the saddle. Select

HTHom: SParam equal to zero

via the **Data Browser** window.

In the **Continuer** window, set **MaxStepsize** to 0.5, see in the right panel of Figure 7.

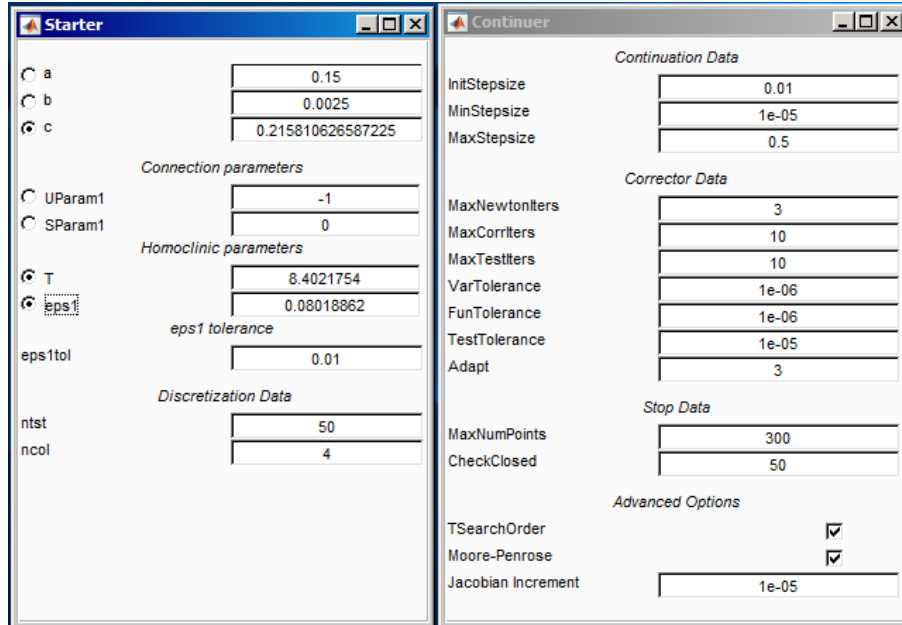


Figure 7: **Starter** and **Continuer** windows for the homotopy towards the saddle.

In the **Starter** window, **SParam1** now equals to zero, while the parameter **c** is adjusted. Activate parameters **c**, **T**, and **eps1** there. Set **eps1tol** equal to 0.01; this will be used as the target distance **eps1** from the end-point to the saddle.

Open a **Numeric** window to monitor the values of the active parameters. Clean the **Plot2D** window and **Compute|Forward**. You should get Figure 8, where the last computed segment is again labeled by **HTHom**. The message

```
eps1 small enough
```

appears indicating that a good approximation of the homoclinic orbit is found. The begin- and the end-points are now both located near the saddle (at distance 0.01). The **Numeric** window at the last computed point is presented in Figure 9. It can be seen that **eps1** became 0.01, while the time-interval **T** increased to 36.6206. Stop the continuation.

### 1.3 Continuation of the homoclinic orbit

Select the just computed point

```
HTHom: eps1 small enough
```

via **Control|View Result** and **Data Browser** windows as initial point.

Select **Type|Curve|Homoclinic to Saddle** and check that the curve type is **Hom**, while the initial point is of type **HTHom**.

In the new **Starter** window, activate two system parameters: **a** and **c** as well as the homoclinic parameter **T** (see Figure 10). These parameters will vary along the homoclinic curve, while both **eps0** and **eps1** (the begin- and end-distances to the saddle) will be fixed. Also, opt to detect the singularity **Double Real Stable leading eigenvalue (DRS)** along the homoclinic curve, see Figure 10.

