Dynamics of Pattern Formation in Lateral-Inhibition Type Neural Fields

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2 May 2012

Motivation

- Shun-ichi Amari, Dynamics of Pattern Formation in Lateral-Inhibition Type Neural Fields
- Biological Cybernetics 27, 1977
- Lots of citations
- Nice results

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Neurofield equation

- Continuous model, 1D
- Average membrane potential: $u(x,t)$

$$\tau \frac{\partial u(x,t)}{\partial t} = -u + \int_{-\infty}^{\infty} w(x,y) Z\left(y, t - \frac{|x-y|}{c}\right) \mathrm{d}y + h + s(x,t)$$

Neurofield equation

- Continuous model, 1D
- Average membrane potential: $u(x,t)$
- Connectivity: $w(x,y)$
- Pulse emission rate: $Z(x,t)$

$$\tau \frac{\partial u(x,t)}{\partial t} = -u + \int_{-\infty}^{\infty} w(x,y) Z\left(y, t - \frac{|x-y|}{c}\right) \mathrm{d}y + h + s(x,t)$$

Neurofield equation

- Continuous model, 1D
- Average membrane potential: $u(x,t)$
- Connectivity: $w(x,y)$
- Pulse emission rate: $Z(x,t)$
- Stimulus: $\bar{s} + s(x,t)$
- Equilibrium state $h = \bar{s} - r$, where r resting potential

$$\tau \frac{\partial u(x,t)}{\partial t} = -u + \int_{-\infty}^{\infty} w(x,y) Z\left(y, t - \frac{|x-y|}{c}\right) \mathrm{d}y + h + s(x,t)$$

Assumptions

$$\tau \frac{\partial u(x,t)}{\partial t} = -u + \int_{-\infty}^{\infty} w(x,y) Z\left(y, t - \frac{|x-y|}{c}\right) \mathrm{d}y + h + s(x,t)$$

- No time delay

Assumptions

$$\tau \frac{\partial u(x,t)}{\partial t} = -u + \int_{-\infty}^{\infty} w(x,y) Z\left(y,t\right) \mathrm{d}y + h + s(x,t)$$

- No time delay
- Pulse emission:

$$Z(x,t) = f[u(x,t)] = \begin{cases} 0, & u \leq 0 \\ 1, & u > 0 \end{cases}$$

Assumptions

$$\tau \frac{\partial u(x,t)}{\partial t} = -u + \int_{-\infty}^{\infty} w(x,y) f[u(y,t)] \mathrm{d}y + h + s(x,t)$$

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- Homogeneous field: $w(x,y) = w(x-y)$

Assumptions

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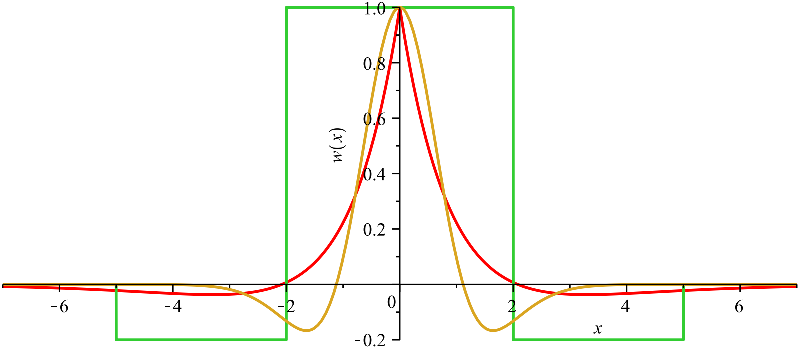
- Homogeneous field: $w(x,y) = w(x-y)$

About w

- w symmetric
- Excitatory connections nearby
- Inhibitory connections larger distance
- Integrable



Content

Conditions

$$\tau\frac{\partial u(x,t)}{\partial t} = -u + \int_{-\infty}^{\infty} w(x-y)f[u(y,t)]\mathrm{d}y + h + s(x,t)$$

- No inhomogeneous input: $s(x,t) = 0$
- $\partial u(x,t)/\partial t = 0$

$$u = \int_{-\infty}^{\infty} w(x-y)f[u(y,t)]\mathrm{d}y + h$$

Excited region

Definition (Excited region)

