Homoclinic saddle to saddle-focus transitions in 3D and 4D ODEs

Yuri A. Kuznetsov

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Joint work with Manu Kalia & Hil Meijer

Universiteit Utrecht

UNIVERSITY OF TWENTE.

Contents

General Methodology

Combine analytic and numerical bifurcation methods !

Homoclinic orbits

- Consider a smooth family of smooth ODE systems

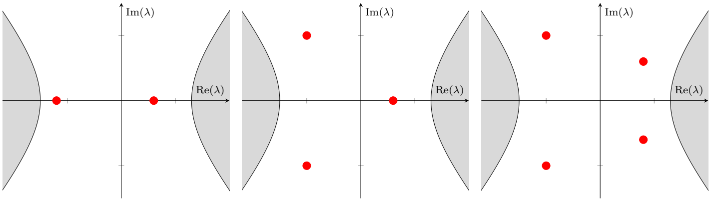
$$\dot{u} = F(u, p), \quad u \in \mathbb{R}^n,\ p \in \mathbb{R}^m$$

- Suppose that at $p_0 \in \mathbb{R}^m$ there is a *hyperbolic equilibrium* $u_0 \in \mathbb{R}^n$.
- Assume that u_0 has a *primary homoclinic orbit* Γ_0, i.e. for a non-equilibrium solution $t \mapsto u(t)$ holds

$$\lim_{t \to \pm\infty} u(t) = u_0$$

- Generically, the corresponding p_0-values occupy a codimension 1 homoclinic *bifurcation manifold* in $\mathbb{R}^m$.

- Leading eigenvalues of u_0 (possibly, after reversing time):



(a) *Saddle* **(b)** *Saddle-focus* **(c)** *Focus-focus*

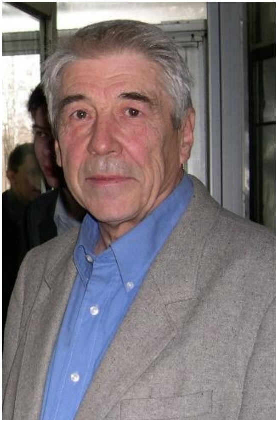
- *Saddle quantity* (*index*):

$$\sigma := \Re(\lambda^s) + \Re(\lambda^u) \qquad \left(\nu := -\frac{\Re(\lambda^s)}{\Re(\lambda^u)}\right)$$

where λ^s and λ^u are leading stable and unstable eigenvalues of u_0.

Limit cycles due to codim 1 homoclinic bifurcations

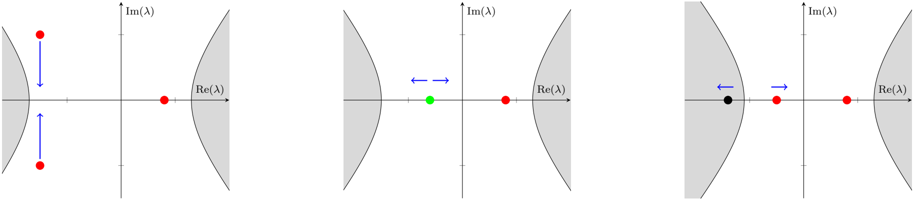
	$\sigma < 0$	$\sigma > 0$
Saddle	one cycle	one cycle
Saddle-focus	one cycle	∞ cycles
Focus-focus	∞ cycles	∞ cycles



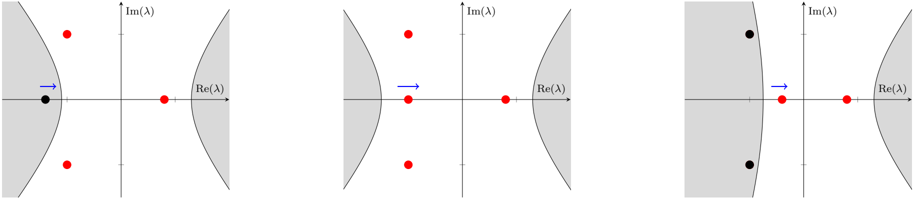
L.P. Shilnikov (1934-2011)

Codim 2 saddle-focus to saddle transitions

2DL (double eigenvalue) Belyakov bifurcation:



3DL bifurcation:



$\mu_1 < 0$ $\mu_1 = 0$ $\mu_1 > 0$

2DL Belyakov bifurcation

L.A. Belyakov
The bifurcation set in a system with a homoclinic saddle curve
Math. Notes **28** (1980), 910-916