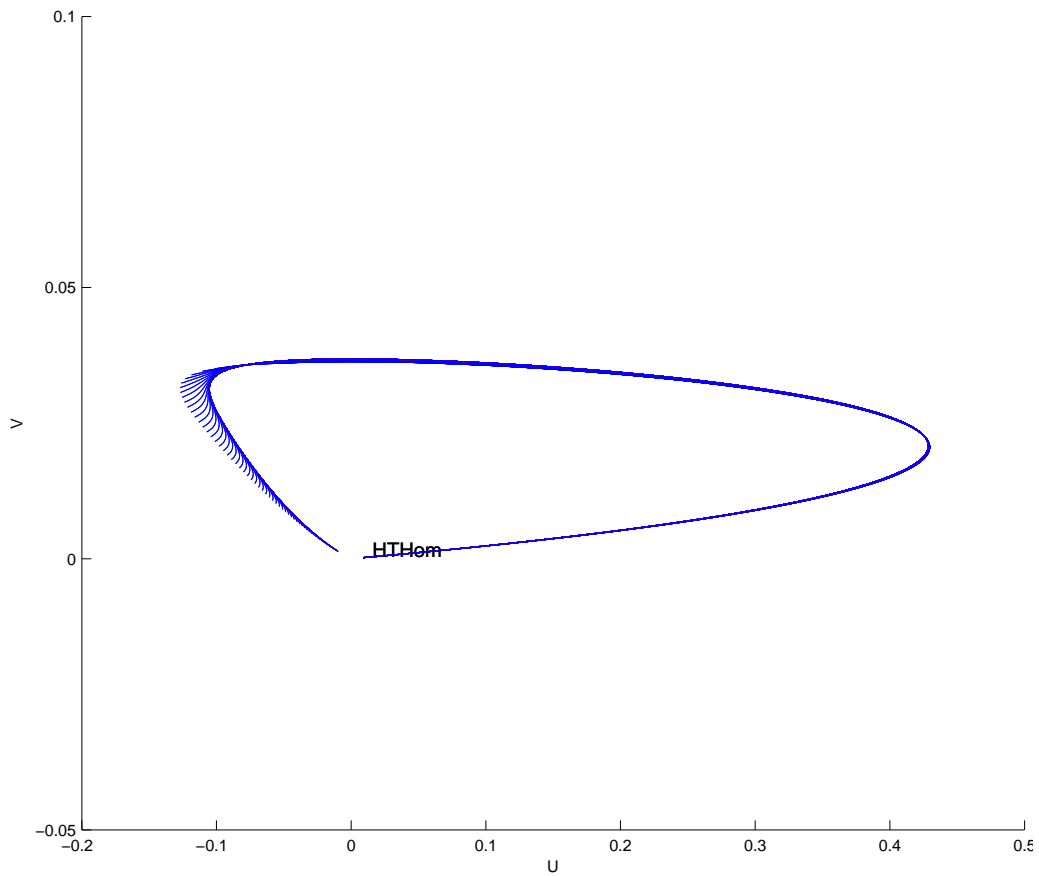


Figure 8: The homotopy results in the manifold segment with both the begin- and the end-points near the saddle.

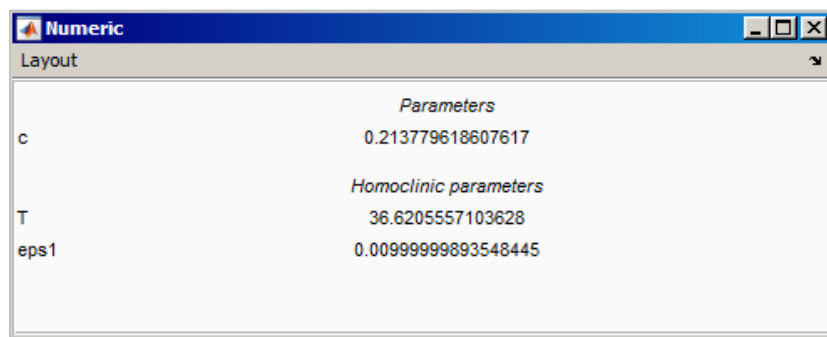


Figure 9: **Numeric** window at the last point of the homotopy towards the saddle.



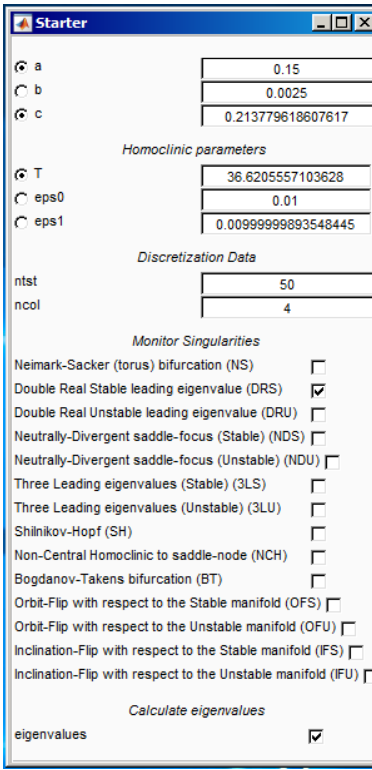


Figure 10: **Starter** window for the two-parameter homoclinic continuation.

To terminate the continuation at  $a = 0$ , add via **Select|System| Manage Userfunctions** a user function **STOP** with the associated label **0** and **res=a** in the edit field of the **User functions** window. Check in the **Starter** window that the user function **STOP** is monitored.

In the **Continuer** window, increase the **MaxStepsize** to 1.