The excited region $R[u]$ of an equilibrium solution u is defined as:

$$R[u] := \{x | u(x) > 0\} = \{x | f(x) = 1\} .$$

$$u = \int_{-\infty}^{\infty} w(x-y) f[u(y,t)] \mathrm{d}y + h = \int_{R[u]} w(x-y) \mathrm{d}y + h$$

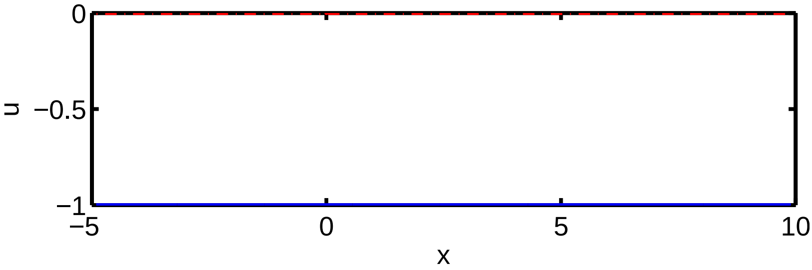
ϕ-solution

Definition (ϕ-solution)

A solution is called ϕ-solution if:

$$R[u] = \emptyset,$$

so $u(x) \leq 0$ for all x.



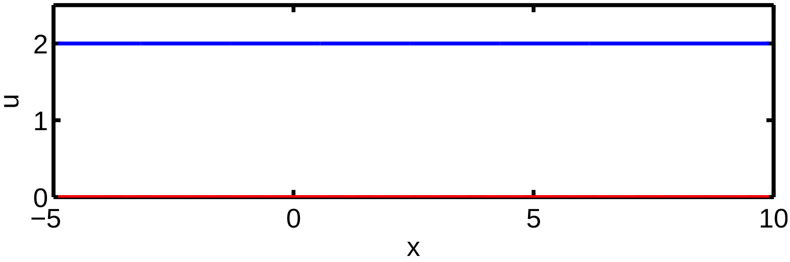
∞-solution

Definition (∞-solution)

A solution is called ∞-solution if:

$$R[u] = (-\infty, \infty),$$

so $u(x) > 0$ *for all* x



a-solution

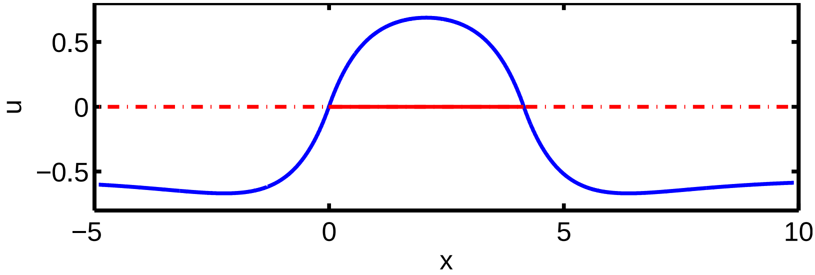
Definition (a-solution)

A solution is called an a-solution if $a_2 - a_1 = a$ *and*

$$R[u] = (a_1, a_2).$$

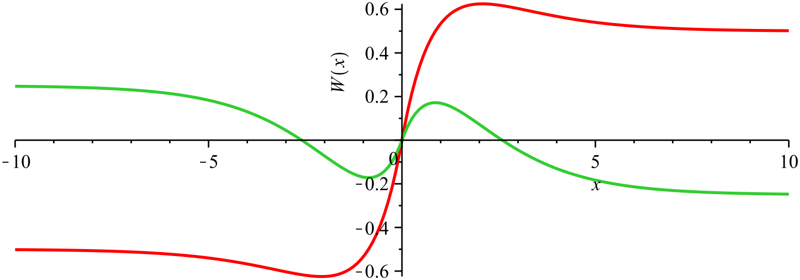
Remark

WLOG assume $R[u] = [0, a]$



W

- $W(x) := \int_0^x w(y)\mathrm{d}y$
- $W(x) = -W(-x)$
- $W_m := \max_{x>0} W(x)$
- $W_\infty := \lim_{x\to\infty} W(x)$



Existence of equilibrium types:

Theorem

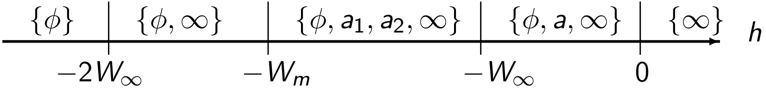
In absence of input:

1. *There exist a* ϕ*-solution* $\iff h < 0$
2. *There exist a* ∞*-solution* $\iff 2W_\infty > -h$
3. *There exist an* a*-solution* $\iff h < 0$ *and* $a > 0$ *satisfies:*

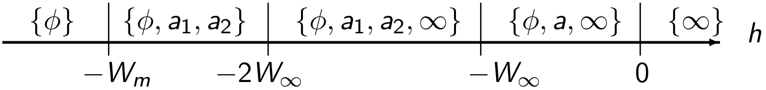
$$W(a) + h = 0$$

Overview equilibria

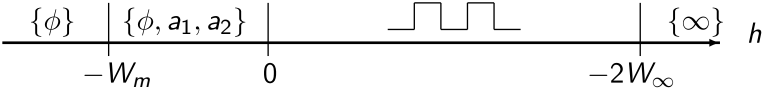
- Case 1a: $2W_\infty > W_m$



- Case 1b: $W_m > 2W_\infty > 0$



- Case 2: $W_\infty < 0$



Boundary movement

- Assume $R[u(x,t)] = (x_1(t), x_2(t))$
- $c_1 := \dfrac{\partial u(x_1,t)}{\partial x} > 0,\ c_2 := -\dfrac{\partial u(x_2,t)}{\partial x} > 0$
- $\dfrac{\mathrm{d}x_1}{\mathrm{d}t} = \dfrac{-1}{\tau c_1}[W(x_2 - x_1) + h],\ \dfrac{\mathrm{d}x_2}{\mathrm{d}t} = \dfrac{1}{\tau c_2}[W(x_2 - x_1) + h]$
- Interval length $a(t) := x_2(t) - x_1(t)$
- $\dfrac{\mathrm{d}a}{\mathrm{d}t} = \dfrac{1}{\tau}\left(\dfrac{1}{c_1} + \dfrac{1}{c_2}\right)[W(a) + h]$

Stability $a(t)$

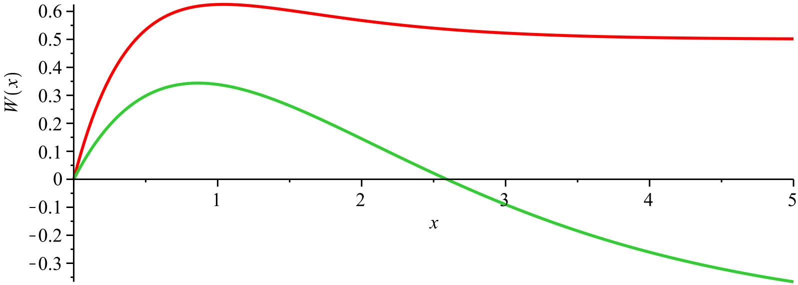
$$\frac{\mathrm{d}a}{\mathrm{d}t} = \frac{1}{\tau}\left(\frac{1}{c_1} + \frac{1}{c_2}\right)[W(a) + h]$$

- Equilibrium length: $W(a_*) + h = 0$
- $\frac{\mathrm{d}^2 a}{\mathrm{d}t^2} = \frac{\mathrm{d}W(a_*)}{\mathrm{d}a} = w(a_*)$
- Stable if $\frac{\mathrm{d}W(a_*)}{\mathrm{d}t} < 0$
- Unstable if $\frac{\mathrm{d}W(a_*)}{\mathrm{d}t} > 0$

Check stability

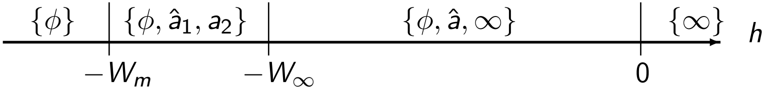
$$\frac{\mathrm{d}a}{\mathrm{d}t} = \frac{1}{\tau}\left(\frac{1}{c_1} + \frac{1}{c_2}\right)[W(a) + h]$$

- Easy check: look at graph W
- Solutions $0 < a_1 < a_2$ then a_1 unstable, a_2 stable
- Finite region grows to ∞-solution needs $W_\infty + h > 0$

