Lecture 4: Bordering technique. Detection of limit and branching points.

4.1 Bordering technique I

Consider a smooth one-parameter family of $N \times N$ matrices $A(s)$, such that $A(0)$ is singular with rank $A(0) = N - 1$.

Lemma 9 *The matrix*

$$M(0) = \begin{pmatrix} A(0) & p \\ q^{\mathrm{T}} & 0 \end{pmatrix},$$

where $A(0)q = A^{\mathrm{T}}(0)p = 0$ *with* $\|q\| = \|p\| = 1$*, is nonsingular.*

Proof:

Suppose that

$$M(0) \begin{pmatrix} X \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

with $X \in \mathbb{R}^N$ and $\beta \in \mathbb{R}$ such that

$$\begin{pmatrix} X \\ \beta \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This is equivalent to the system

$$\left\{ \begin{array}{rcl} A(0)X + \beta p & = & 0, \\ \langle q, X \rangle & = & 0. \end{array} \right. \tag{4.21}$$

Computing the scalar product of the first equation in (4.21) with p, we obtain

$$0 = \langle p, A(0)X + \beta p \rangle = \langle A^{\mathrm{T}}(0)p, X \rangle + \beta \langle p, p \rangle = \beta \|p\|^2 = \beta,$$

where $A^{\mathrm{T}}(0)p = 0$ is taken into acoount. We conclude that $\beta = 0$ and so the first equation in (4.21) actually has the form

$$A(0)X = 0.$$

This implies that $X = \gamma q$ with some $\gamma \in \mathbb{R}$. Substituting $X = \gamma q$ in the second equation of (4.21), we see that

$$\langle q, \gamma q \rangle = \gamma \|q\|^2 = \gamma = 0,$$

yielding $X = 0$. Thus

$$\begin{pmatrix} X \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

a contradiction. Therefore, $M(0)$ is nonsingular. □

Lemma 9 ensures by continuity that the matrix

$$M(s) = \begin{pmatrix} A(s) & p \\ q^{\mathrm{T}} & 0 \end{pmatrix} \tag{4.22}$$

is nonsingular for all s with $|s|$ sufficiently small. For such values of s, introduce the nonsingular **bordered system**:

$$M(s) \begin{pmatrix} w \\ g \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{4.23}$$

At $s = 0$, the explicit solution to this system is obvious:

$$\begin{pmatrix} w(0) \\ g(0) \end{pmatrix} = \begin{pmatrix} q \\ 0 \end{pmatrix}.$$

Thus, $g(0) = 0$. If

$$\begin{pmatrix} w \\ g \end{pmatrix} = \begin{pmatrix} w(s) \\ g(s) \end{pmatrix}$$

is the solution of (4.23), then Cramer's rule gives

$$g(s) = \frac{\det A(s)}{\det M(s)} , \tag{4.24}$$

implying that $g(s)$ is as smooth as $A(s)$. The following lemma shows how the derivative $\dot{g}(0)$ can be computed explicitly.

Lemma 10 *It holds that*

$$\dot{g}(0) = -\langle p, \dot{A}(0)q \rangle \ .$$

Proof:

Differentiating (4.23) w.r.t. s yields

$$\dot{M}(s) \begin{pmatrix} w(s) \\ g(s) \end{pmatrix} + M(s) \begin{pmatrix} \dot{w}(s) \\ \dot{g}(s) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

implying

$$M(0) \begin{pmatrix} \dot{w}(0) \\ \dot{g}(0) \end{pmatrix} = -\dot{M}(0) \begin{pmatrix} w(0) \\ g(0) \end{pmatrix}.$$

Thus,

$$M(0) \begin{pmatrix} \dot{w}(0) \\ \dot{g}(0) \end{pmatrix} = -\dot{M}(0) \begin{pmatrix} q \\ 0 \end{pmatrix}. \tag{4.25}$$