Change the layout of the **Plot2D** window via **MatCont|Layout**: Select **a** and **c** as the abscissa and ordinate with the visibility ranges

```
Abscissa:      0          0.3
Ordinate:     0          0.8
```

Now you are ready to start the continuation. **Compute|Forward** and **Backward**, resume computations at the DRS-points, and stop when a zero of the user-defined function **STOP** is detected. Two DRS-points will be located, where the equilibrium undergoes the saddle-to-saddle-focus transition:

```
Double real stable leading eigenvalues, parameters = 0.131821 and 0.217156.
Double real stable leading eigenvalues, parameters = 0.107654 and 0.532073.
```

These are codim 2 bifurcation points  $A_{1,2}$  introduced in Section 1.

Delete all previously computed curves except the last two, namely

```
HTHom_Hom(1)
HTHom_Hom(2)
```

and click **MatCont|Redraw Diagram**. This should produce Figure 11.

To verify that that all computed points indeed correspond to homoclinic orbits, open a **3Dplot** window and select **U,W** and **V** as variables along the coordinate axes with the visibility limits

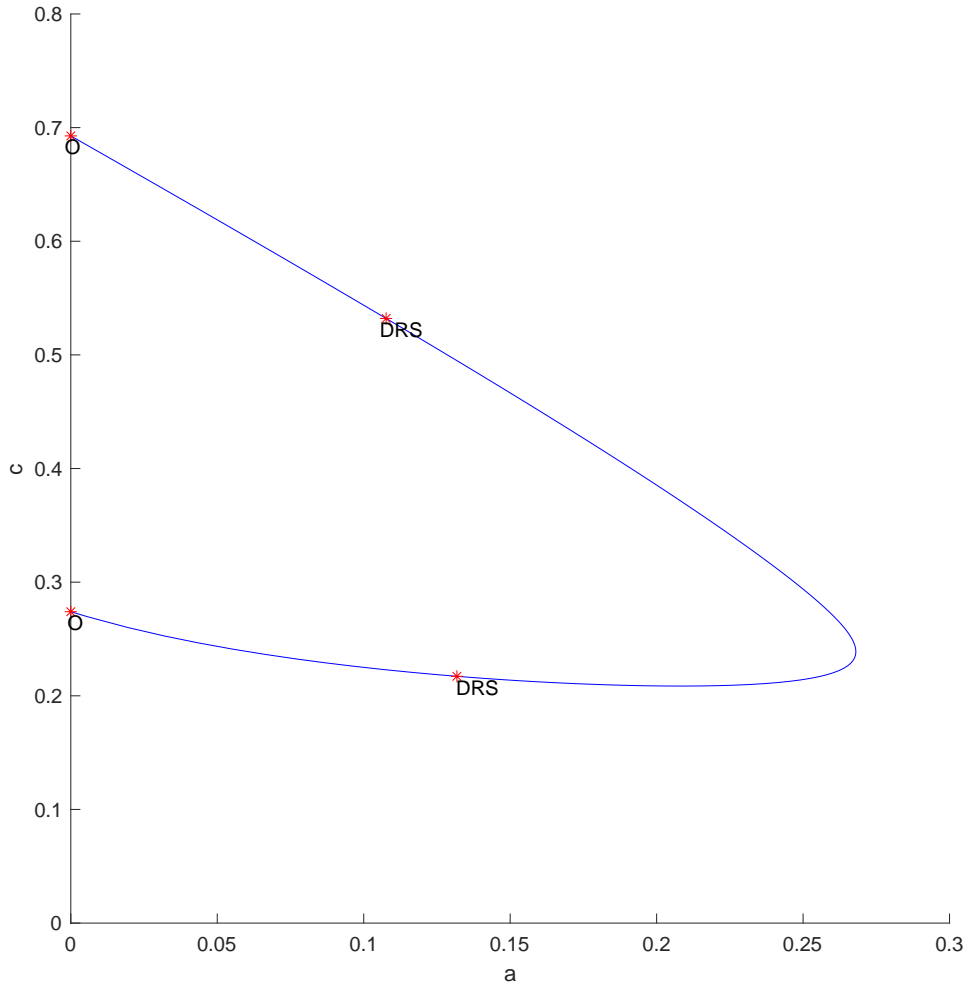


Figure 11: The homoclinic bifurcation curve in the  $(a, c)$ -plane. The saddle to saddle-focus transitions  $A_{1,2}$  are labeled by **DRS**, while the labels **0** indicate  $a = 0$ , where the continuation is terminated. The values of  $c$  at fixed  $a$  are possible propagation velocities of the impulses.

Abscissa:	-0.4	1.2
Ordinate:	-0.2	0.2
Applicate:	-0.01	0.2

respectively. **MatCont|Redraw Diagram** in this new window should produce Figure 12.

