Numerical Bifurcation Analysis of Maps

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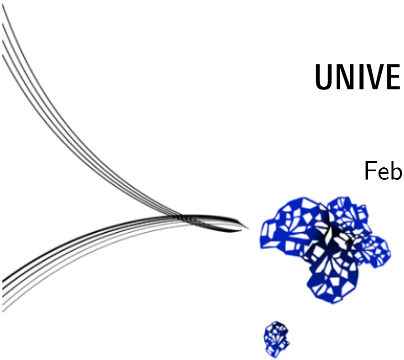
February 16/18 2010



Aim of the course

codim 2 bifurcations are organizing centers in bifurcation diagrams

Part 1 Analysis of codim 2 bifurcations:
Normal forms, Center Manifolds, Unfoldings
$\rightarrow$ Get a feeling of dynamical behaviour.

Part 2 Bifurcations of invariant tori:
KAM Resonance tongues, Bubble analysis, homoclinic bifurcations.
$\rightarrow$ Get a feeling of fine details near torus bifurcations

Infinite sequences of bifurcations

Setting

Consider a map

$$F : x \mapsto F(x, \alpha) \in \mathbb{R}^n, \qquad x \in \mathbb{R}^n, \quad \alpha \in \mathbb{R}^m.$$

Study dynamics near a fixed point of the k-th iterate of the map.
Fixed points satisfy $F(x^0, \alpha^0)^k - x^0 = 0$ and have multipliers

$$\{\mu_1, \mu_2, \dots, \mu_n\} = \sigma(A),$$

where $A = F_x(x^0, \alpha^0)$.
k is the period of the fixed point.
W.l.o.g. $k = 1$, $x_0 = 0$, $\alpha_0 = 0$.

Notation

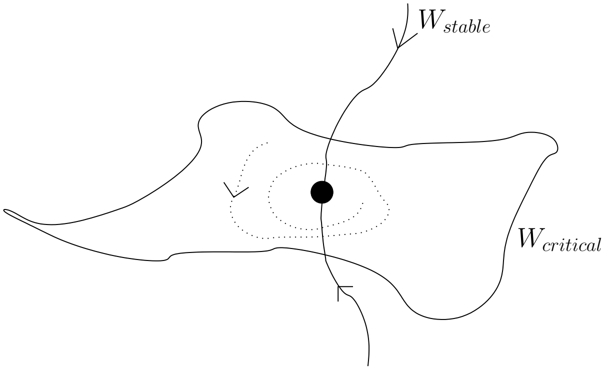
- variables $x \in \mathbb{R}$ and $z \in \mathbb{C}$
- multi-index For a multi-index ν we have $\nu = (\nu_1, \nu_2, \ldots, \nu_n), \nu_i \in \mathbb{Z}_{\geq 0}$
  $\nu! = \nu_1!\nu_2! \ldots \nu_n!$,
  $|\nu| = \nu_1 + \nu_2 + \ldots + \nu_n$ and
  $\tilde{\nu} \leq \nu$ if $\tilde{\nu_i} \leq \nu_i$ for all $i = 1, \ldots, n$.
- $\langle u, v \rangle = \bar{u}^T v$ is the standard scalar product in $\mathbb{C}^n$ (or $\mathbb{R}^n$).

Center Manifold: 1

Decompose phase space(W) near steady solution:

$$W = W_u \oplus W_s \oplus W_c$$

Manifolds W_i invariant under the mapping F.



Center Manifold: 2

Center Manifold W_c ($\mod(\mu) = 1$)

Bifurcations occur on W_c.
Normal form determines locally properties of the solutions.

Check:

1. Nondegenerate : Coefficients nonzero?

   Predict the presence of heteroclinic/homoclinic structures and invariant circles.

2. Transversal : Depends on parameters

   Transversality allows to switch to new branches.

Center Manifold: Invariance

$$
\begin{array}{ccc}
\mathbb{R}^{n_0} \ni w & \xrightarrow{\quad H \quad} & u \in \mathbb{R}^n \\
G \big\downarrow & & \big\downarrow F \\
\dot{w} & \xrightarrow[\quad H \quad]{} & \dot{u}
\end{array}
$$

HOMOLOGICAL EQUATION:

$$F(H(w)) = H(G(w))$$

where F Critical Map, G Normal Form
Center Manifold $x = H(w)$

Center Manifold Reduction: Ansatz

Let

$$\begin{aligned} F(x) &= Ax + \frac{1}{2}B(x,x) + \frac{1}{6}C(x,x,x) \\ &\quad + \frac{1}{24}D(x,x,x,x) + \frac{1}{120}E(x,x,x,x,x) + \cdots \end{aligned}$$

and expand the functions G, H into Taylor series with unknown coefficients,

$$G(w) = \sum_{|\nu|\geq 1} \frac{1}{\nu!} g_\nu w^\nu, \quad H(w) = \sum_{|\nu|\geq 1} \frac{1}{\nu!} h_\nu w^\nu,$$

Center Manifold Reduction:Equations

Insert this into the homological equation and collect the coefficients of the w^ν-terms in the homological equation. This gives a linear system for h_ν :

$$L_\nu h_\nu = R_\nu.$$

where $L_\nu = (A - \mu^\nu I)$ with the multipliers μ.
Singular if $\mu^\nu = 1$. Interpretation: These terms are needed in the normal form.