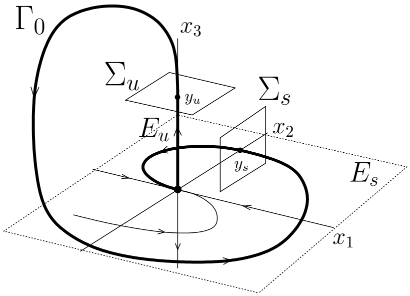
Yu.A. Kuznetsov, O. De Feo, and S. Rinaldi
Belyakov homoclinic bifurcations in a tritrophic food chain model
SIAM J. App. Math. **62** (2001), 462-487

3D ODE system near the equilibrium $x=0$:

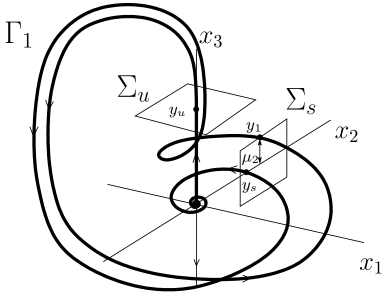
$$\begin{cases} \dot{x}_1 & = & \gamma(\mu)x_1 + x_2 + f_1(x,\mu) \\ \dot{x}_2 & = & -\mu_1 x_1 + \gamma(\mu)x_2 + f_2(x,\mu) \\ \dot{x}_3 & = & \beta(\mu)x_3 \end{cases}$$

where $\mu = (\mu_1, \mu_2) \in \mathbb{R}^2$ and $\lambda^s_{1,2} = \gamma(0) < 0, \ \lambda^u = \beta(0) > -\gamma(0)$.

Geometric setting



Homoclinic orbit at $\mu = 0$



Double homoclinic orbit

Model Poincaré maps

- 2D Model Poincaré Map on Σ_s:

$$\begin{pmatrix} x_2 \\ x_3 \end{pmatrix} \mapsto \begin{pmatrix} 1 + \frac{A}{\sqrt{\mu_1}} x_2 x_3^\nu \sin\left(-\frac{\sqrt{\mu_1}}{\beta}\ln x_3 + \Theta\right) \\ \mu_2 + \frac{C}{\sqrt{\mu_1}} x_2 x_3^\nu \sin\left(-\frac{\sqrt{\mu_1}}{\beta}\ln x_3\right) \end{pmatrix}$$

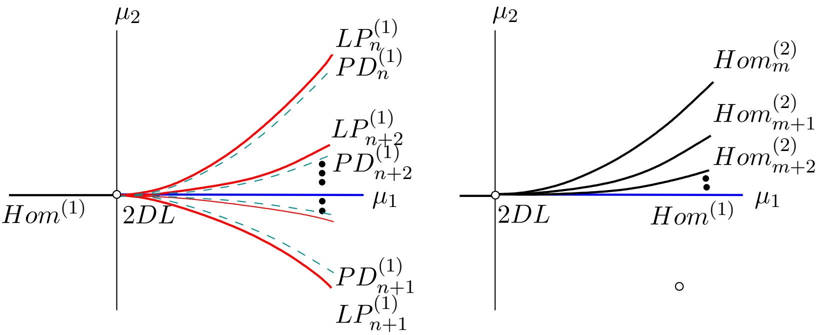
where

$$\nu = -\frac{\gamma}{\beta} < 1 \quad (\text{chaotic case})$$

- 1D Model Poincaré Map:

$$x_3 \mapsto \mu_2 + \frac{C}{\sqrt{\mu_1}} x_3^\nu \sin\left(-\frac{\sqrt{\mu_1}}{\beta}\ln x_3\right)$$

(Partial) bifurcation diagram ($\nu < 1$)



$$LP_n^{(1)} \text{ and } PD_n^{(1)} \quad : \quad \mu_2^{(n)} = \frac{(-1)^n C}{e\gamma} \exp\left(\frac{\gamma \pi n}{\sqrt{\mu_1}}\right)(1 + o(\mu_1))$$

$$Hom_m^{(2)} \quad : \quad \mu_2^{(m)} = \exp\left(-\frac{\beta \pi m}{\sqrt{\mu_1}}\right)(1 + o(\mu_1))$$

3DL bifurcation

M. Kalia, Yu.A. Kuznetsov, and H.G.E. Meijer
Homoclinic saddle to saddle-focus transitions in 4D systems
arXiv:1712.03212[submitted to Nonlinearity]