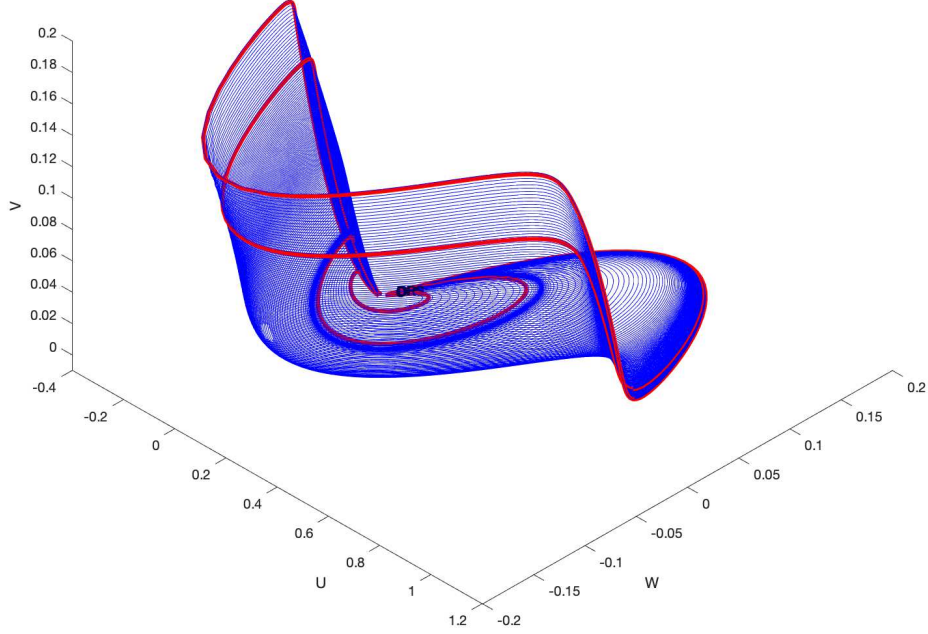


Figure 12: The family of homoclinic orbits in the phase space of system (2) for  $b = 0.0025$ .

## 2 The Hopf-Hopf normal form with broken symmetry

We start with the truncated normal form of the Hopf-Hopf codimension-2 bifurcation in polar coordinates:

$$\begin{cases} \dot{r}_1 = r_1(\mu_1 + p_{11}r_1^2 + p_{12}r_2^2 + s_1r_2^4) \\ \dot{r}_2 = r_2(\mu_2 + p_{21}r_1^2 + p_{22}r_2^2 + s_2r_1^4) \\ \dot{\varphi}_1 = \omega_1 \\ \dot{\varphi}_2 = \omega_2. \end{cases}$$

### 2.1 System specification

We rewrite this system in the cartesian  $(x_1, y_1, x_2, y_2)$ -coordinates (replacing  $\mu$  by  $u$  and  $\omega$  by  $w$  for simplicity in MATCONT) and add order-6 terms so that we obtain the following equations:

$$\begin{cases} \dot{x}_1 = x_1(u_1 + p_{11}(x_1^2 + y_1^2) + p_{12}(x_2^2 + y_2^2) + s_1(x_2^2 + y_2^2)^2) - y_1w_1 + \mathbf{3y}_1^6 \\ \dot{y}_1 = y_1(u_1 + p_{11}(x_1^2 + y_1^2) + p_{12}(x_2^2 + y_2^2) + s_1(x_2^2 + y_2^2)^2) + x_1w_1 - \mathbf{2x}_1^6 \\ \dot{x}_2 = x_2(u_2 + p_{21}(x_1^2 + y_1^2) + p_{22}(x_2^2 + y_2^2) + s_2(x_1^2 + y_1^2)^2) - y_2w_2 - \mathbf{7y}_1^6 \\ \dot{y}_2 = y_2(u_2 + p_{21}(x_1^2 + y_1^2) + p_{22}(x_2^2 + y_2^2) + s_2(x_1^2 + y_1^2)^2) + x_2w_2 + \mathbf{x}_1^6. \end{cases}$$

The initial parameter values are:  $u_1 = 9.7, u_2 = -50, p_{11} = 1, p_{12} = 1.5, p_{21} = -2, p_{22} = -1, s_1 = 1.3, s_2 = 1.7, w_1 = 0.001, w_2 = 0.00235$ .

## 2.2 Location of a connecting orbit by homotopy

### 2.2.1 Approximating the unstable manifold by integration

Point  $(0, 0, 0, 0)$  is an equilibrium of this system with the eigenvalues

$$\begin{pmatrix} 9.7 + 0.001i \\ 9.7 - 0.001i \\ -50 + 0.00235i \\ -50 - 0.00235i \end{pmatrix},$$

so we have a two-dimensional unstable manifold and a two-dimensional stable manifold where in both cases the eigenvalues consist of a conjugate complex pair. This gives us an interesting example for the application of the homotopy method.