- Iterative solutions for higher order terms.
- Critical coefficients come from singular systems.
- If necessary singular systems are solved by bordered systems.
- Parameters can be included in this reduction process
- Method by Elphick et.al.(1987)

Center Manifold Reduction: ODE's

$$
\begin{array}{ccc}
\mathbb{R}^{n_0} \ni w & \xrightarrow{\quad H \quad} & u \in \mathbb{R}^n \\
G \downarrow & & \downarrow F \\
\dot{w} & \xrightarrow[\quad H \quad]{} & \dot{u}
\end{array}
$$

Center Manifold W_c ($\Re(\lambda) = 0$)
Homological equation:

$$F(H(w)) = (D_w H)G(w)$$

$L_\nu = (A - \langle \nu, \lambda \rangle I)$

Vectorfield Approximation

Observation: composition $A \circ F$ is close to the identity.
Theorem (Takens,Neimark): Suppose $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ is a diffeomorphism and $D\Phi(0)$ has all eigenvalues on the unit circle. Denote by S the semi-simple part of $D\Phi(0)$. Then there exists a diffeomorphism Ψ and a vectorfield X such that

$$\Psi \circ \Phi \circ \Psi^{-1} = \phi_X(t = 1) \circ S$$

in the sense of Taylor series.
Proof: Global Analysis of Dynamical Systems: Festschrift dedicated to Floris Takens for his 60th birthday. Eds. H.W Broer, B. Krauskopf G. Vegter, see Thm 4.6 p20.
Remark:

- Φ is the time-1 map of the flow of the vectorfield X.
- parameters can be included.

Reduced ODEs for codim 2 bifurcations

- Cusp $\dot{x} = \beta_1 + \beta_2 x + x^3$
- Bautin $\dot{x} = x(\beta_1 + \beta_2 x^2 + x^4)$
- Bogdanov-Takens $\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \end{pmatrix} \begin{pmatrix} x_2 \\ \beta_1 + \beta_2 x_1 + x_1^2 - x_1 x_2 \end{pmatrix}$
- Zero-Hopf $\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \end{pmatrix} \begin{pmatrix} \beta_1 + x_1^2 + s x_2^2) \\ x_2(\beta_2 + \theta x_1 + x_2^2 \end{pmatrix}$
- Double Hopf $\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \end{pmatrix} \begin{pmatrix} x_1(\beta_1 - x_1^2 - \theta x_2^2) \\ x_2(\beta_2 - \delta x_1^2 \pm x_2^2) \end{pmatrix}$

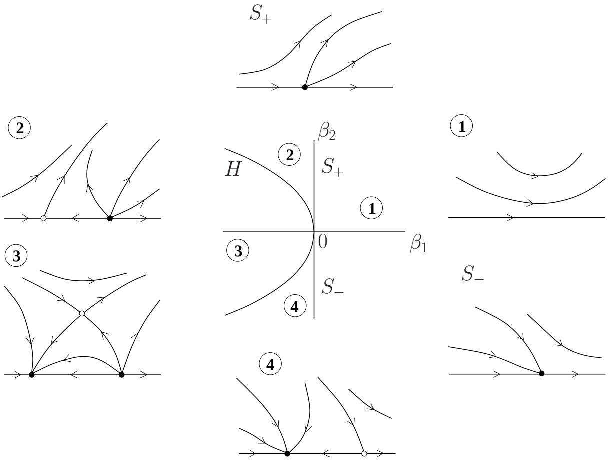
Fold-Hopf: Normal form

$$\begin{pmatrix} \dot{x} \\ \dot{z} \end{pmatrix} \begin{pmatrix} \beta_1 + x^2 + s|z|^2) \\ (\beta_2 + i\omega)z + (\theta + i\vartheta)xz + x^2 z \end{pmatrix}$$