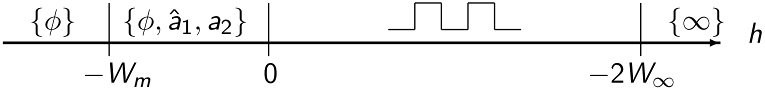


Overview stability equilibria

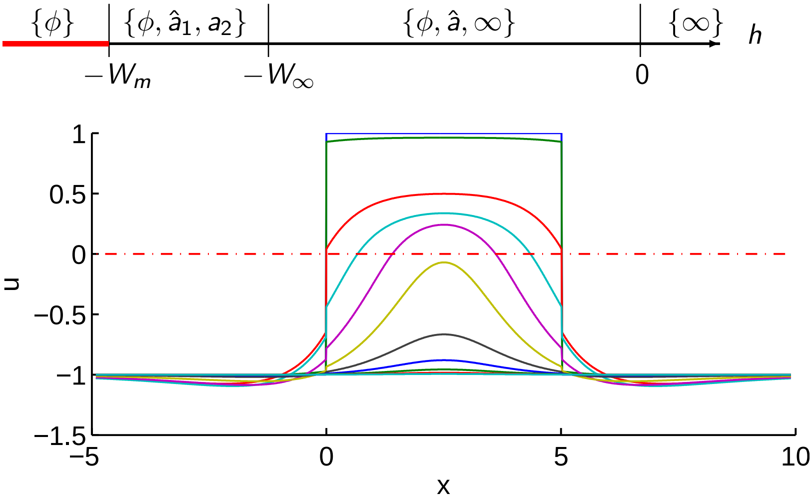
- Remove ∞-solutions with $W_\infty + h < 0$
- Case 1: $W_\infty > 0$