Since the eigenvalues with smallest positive real part constitute a conjugate complex pair, both `UParam1` and `UParam2` play a role in the initialization, we set them equal to  $-1$  and  $1$ , respectively. The unstable parameters are now automatically rescaled: `UParam1` =  $-0.7071$ , `UParam2` =  $0.7071$ . For the value of `epsilon_0` we take  $1.4142e-4$ . Therefore we start time integration from the point

$$x(0) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + 1.4142e-4(-0.7071q_{0,1}^{(0)} + 0.7071q_{0,2}^{(0)}),$$

where  $q_0^1 = (1, 0, 0, 0)$  and  $q_0^2 = (0, 1, 0, 0)$  span the two-dimensional eigenspace corresponding to the eigenvalues  $9.7 \pm 0.001i$ .

We start by selecting **Type|Initial point|Equilibrium** and **Type|Curve|ConnectionSaddle** in the `MATCONT` main panel. After setting the integration interval to  $1.04$  we get the **Starter** and **Integrator** windows shown in Figure 13. We note that the value for `eps0` can be input as `sqrt(2)*1.0e-4`.

When we execute time-integration, we obtain Figure 14.

### 2.2.2 Homotopy towards the stable eigenspace

From Figure 14 we guess that initially `eps1` increases, then it decreases, and we take the point where `eps1` stops decreasing as endpoint of the initial connecting orbit. To this end we click the **View Result** button in the **Control** window to open the **Data Browser**. We then select the **HTHom Select Connection** point. For the discretization we use 50 mesh intervals and 4 collocation points. This opens the **Starter** window for an `init_HTHom_HTHom` initializer. We see that none of the `SParam` values equals zero, but they are quite small;  $-0.039964$  and  $-0.751048$ , respectively. Following the outline in Appendix we activate both unstable homoclinic parameters, both stable homoclinic parameters and `eps1`. The **Starter** is shown in Figure 15 (left).

During the continuation, we detect a point where `SParam1` vanishes. We stop the continuation here and select the **HTHom SParam equal to zero** point via the **View Result** button in the **Control** window. This opens the **Starter** window for the second `init_HTHom_HTHom` initializer. We see that `SParam1` is zero, `SParam2` equals  $-0.749078$ . Following the outline in Appendix, we activate `SParam2` and `eps1` and the system parameter `u2`. The **Starter** is shown in Figure 15 (right).

In this continuation (**Forward** and **Extend**) we detect a point where the `SParam2` also vanishes. Here `eps1` =  $1.2457$ . The next continuation (**Backward** and **Extend** many times), with `T`, `eps1` and one system parameter free, again we take `u2`, makes `eps1` smaller as needed. This gives us the end-values: `T` =  $0.66932$ , `eps1` =  $0.01$ , `u2` =  $-14.395$ . Now we can start the continuation of homoclinic orbits (**Type|Curve|Homoclinic to Saddle**), with `T` and `eps1` as free

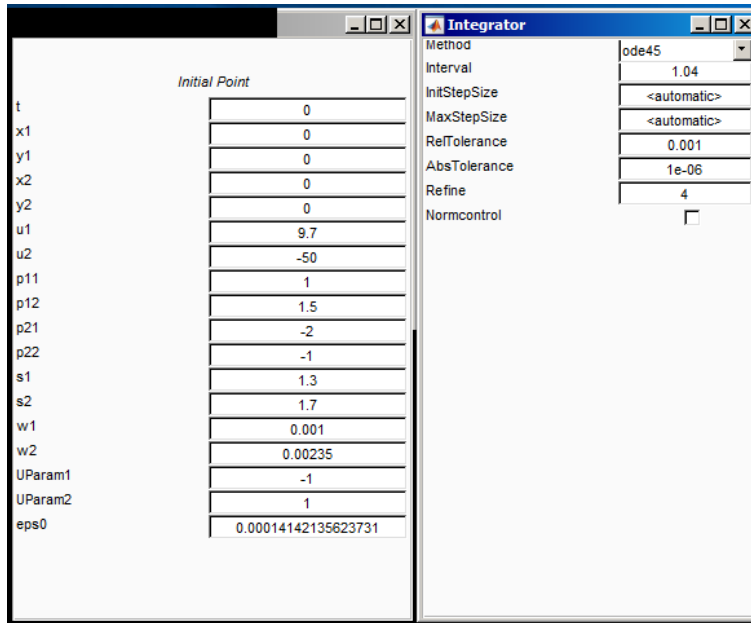


Figure 13: **Starter** and **Integrator** windows for the integration of the unstable manifold.