Further notice that the transposed matrix

$$M^{\mathrm{T}}(0) = \begin{pmatrix} A^{\mathrm{T}}(0) & q \\ p^{\mathrm{T}} & 0 \end{pmatrix}$$

is also nonsingular, so that the linear system

$$M^{\mathrm{T}}(0) \begin{pmatrix} \varphi \\ h \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{4.26}$$

has the unique solution, namely

$$\begin{pmatrix} \varphi \\ h \end{pmatrix} = \begin{pmatrix} p \\ 0 \end{pmatrix}.$$

Computing now the scalar product of this solution with both sides of (4.25), we obtain

$$\left\langle \begin{pmatrix} p \\ 0 \end{pmatrix}, M(0) \begin{pmatrix} \dot{w}(0) \\ \dot{g}(0) \end{pmatrix} \right\rangle = -\left\langle \begin{pmatrix} p \\ 0 \end{pmatrix}, \dot{M}(0) \begin{pmatrix} q \\ 0 \end{pmatrix} \right\rangle$$

or

$$\left\langle M^{\mathrm{T}}(0) \begin{pmatrix} p \\ 0 \end{pmatrix}, \begin{pmatrix} \dot{w}(0) \\ \dot{g}(0) \end{pmatrix} \right\rangle = -\left\langle \begin{pmatrix} p \\ 0 \end{pmatrix}, \dot{M}(0) \begin{pmatrix} q \\ 0 \end{pmatrix} \right\rangle .$$

Taking into account (4.26), we see that

$$\dot{g}(0) = -\left\langle \begin{pmatrix} p \\ 0 \end{pmatrix}, \dot{M}(0) \begin{pmatrix} q \\ 0 \end{pmatrix} \right\rangle .$$

Since

$$\dot{M}(0) = \begin{pmatrix} \dot{A}(0) & 0 \\ 0 & 0 \end{pmatrix},$$

we get

$$\dot{g}(0) = -\langle p, \dot{A}(0) q \rangle .$$

This complets the proof. □

4.2 Detection of local bifurcations

Consider a system of ODEs depending on one parameter

$$\dot{u} = f(u, \alpha), \quad u \in \mathbb{R}^n, \ \alpha \in \mathbb{R}, \tag{4.27}$$

where $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is smooth. The continuation of a branch in its equilibrium manifold leads to ALCP (3.7) with

$$x = \begin{pmatrix} u \\ \alpha \end{pmatrix} \in \mathbb{R}^{n+1}$$

and $F(x) = f(u, \alpha)$. Assume that this branch is parametrized by $u = u(s)$ and $\alpha = \alpha(s)$ and that $s = 0$ corresponds to either a quadratic limit point w.r.t. α or a simple branching point of (3.7). Ee will construct a **regular test-function** $\Psi(s)$ to detect each bifurcation, i.e. a smooth scalar function satisfying

$$\Psi(0) = 0, \quad \dot{\Psi}(0) \neq 0.$$

4.2.1 Limit point detection

Assume that $s = 0$ corresponds to a limit point w.r.t. α. We can also select such a parametrization of the equilibrium branch near the limit point by s that the tangent vector at $s = 0$ will have the form

$$\begin{pmatrix} \dot{u}(0) \\ \dot{\alpha}(0) \end{pmatrix} = \begin{pmatrix} q \\ 0 \end{pmatrix},$$

with $q \in \mathbb{R}^n$ satisfying

$$A(0)q = 0, \quad \|q\| = 1,$$

where $A(0) = f_u(u(0), \alpha(0))$. Introduce

$$\Psi_{\mathrm{LP}}(s) = g(s),$$

where $g(s)$ is defined by solving the bordered system (4.23). In that system, matrix $M(s)$ is given by (4.22) with $A(s) = f_u(u(s), \alpha(s))$, vector q is defined above, and $p \in \mathbb{R}^n$ satisfies $A^{\mathrm{T}}(0)p = 0$, $\|p\| = 1$.