4D ODE C^1 orbitally equivalent system near the equilibrium $x = 0$:

$$\begin{cases} \dot{x}_1 &= \gamma(\mu)x_1 - x_2, \\ \dot{x}_2 &= x_1 + \gamma(\mu)x_2, \\ \dot{x}_3 &= \big(\gamma(\mu) - \mu_1\big)\, x_3, \\ \dot{x}_4 &= \beta(\mu)x_4, \end{cases}$$

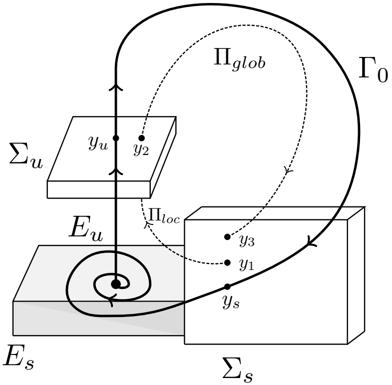
where $\mu = (\mu_1, \mu_2) \in \mathbb{R}^2$ and

$$\Re(\lambda^s_{1,2,3}) = \gamma(0) < 0, \quad \lambda^u = \beta(0) > -\gamma(0)$$

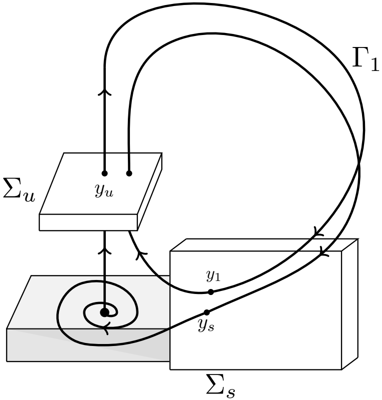
so that

$$\nu = -\frac{\gamma}{\beta} < 1 \quad (\text{chaotic case})$$

Geometric setting



Homoclinic orbit



Double homoclinic orbit

Model Poincaré maps

- 3D Model Poincaré Map on Σ_s:

$$G : \begin{pmatrix} x_1 \\ x_2 \\ x_4 \end{pmatrix} \mapsto \begin{pmatrix} 1 + \alpha_1 x_1 x_4^{\nu} \cos\left(-\frac{1}{\beta}\ln x_4 + \phi_1\right) + \alpha_2 x_2 x_4^{\nu+\mu_1/\beta} \\ 1 + \alpha_3 x_1 x_4^{\nu} \sin\left(-\frac{1}{\beta}\ln x_4 + \phi_2\right) + \alpha_4 x_2 x_4^{\nu+\mu_1/\beta} \\ \mu_2 + C_1 x_1 x_4^{\nu} \sin\left(-\frac{1}{\beta}\ln x_4\right) + C_2 x_2 x_4^{\nu+\mu_1/\beta} \end{pmatrix}$$

- 1D Model Poincaré Map:

$$F : x_4 \mapsto \mu_2 + C_1 x_4^{\nu} \sin\left(-\frac{1}{\beta}\ln x_4\right) + C_2 x_4^{\nu+\mu_1/\beta}$$

Lemma (Bifurcations of 1D model map)

Generically, the model map F with $\nu < 1$ has an infinite number of fold curves $LP_n^{(1)}$ accumulating to the half axis $\mu_2 = 0$ with $\mu_1 \geq 0$. Each curve resembles a 'horn' with a cusp point

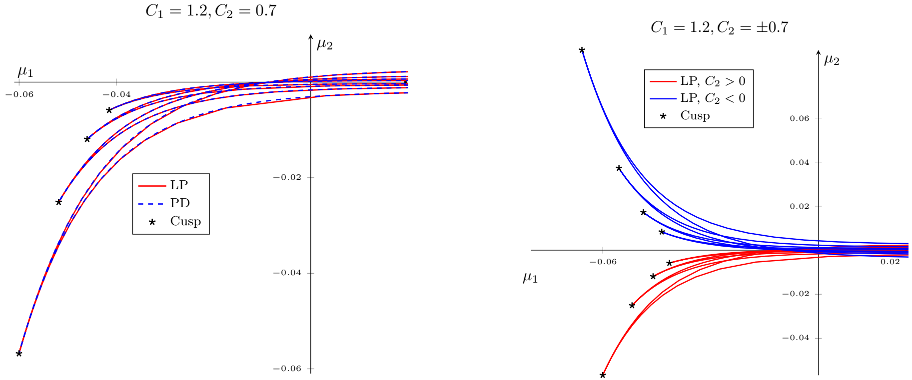
$$CP_n^{(1)} : \begin{pmatrix} \mu_1^{(n)} \\ \mu_2^{(n)} \end{pmatrix} = \begin{pmatrix} \frac{1}{4\pi n}\left[\ln a + O\left(\frac{1}{n}\right)\right] \\ -e^{-\beta\nu(2\pi n+\theta+\phi)} \frac{\operatorname{sign}(C_2)C_1}{\beta\nu\sqrt{1+\beta^2\nu^2}} a^{-(\theta+\phi)/4\pi n} + O\left(\frac{1}{\sqrt{n}}\right) \end{pmatrix}$$