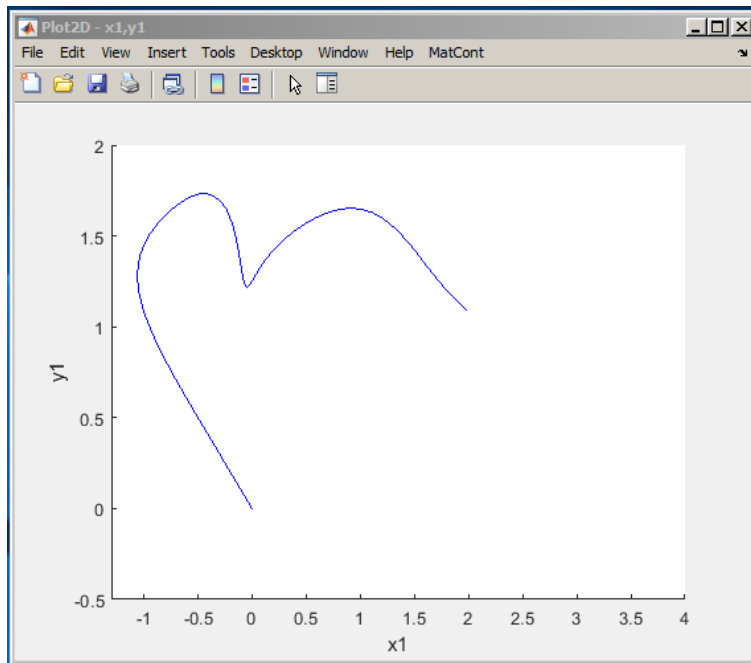


Figure 14: A segment of the unstable manifold of the saddle at the initial parameter values.

homoclinic parameters and  $u_1$  and  $u_2$  as free system parameters. The result of this two-parameter continuation is given in Figure 16.

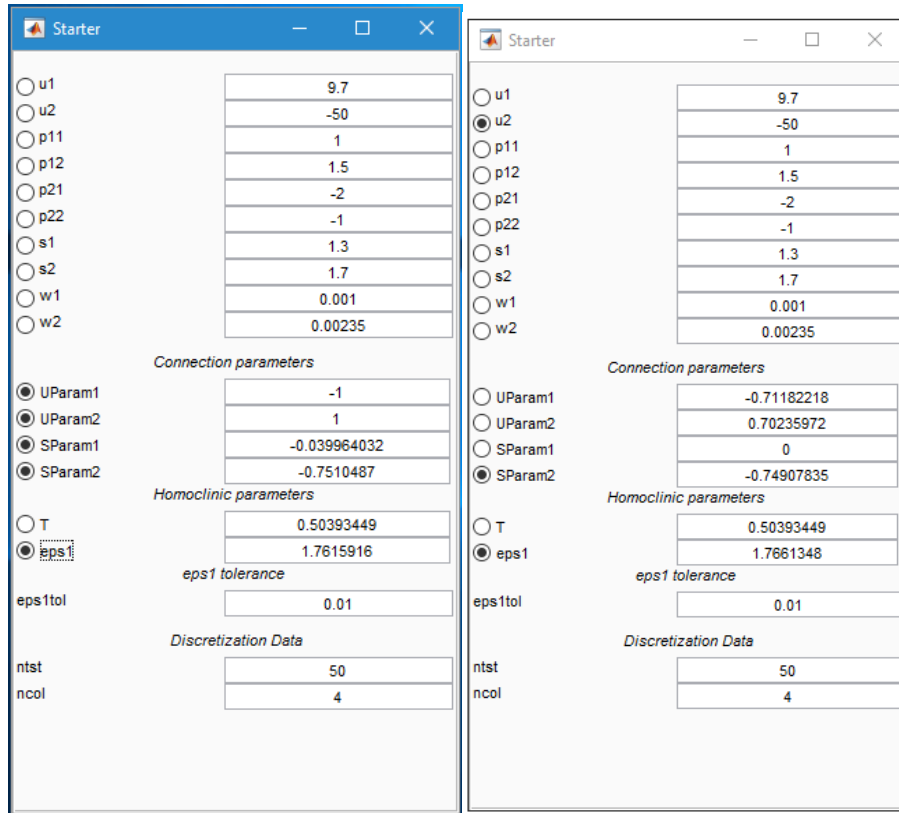


Figure 15: **Starter** windows for the first (left) and second (right) continuation of  $\text{HTHom}$  connections.

### 3 Additional Problems

A. Consider the *Lorenz system*

$$\begin{cases} \dot{x} &= \sigma(-x + y), \\ \dot{y} &= rx - y - xz, \\ \dot{z} &= -bz + xy, \end{cases}$$

with the standard parameter value  $b = \frac{8}{3}$ . Use MATCONT to analyse its homoclinic bifurcations:

1. Locate at  $\sigma = 10$  the bifurcation parameter value  $r_{\text{Hom}}$  corresponding to the primary orbit homoclinic to the origin. *Hint:* Use homotopy starting from  $r = 15.5$ .
2. Compute the primary homoclinic bifurcation curve in the  $(r, \sigma)$ -plane for  $b = \frac{8}{3}$ . Try to reach  $r = 100$  and  $\sigma = 100$ .
3. Locate and continue in the same  $(r, \sigma)$ -plane several secondary homoclinic to the origin orbits in the Lorenz system. *Hint:* These orbits make turns around both nontrivial equilibria. The simplest one can be found starting from  $(\sigma, r) = (10, 55)$ .