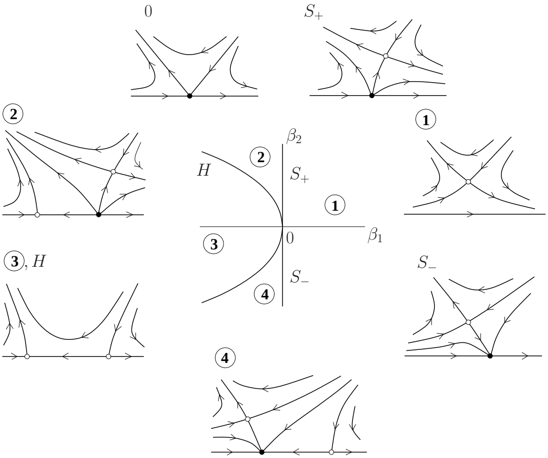
Introduce cylindrical coordinates $(x, z) = (x_1, x_2 e^{i\phi})$, scalings then give amplitude system $\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \end{pmatrix} \begin{pmatrix} \beta_1 + x_1^2 + s x_2^2) \\ x_2(\beta_2 + \theta x_1 + x_2^2 \end{pmatrix}$ Bifurcation curves:

- fold $\beta_1 = 0$
- Hopf $\beta_1 = -\left(\frac{\beta_2}{\theta}\right)^2$
- Torus If $s\theta < 0$, $\beta_2 = 0, \theta\beta_1 < 0$.
- Heteroclinic If $s < 0 < \theta$, $\beta_2 = \frac{\theta}{3\theta - 2}\beta_1$, $\beta_1 < 0$.

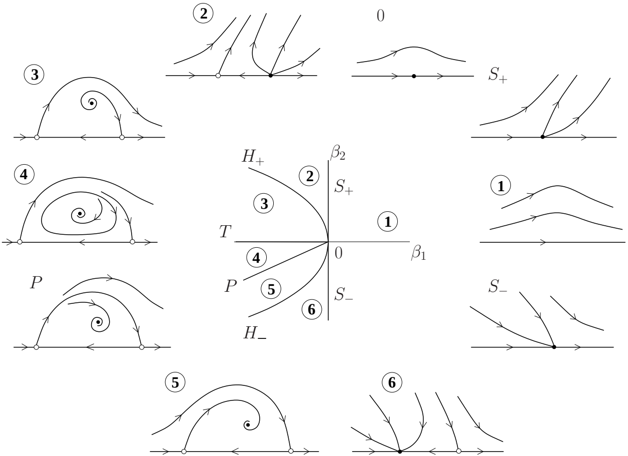
Fold-Hopf Unfolding $s = 1, \theta > 0$



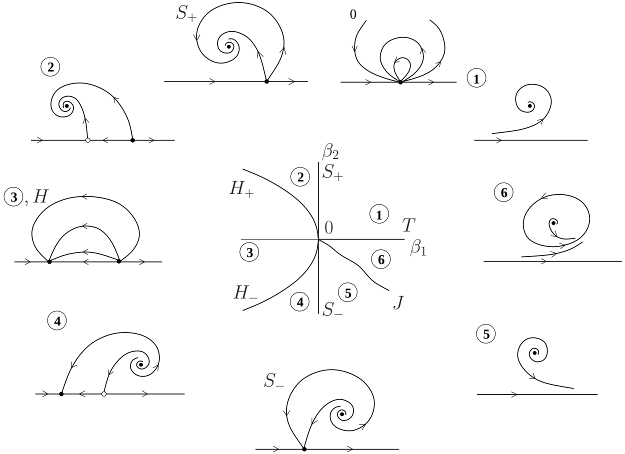
Fold-Hopf Unfolding $s = -1, \theta < 0$