Theorem 5 *At a quadratic limit point holds*

$$\Psi_{\mathrm{LP}}(0) = 0 \quad \text{and} \quad \dot{\Psi}_{\mathrm{LP}}(0) \neq 0.$$

Proof:

Clearly, $\Psi_{\mathrm{LP}}(0) = g(0) = 0$. Using Lemma 10, we obtain

$$\dot{g}(0) = -\langle p, \dot{A}(0)q \rangle = -\langle p, f_{uu}(u(0), \alpha(0))[q, q] \rangle = -\langle p, B(q, q) \rangle.$$

Since $\langle p, B(q, q) \rangle \neq 0$ at a quadratic limit point, $\dot{\Psi}_{\mathrm{LP}}(0) = \dot{g}(0) \neq 0$. □

4.2.2 Branching point detection

Suppose that $s = 0$ corresponds to a simple branching point of ALCP (3.7) in the solution branch Γ_1 parametrized by $x^{(1)}(s)$ such that

$$\|\dot{x}^{(1)}(0)\| = 1.$$

As in Theorem 3, define the $(N + 1) \times (N + 1)$ matrix

$$D(s) = \begin{pmatrix} F_x(x^{(1)}(s) \\ [\dot{x}^{(1)}(s)]^{\mathrm{T}} \end{pmatrix}$$

and introduce

$$\Psi_{\mathrm{BP}}(s) = g(s),$$

where $g(s)$ is still defined by solving the bordered system (4.23) but now

$$M(s) = \begin{pmatrix} D(s) & P \\ Q^{\mathrm{T}} & 0 \end{pmatrix}$$

with vectors $Q, P \in \mathbb{R}^{N+1}$ satisfying $D(0)Q = D^{\mathrm{T}}(0)P = 0$ and $\|Q\| = \|P\| = 1$, so that $M(s)$ is a $(N+2)\times(N+2)$ nonsingular matrix for small $|s|$.

Theorem 6 *At a simple branching point holds*

$$\Psi_{\mathrm{BP}}(0) = 0 \quad \text{and} \quad \dot{\Psi}_{\mathrm{BP}}(0) \neq 0.$$

Proof:

We have already seen in the proof of Theorem 3 that matrix $D(0)$ is singular. Its null-space $N(D(0))$ is one-dimensional and is spanned by $Q = q^{(2)}$. Thus, $g(0) = 0$.

The null-space $N(D^{\mathrm{T}}(0)$ is also one-dimensional and spanned by

$$P = \begin{pmatrix} \varphi \\ 0 \end{pmatrix} \in \mathbb{R}^{N+1},$$

where $J^{\mathrm{T}}\varphi = 0$ and $\|\varphi\| = 1$ implying $\|P\| = 1$.

Now, Lemma 10 allows us to write

$$\dot{g}(0) = -\langle P, \dot{D}(0)Q\rangle \ .$$

Since $Q = q^{(2)}$ and $\dot{x}^{(1)}(0) = q^{(1)}$, we have

$$\dot{D}(0)Q = \dot{D}(0)q^{(2)} = \begin{pmatrix} F_{xx}[\dot{x}^{(1)}(0), q^{(2)}] \\ [\ddot{x}^{(1)}(0)]^{\mathrm{T}} q^{(2)} \end{pmatrix} = \begin{pmatrix} B(q^{(1)}, q^{(2)}) \\ [\ddot{x}^{(1)}(0)]^{\mathrm{T}} q^{(2)} \end{pmatrix},$$

so that

$$\langle P, \dot{D}(0)q^{(2)}\rangle = \left\langle \begin{pmatrix} \varphi \\ 0 \end{pmatrix}, \begin{pmatrix} B(q^{(1)}, q^{(2)}) \\ [\ddot{x}^{(1)}(0)]^{\mathrm{T}} q^{(2)} \end{pmatrix} \right\rangle = \langle \varphi, B(q^{(1)}, q^{(2)})\rangle.$$

This gives

$$\dot{g}(0) = -\langle \varphi, B(q^{(1)}, q^{(2)})\rangle = -b_{12} \neq 0,$$

because the branching point is simple. □