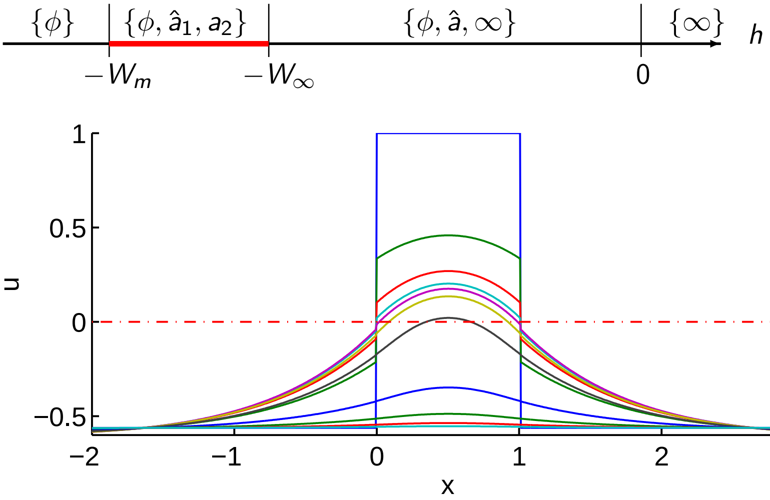
- Case 2: $W_\infty < 0$



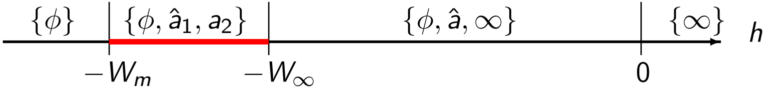
Case 1: $h < -W_m$

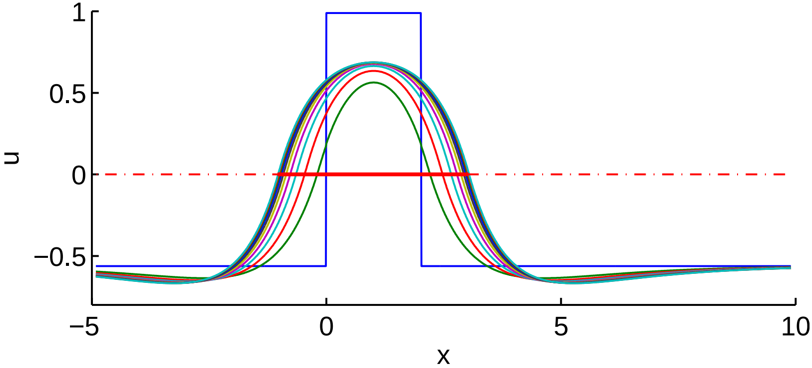


Case 1: $-W_m < h < -W_\infty$

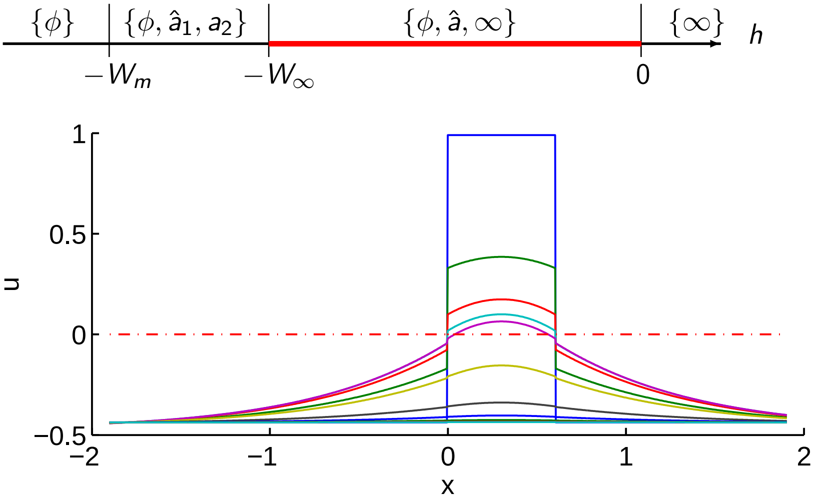


Case 1: $-W_m < h < -W_\infty$

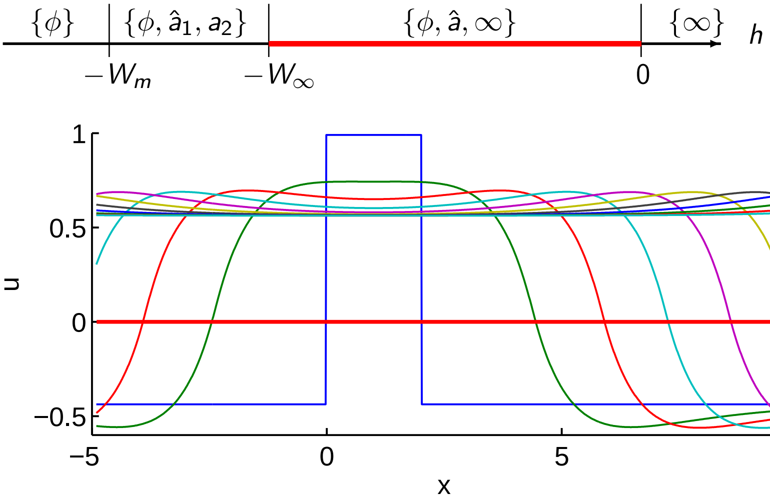




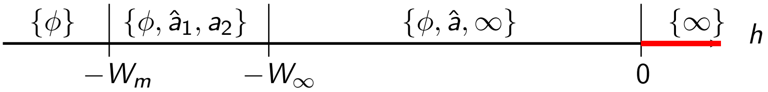
Case 1: $-W_\infty < h < 0$

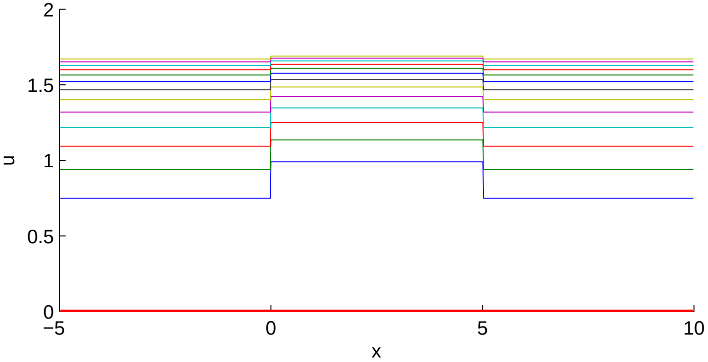


Case 1: $-W_\infty < h < 0$



Case 1: $h > 0$


$\{\phi\}$
$\{\phi, \hat{a}_1, a_2\}$
$\{\phi, \hat{a}, \infty\}$
$\{\infty\}$
h
$-W_m$
$-W_\infty$
0