Fold-Hopf Unfolding $s = -1, \theta > 0$



Fold-Hopf Unfolding $s = 1, \theta < 0$

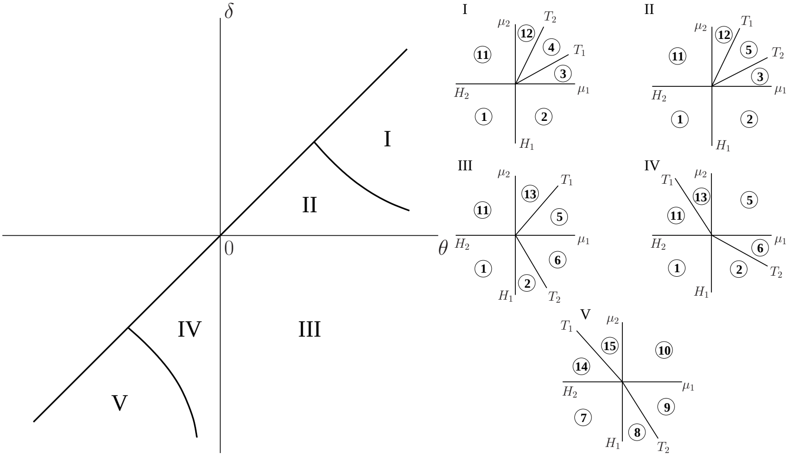


Double Hopf:Normal form

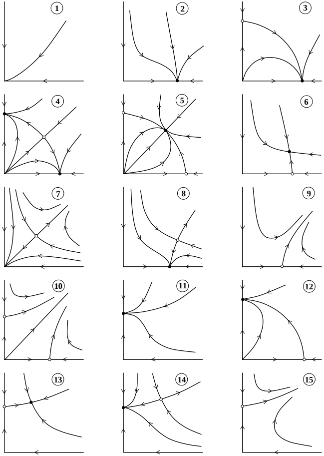
$$\begin{pmatrix} \dot{w_1} \\ \dot{w_2} \end{pmatrix} = \begin{pmatrix} (i\omega_1(\beta)+\beta_1)w_1 + f_{2100}w_1|w_1|^2 + f_{1011}w_1|w_2|^2 \\ (i\omega_2(\beta)+\beta_2)w_2 + g_{1110}w_2|w_1|^2 + g_{0021}w_2|w_2|^2 \end{pmatrix} + O(\|(w_1,w_2)\|^4). \quad (1)$$

There are always two curves of Neimark-Sacker bifurcations.

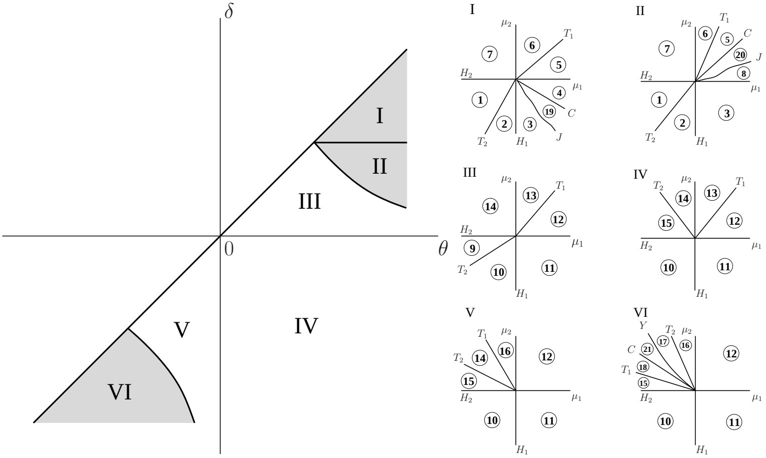
Simple case:parameter diagrams



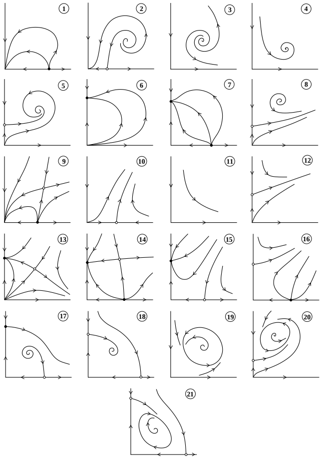
Simple case:phase portraits



Difficult case:parameter diagrams



Difficult case:phase portraits



Fold and Period-doubling

1. *Fold*: The fixed point has a simple eigenvalue $\lambda_1 = 1$ and no other eigenvalues on the unit circle, while the restriction of F to a one-dimensional center manifold at the critical parameter value has the form

$$\xi \mapsto \xi + \frac{1}{2} a \xi^2 + O(\xi^3), \tag{2}$$

where $a \neq 0$. When the parameter crosses the critical value, two fixed points coalesce and disappear. If $Av = F_x v$ and $B(u,v) = F_{xx}[u,v]$ are evaluated at the critical fixed point, then

$$a = \langle q^*, B(q,q) \rangle, \tag{3}$$

where $Aq = q,\ A^T q^* = q^*$, and $\langle q^*, q \rangle = 1$.

2. *Flip*: The fixed point has a simple eigenvalue $\lambda_1 = -1$ and no other eigenvalues on the unit circle, while the restriction of (**??**) to a one-dimensional center manifold at the critical parameter value can be transformed to the normal form

$$\xi \mapsto -\xi + \frac{1}{6} b \xi^3 + O(\xi^4), \tag{4}$$

where $b \neq 0$. When the parameter crosses the critical value, a cycle of period

Neimark-Sacker

The fixed point has simple critical eigenvalues $\lambda_{1,2} = e^{\pm i\theta_0}$ and no other eigenvalues on the unit circle. Assume that

$$e^{iq\theta_0} - 1 \neq 0, \quad q = 1, 2, 3, 4 \text{ (no strong resonances).}$$

Then, the restriction of (**??**) to a two-dimensional center manifold at the critical parameter value can be transformed to the normal form

$$\eta \mapsto \eta e^{i\theta_0}\left(1 + \frac{1}{2}d|\eta|^2\right) + O(|\eta|^4), \tag{6}$$

where η is a complex variable and d is a complex number. Further assume that

$$c = \text{Re } d \neq 0.$$

Under the above assumptions, a unique *closed invariant curve* around the fixed point appears when the parameter crosses the critical value. One has the following expression for d:

$$d = \frac{1}{2}e^{-i\theta_0}\langle v^*, C(v,v,\bar{v}) + 2B(v,(I_n - A)^{-1}B(v,\bar{v})) + B(\bar{v},(e^{2i\theta_0}I_n - A)^{-1}B(v,v))\rangle, \tag{7}$$

where $Av = e^{i\theta_0}v$, $A^T v^* = e^{-i\theta_0}v^*$, and $\langle v^*, v\rangle = 1$.