where

$$a = \frac{\beta^2\nu^2}{1+\beta^2\nu^2}\frac{C_2^2}{C_1^2}, \quad \sin\phi = (1+\beta^2\nu^2)^{-1/2}, \quad \theta = \begin{cases} \pi/2, & \text{if } C_2 < 0, \\ 3\pi/2, & \text{if } C_2 > 0. \end{cases}$$

Moreover, there exists an infinite number of period-doubling curves $PD_n^{(1)}$ having – away from the cusp points $CP_n^{(1)}$ – the same asymptotic properties as the fold bifurcation curves $LP_n^{(1)}$.

Primary LP and PD curves of 1D map F

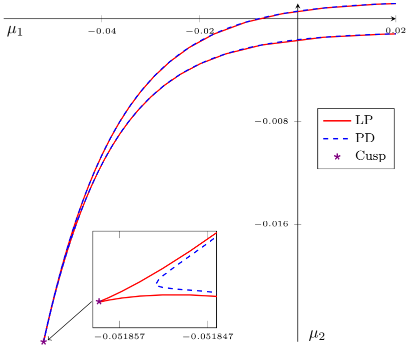


$$\beta = 0.2, \quad \nu = 0.5$$

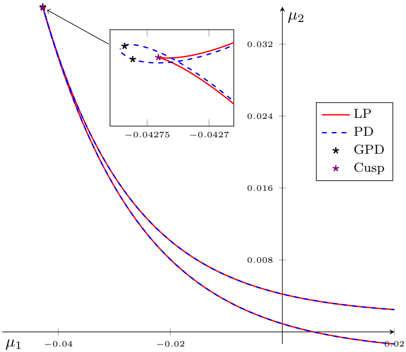
Spring and saddle areas

Depending on (C_1, C_2), curves $PD_n^{(1)}$ could either be smooth or develop small loops around the corresponding cusp points

$C_1 = 1.2, C_2 = 0.7$ (Saddle area)



$C_1 = 0.8, C_2 = -1.1$ (Spring area)



Theorem (Bifurcations of 3D model map)

Generically, the model map G with $\nu < 1$ *has an infinite number of fold and period-doubling curves accumulating to the half axis* $\mu_2 = 0$ *with* $\mu_1 \geq 0$ *and having the same asymptotic properties as the curves* $LP_n^{(1)}$ *and* $PD_n^{(1)}$ *above.*

Moreover, there exists an infinite sequence of 'parabolas' $Hom_m^{(2)}$ *that accumulate onto the half axis* $\mu_2 = 0$ *with* $\mu_1 \geq 0$ *and correspond to the double homoclinic orbits. The turning points of the parabolas are*

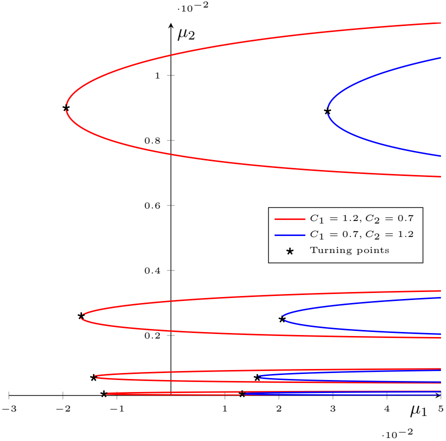
$$\begin{pmatrix} \mu_1^{(m)} \\ \mu_2^{(m)} \end{pmatrix} = \begin{pmatrix} \frac{1}{4\pi m}\left(\ln\left(\frac{C_2^2}{C_1^2}\right) + O\left(\frac{1}{m}\right)\right) \\ e^{-\beta(2\pi m+\theta)}\left(1 + O\left(\frac{1}{m}\right)\right) \end{pmatrix}$$