B. Study with MATCONT homoclinic bifurcations in the *adaptive control system of Lur'e type*:

$$\begin{cases} \dot{x} &= y, \\ \dot{y} &= z, \\ \dot{z} &= -\alpha z - \beta y - x + x^2, \end{cases}$$

where  $\alpha$  and  $\beta$  are parameters.

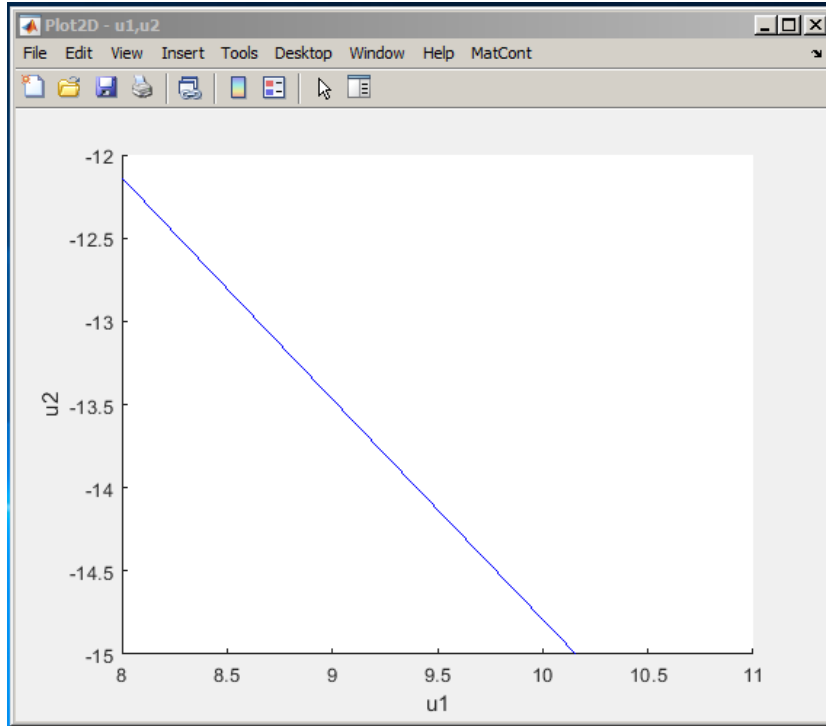


Figure 16: Continuation of homoclinic connections in the  $(u1, u2)$  parameter space.

- C. 1. Extend the system in §2.1 by adding a new parameter  $tst$ , a new state variable  $Z$  and a new state equation

$$Z' = tst * Z - (x1^4 + y1^4 + x2^4 + y2^4)$$

With fixed  $tst = 12$  compute and continue a homoclinic orbit of the new system along the lines for the system in §2.1, using the same system parameters. (Hint: this will involve a continuation with two free unstable homoclinic parameters, a continuation with three free unstable homoclinic parameters, and two continuations with no free unstable homoclinic parameters).

2. Study the previous system for the fixed value  $tst = 8$ . Explain why the system now behaves differently.

## Appendix: Outline of the homotopy method for homoclinic connections to hyperbolic equilibria

We consider autonomous systems of ODEs depending on at least two parameters

$$\dot{u} = f(u, \alpha), \quad u \in \mathbb{R}^n, \alpha \in \mathbb{R}^m. \quad (3)$$

Here  $n \geq 2$  and  $m \geq 2$ . If  $m > 2$ , then at most two parameters are active at any time but the choice of the active parameters can be changed between the steps of the algorithm.

We will initialize homoclinic orbits to hyperbolic equilibria, continue them under variation of two system parameters and detect some codim 2 bifurcations. Details can be found in a recent publication<sup>3</sup>.

<sup>3</sup>V. De Witte, W. Govaerts, Yu.A. Kuznetsov, and M. Friedman. Interactive initialization and continuation of homoclinic and heteroclinic orbits in MATLAB. *ACM Trans. Math. Software* **38** (2012), 18:1-18:34.

Let  $u_0$  be a hyperbolic equilibrium of (3) with  $n_u$  eigenvalues with a positive real part and  $n_s$  eigenvalues with a negative real part, so  $n_u \geq 1$ ,  $n_s \geq 1$  and  $n_u + n_s = n$ . Let the unstable eigenvalues  $\mu_1, \dots, \mu_{n_u}$  be ordered by increasing values of their real parts. We assume that the critical eigenvalues, i.e. the unstable eigenvalues with the smallest real part, consist of either a simple real eigenvalue or a single conjugate pair of complex eigenvalues.