List of local codim 2 bifurcations for maps

(1) $\mu_1 = 1, b = 0$ **(cusp)**
(2) $\mu_1 = -1, c = 0$ **(generalized flip)**
(3) $\mu_{1,2} = e^{\pm i\theta_0}, \mathrm{Re}\,[e^{-i\theta_0}c_1] = 0$ (Chenciner bifurcation)
(4) $\mu_1 = \mu_2 = 1$ (1:1 resonance)
(5) $\mu_1 = \mu_2 = -1$ **(1:2 resonance)**
(6) $\mu_{1,2} = e^{\pm i\theta_0}, \theta_0 = \frac{2\pi}{3}$ (1:3 resonance)
(7) $\mu_{1,2} = e^{\pm i\theta_0}, \theta_0 = \frac{\pi}{2}$ (1:4 resonance)
(8) $\mu_1 = 1, \mu_2 = -1$ **(fold-flip)**
(9) $\mu_1 = 1, \mu_{2,3} = e^{\pm i\theta_0}$ ("fold-Hopf for maps")
(10) $\mu_1 = -1, \mu_{2,3} = e^{\pm i\theta_0}$ ("flip-Hopf for maps")
(11) $\mu_{1,2} = e^{\pm i\theta_1}, \mu_{3,4} = e^{\pm i\theta_2}$ ("Hopf-Hopf for maps")

Cusp

The critical normal form is

$$w \mapsto G(w) = w + \left(\frac{1}{2}bw^2\right) + \frac{1}{6}cw^3 + \cdots$$

on the center manifold

$$H(w) = wh_1 + \frac{w^2}{2}h_2 + \frac{w^3}{6}h_3 + \cdots$$

The first three terms of the expansion are given by

$$\begin{array}{ll} w: & (A-I)h_1 = 0 \\ w^2: & (A-I)h_2 = bh_1 - B(h_1,h_1) \\ w^3: & (A-I)h_3 = ch_1 - C(h_1,h_1,h_1) - 3B(h_1,h_2) \end{array}$$

Cusp

So we first obtain the eigenvectors such that

$$Aq = q, A^T p = p, \langle p, q \rangle = 1,$$

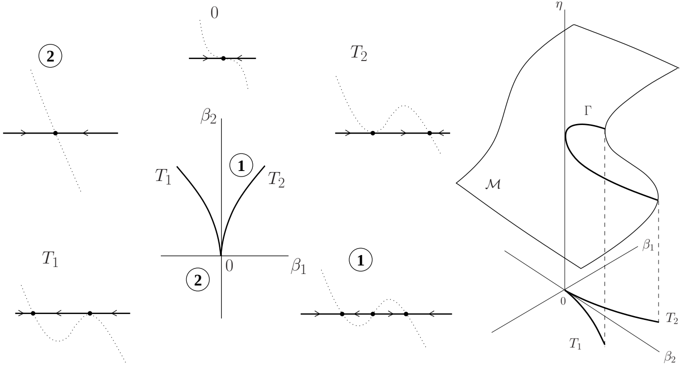
Then higher order terms give

$$\begin{gathered} b = \langle p, B(q,q) \rangle = 0, \\ h_2 = -(A - I_n)^{INV} B(q,q), \end{gathered}$$

and finally the critical normal form coefficient

$$c = \langle p, C(q,q,q) + 3B(q,h_2) \rangle$$

Cusp:Unfolding



Degenerate Period-Doubling

$$Aq = -q, A^T p = -p, \langle p, q \rangle = 1, \quad c = 0$$

The critical normal form

$$w \mapsto G(w) = -w + \left(\frac{1}{6} c w^3\right) + \frac{1}{120} g w^5 + \cdots$$

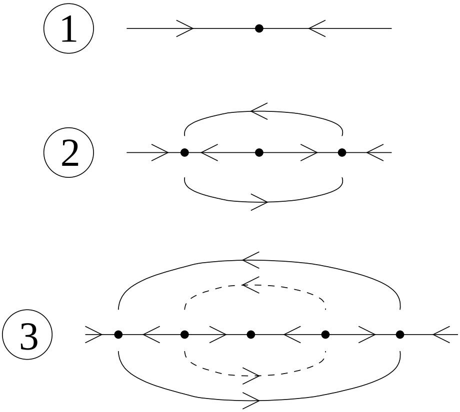
$$H(w) = wq + \frac{w^2}{2} h_2 + \frac{w^3}{6} h_3 + \frac{w^4}{24} h_4 + \frac{w^5}{120} h_5 + \cdots$$