where

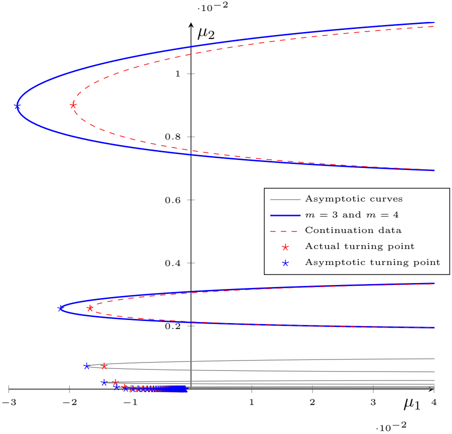
$$\theta = \begin{cases} \pi/2, & \text{if } C_2 < 0, \\ 3\pi/2, & \text{if } C_2 > 0. \end{cases}$$

Secondary homoclinic curves defined by 3D map G

Continuations: $m = 3, 4...6$



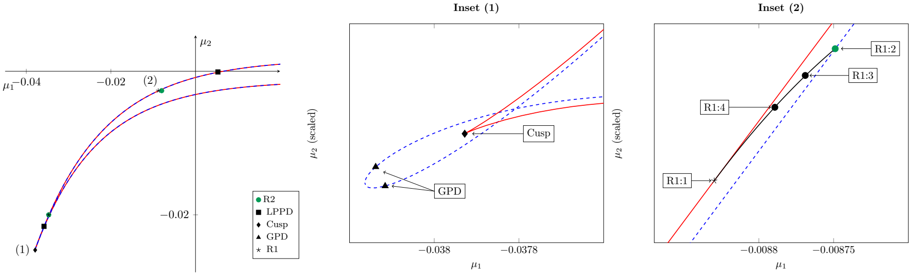
$C_1 = 1.2,\ C_2 = 0.7,\ m = 3, 4..., 90$



$$\beta = 0.2, \quad \nu = 0.5$$

Codim 2 bifurcations in 3D map G

$$\nu = 0.5, \beta = 0.2$$



$$C_1 = 0.8,\ \ C_2 = 1.2,\ \ \alpha_1 = 0.8,\ \ \alpha_2 = 1.3,\ \ \alpha_3 = 0.6,\ \ \alpha_4 = 1.1$$

$$\phi_1 = \phi_2 = \pi/6$$

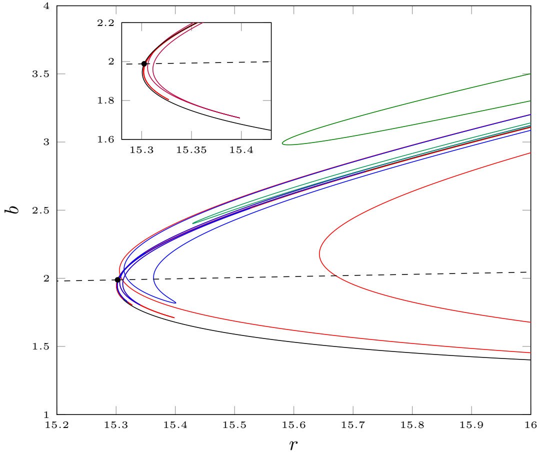
An example: Modified Lorenz-Stenflo equations

L. Stenflo
Generalized Lorenz equations for acoustic-gravity waves in the atmosphere
Physica Scripta **53**(1):83, 1996

$$\begin{cases} \dot{x} &=& \sigma(y-x)+su \\ \dot{y} &=& rx-xz-y+\epsilon_1 z \\ \dot{z} &=& xy-bz \\ \dot{u} &=& -x-\sigma u+\epsilon_2 y \end{cases}$$

$$\sigma=0.1,\ s=33,\ \epsilon_1=0.1,\ \epsilon_2=0.3$$

Chaotic 3DL bifurcation in modified LS model



$$3DL: (r, b) \approx (15.302531, 1.9884) \qquad (\nu \approx 0.71605 < 1)$$

Open questions

- Exact condition for the Saddle/Spring area transition in 2D model map.
- Analytic proof of strong resonances in 3D map.
- Can C^k-linearization with $k > 1$ and related non-resonance conditions be avoided ?
- Can the use of Homoclinic Center Manifold Theorem and related gap conditions be avoided as well ?
- More realistic examples.
- What happens in the volume-preserving case, where 3DL has codim 1 and is always chaotic ?