The homotopy method is based on the use of an orthonormal base  $(q_0^1, \dots, q_0^{n_u})$  of the right unstable eigenspace and an orthonormal base  $(p_0^1, \dots, p_0^{n_u})$  of the left unstable eigenspace of the Jacobian  $f_u(u_0)$ . These bases are constructed so that if  $\Re(\mu_i) < \Re(\mu_{i+1})$  for  $1 \leq i \leq n_u - 1$  then  $(q_0^1, \dots, q_0^i)$  span the generalized right eigenspace of the first  $i$  unstable eigenvalues, and similarly for the left eigenspaces.

The homotopy method consists of three stages, namely:

1. Stage 1: Approximating the unstable manifold by integration.
2. Stage 2: Homotopy towards the stable eigenspace of the saddle.
3. Stage 3: Homotopy of the end-point towards the saddle.

Stage 1 consists of a time integration that starts from a point of the form

$$u_0 + \text{eps0}(\text{Uparam1 } q_0^1 + \text{Uparam2 } q_0^2) / \|\text{Uparam1 } q_0^1 + \text{Uparam2 } q_0^2\|$$

in the unstable eigenspace at distance **eps0** from  $u_0$ . The integration interval, **eps0**, **Uparam1**, and **Uparam2** are user-defined. The **UParam** parameters are called *unstable homoclinic parameters*.

If  $\mu_1$  is real, then the term with **Uparam2** is not used and **Uparam1** must be nonzero. In the other case **Uparam1** and **Uparam2** must not be both zero.

Typically the computed orbit does not look like a homoclinic connection but the algorithm chooses the point where the distance to  $u_0$  stops decreasing for the last time as the endpoint  $u(\text{end})$  of the first approximation to a homoclinic connection.

Stage 2 is the most complicated one. The inner products

$$\text{SParam } i = \langle u(\text{end}) - u_0, p_0^i \rangle, i = 1 \dots n_u,$$

are called the *stable homoclinic parameters*. The end point  $u(\text{end})$  is in the stable eigenspace if all stable parameters vanish. The basic idea in Stage 2 is that each continuation step should annihilate (at least) one stable homoclinic parameter (not necessarily in any a priori given order). The distance

$$\text{eps1} = \|u(\text{end}) - u(0)\|$$

is also used as a parameter in Stage 2 and Stage 3.

If  $n_u = 1$  then Stage 2 consists of a single continuation in which the active parameters are **SParam1**, **eps1**, and one user-chosen system parameter. **Uparam1** is fixed at  $\pm 1$ . Stage 2 is successful if a point is found where **SParam1** is zero.

If  $n_u > 1$  then in the first continuation step the free parameters consist of **Uparam1**, **Uparam2**, all  $n_u$  stable homoclinic parameters and **eps1**. The continuation involves the normalization equation

$$\text{Uparam1}^2 + \text{Uparam2}^2 = 1$$

and the initial point of the orbit is given by

$$u_0 + \text{eps0}(\text{Uparam1 } q_0^1 + \text{Uparam2 } q_0^2)$$

This step is successful if a point is found where one of the stable homoclinic parameters vanishes.

The next step starts from this point. First, all stable homoclinic parameters below a threshold are set to zero. Let  $k$  be the number of zero stable homoclinic parameters. If  $k < n_u - 1$  then in the next continuation the free parameters consist of **Uparam**  $i$ ,  $1 \leq i \leq k + 2$ , the  $n_u - k$  nonzero stable homoclinic parameters and **eps1**. The new orbits start from

$$u_0 + \text{eps0}(\text{Uparam1 } q_0^1 + \dots + \text{Uparam}(k + 2) q_0^{k+2})$$



The continuation again involves the normalization equation

$$\text{UParam1}^2 + \dots + \text{UParam}(k+2)^2 = 1,$$

and the new unstable homoclinic parameters are initially zero.

If  $k = n_u - 1$  then in the next continuation the remaining nonzero stable homoclinic parameter, **eps1** and one user-chosen system parameter are free. The new orbits start from

$$u_0 + \text{eps0}(\text{Uparam1} q_0^1 + \dots + \text{Uparam}(n_u) q_0^{n_u}).$$

Stage 2 is succesful when at the end all stable homoclinic parameters are zero.

Stage 3 consists of a single continuation in which **eps1**, the ‘period’ parameter **T** (the length of the final approximating orbit will be  $2T$ ), and one user-chosen system parameter are free.

Stage 3 is succesful when **eps1** attains a user-chosen threshold **eps1tol**.

From this point the continuation of orbits homoclinic to a hyperbolic saddle can be started. In such a continuation, two system parameters must be free, and also one of the three homoclinic parameters **eps0**, **eps1**, or **T**.