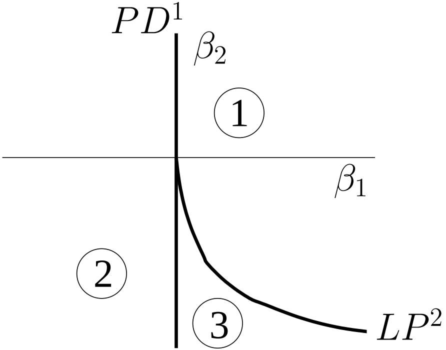
where

$$\begin{array}{rcl} h_2 = & -(A - I_n)^{-1} & B(q,q) \\ h_3 = & -(A + I_n)^{INV} & [C(q,q,q) + 3B(q,h_2)] \\ h_4 = & -(A - I_n)^{-1} & [4B(q,h_3) + 3B(h_2,h_2) + \\ & & 6C(q,q,h_2) + D(q,q,q,q)] \end{array}$$

$$\begin{array}{rcl} g & = & \langle p, 5B(q,h_4) + 10B(h_2,h_3) + \\ & & 10C(q,q,h_3) + 15C(q,h_2,h_2) + \\ & & 10D(q,q,q,h_2) + E(q,q,q,q,q) \rangle \end{array}$$

Degenerate Period-Doubling:Unfolding





1:2 Resonance: normalization

The normal form G (including parameters) is:

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} -x+y \\ \beta_1 + (-1+\beta_2)y + c_1 x^3 + d_1 x^2 y \end{pmatrix} + \cdots$$

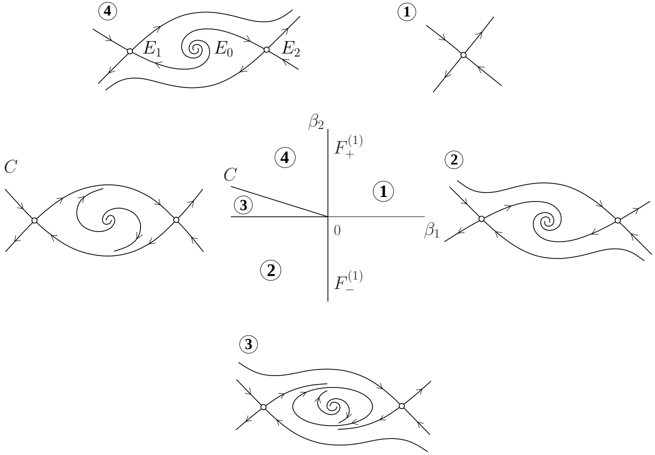
If $c_1 < 0$ a codim 1 branch of Neimark-Sacker bifurcation of double period emanates.

Asymptotic expression of the new branch

$$H^2 : (x^2, y, \beta_1, \beta_2) = \left(-\frac{1}{c_1}, 0, 1, \left(2 + \frac{d_1}{c_2}\right)\right)\varepsilon$$

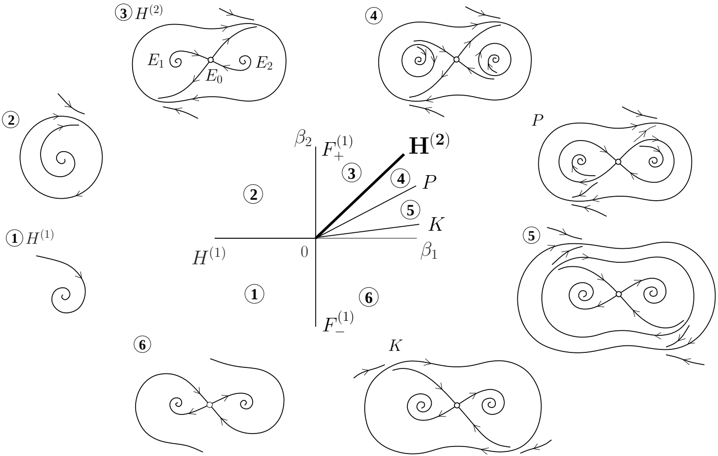
Unfolding $c_1 > 0$

No new local branches



Unfolding $c_1 < 0$:

New codim 1 branch H^2 (local bifurcation)



1:2 Resonance: normalization

Introduce (generalized) eigenvectors:

$$
\begin{gathered}
Aq_0 = -q_0, Aq_1 = -q_1 + q_0, \\
A^T p_0 = -p_0, A^T p_1 = -p_1 + p_0, \\
\langle p_0, q_1 \rangle = \langle p_1, q_0 \rangle = 1, \quad \langle p_0, q_0 \rangle = \langle p_1, q_1 \rangle = 0.
\end{gathered}
$$

Collecting the quadratic terms we get

$$
\begin{aligned}
(A - I_n)h_{20} &= -B(q_0, q_0) \\
(A - I_n)h_{11} &= -B(q_0, q_1) - h_{20} \\
(A - I_n)h_{02} &= -B(q_1, q_1) - 2h_{11} + h_{20}
\end{aligned}
$$

These are all solvable, since $\lambda = 1$ is not an eigenvalue of A.

1:2 Resonance: Cubic normalization

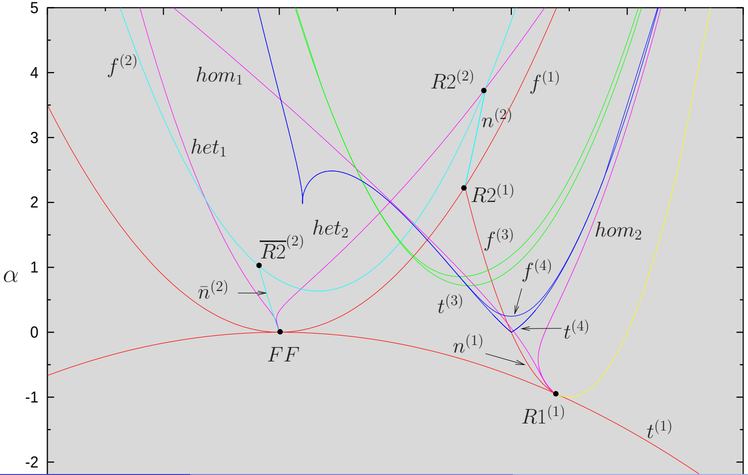
We only need cubic terms to find the coefficients.

$$
\begin{aligned}
c_1 = & \quad \langle p_0, C(q_0, q_0, q_0) + 3B(q_0, h_{20}) \rangle, \\
d_1 = & \quad \langle p_0, C(q_0, q_0, q_1) + B(q_1, h_{20}) + 2B(q_0, h_{11}) \rangle \\
& \quad + \langle p_1, C(q_0, q_0, q_0) + 3B(q_0, h_{20}) \rangle
\end{aligned}
$$

Non-degenerate if $c_1 \neq 0$ and $d_1 + c_1 \neq 0$.

Example I: GHM

$$\left\{ \begin{array}{c} x \\ y \end{array} \right) = \left\{ \begin{array}{c} y \\ a - b * x - y * y + r * x * y \end{array} \right)$$



Example II: Adaptive control

Golden&Ydstie(1988):

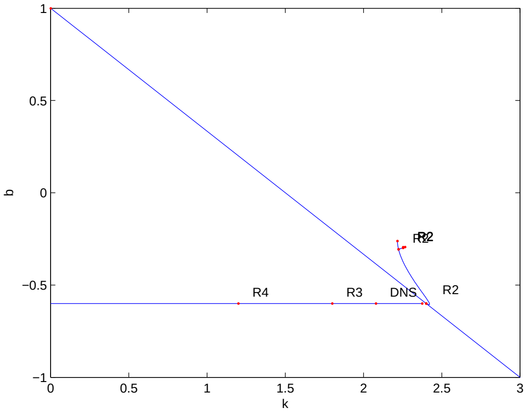
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} y \\ bx + k + yz \\ z - \frac{ky}{c+y^2}(bx + k + yz - 1) \end{pmatrix}$$

Unique fixed point

$$x = y = 1, z = 1 - b - k.$$

Loses stability by Period-Doubling or Neimark-Sacker bifurcation.

Example II: Bifurcation Diagram



c=.5

Fold-flip

The hypernormal form is:

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x + \frac{1}{2}a_0 x^2 + \frac{1}{2}b_0 y^2 + \frac{1}{6}c_0 x^3 + \frac{1}{2}d_0 x y^2 \\ -y + xy \end{pmatrix} + \cdots$$

Nondegeneracy conditions:

$$a_0 \neq 0, b_0 \neq 0$$

and

$$b_0 c_0 - a_0^2 b_0 - 3a_0 b_0 - a_0 d_0 \neq 0.$$

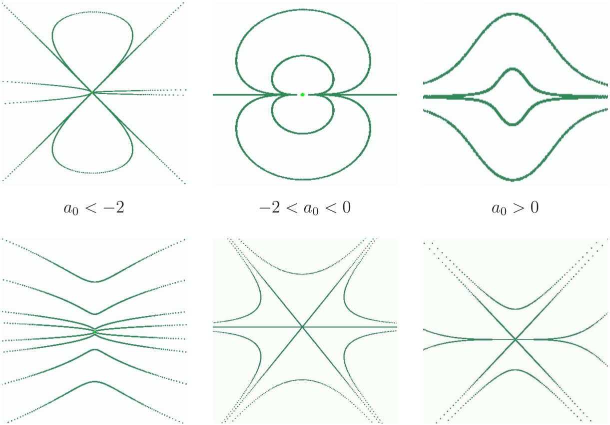
Approximating vectorfield

$$X(x,\mu) = \begin{pmatrix} \mu_1 + \left(-\frac{1}{2}a_0\mu_1 + \mu_2\right)x_1 + \frac{1}{2}a_0 x_1^2 + \frac{1}{2}b_0 x_2^2 + d_1 x_1^3 + d_2 x_1 x_2^2 \\ \frac{1}{2}\mu_1 x_2 - x_1 x_2 + d_3 x_1 x_2^2 + d_4 x_2^3 \end{pmatrix} \tag{8}$$

with

$$d_1 = \frac{1}{6}\left(c_0 - \frac{3}{2}a_0^2\right),\ d_2 = \frac{1}{2}\left(d_0 + \frac{1}{2}b_0(2-a_0)\right),\ d_3 = \frac{1}{4}(a_0 - 2),\ d_4 = \frac{1}{4}b_0.$$

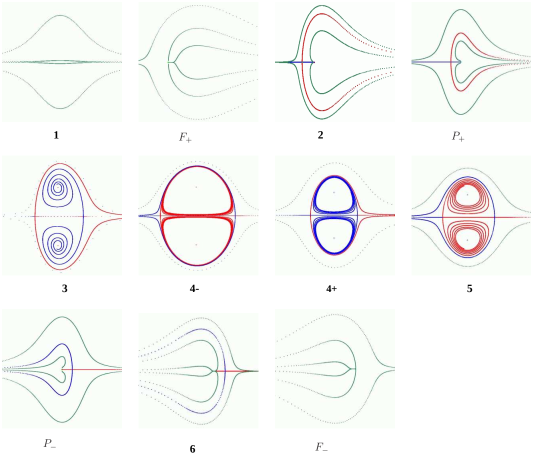
Fold-Flip: Critical Phase portraits



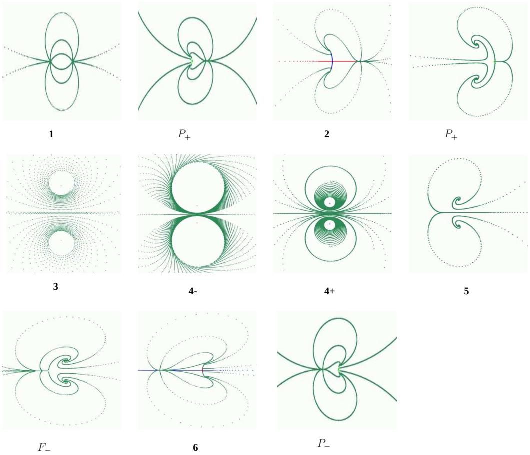
$b_0 > 0$: $a_0 < -2$, $-2 < a_0 < 0$, $a_0 > 0$

$b_0 < 0$: $a_0 < -2$, $-2 < a_0 < 0$, $a_0 > 0$

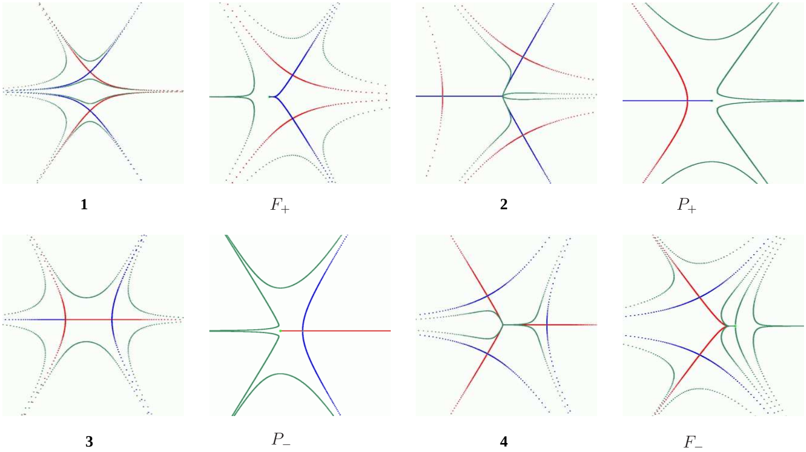
Fold-Flip: Case $a_0, b_0 > 0$



Fold-Flip: Case $a_0 < 0 < b_0, 0$



Fold-Flip: Case $a_0 > 0 > b_0$



Fold-Flip: Case $a_0, b_0 < 0$