2
1.5
1
0.5
0
u
−5
0
5
10
x

Content

Assumptions

- Space dependent input: $s(x) = s(x, t)$
- Small: $\epsilon s(x)$
- Assume a_0-solution, stable equilibrium
- Solution of form: $u(x, t) = u_0(x) + \epsilon u_1(x, t) + O(\epsilon^2)$

Boundary movement

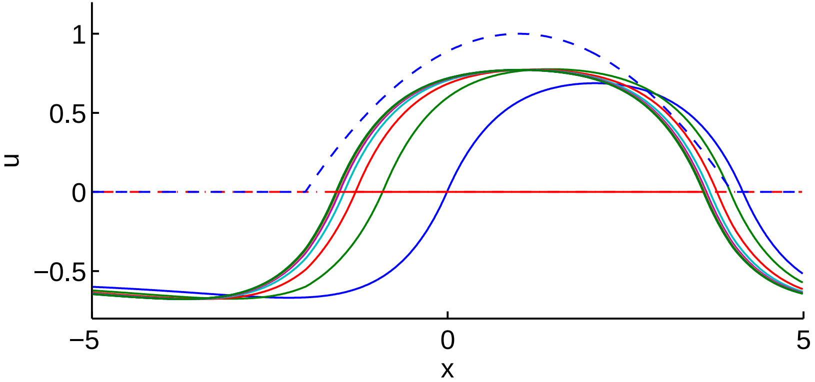
- $$\frac{\mathrm{d}x_1}{\mathrm{d}t} = \frac{-1}{\tau c_1}\left[W(x_2 - x_1) + h + \epsilon s(x_1)\right],$$
  $$\frac{\mathrm{d}x_2}{\mathrm{d}t} = \frac{1}{\tau c_1}\left[W(x_2 - x_1) + h + \epsilon s(x_2)\right]$$
- Symmetry: $c_1 = c + O(\epsilon)$, $c_2 = c + O(\epsilon)$
- Assume $a(t) = a_0 + \epsilon a_1 + O(\epsilon^2)$
- Center speed $\frac{1}{2}\frac{\mathrm{d}x_1 + x_2}{\mathrm{d}t} = \frac{\epsilon}{2\tau c}\left[s(x_2) - s(x_1)\right]$
- Length change:
  $$\frac{\mathrm{d}x_2 - x_1}{\mathrm{d}t} = \epsilon\frac{\mathrm{d}a_1}{\mathrm{d}t} = \frac{\epsilon}{\tau c}\left[2w(a_0)a_1 + s(x_1) + s(x_2)\right]$$

Interpertation

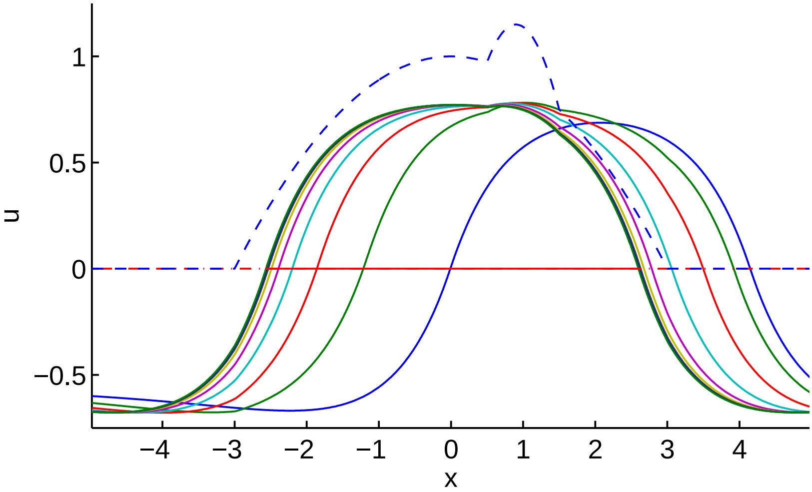
$$\frac{1}{2}\frac{\mathrm{d}x_1 + x_2}{\mathrm{d}t} = \frac{\epsilon}{2\tau c}\left[s(x_2) - s(x_1)\right]$$
$$\frac{\mathrm{d}a_1}{\mathrm{d}t} = \frac{\epsilon}{\tau c}\left[2w(a_0)a_1 + s(x_1) + s(x_2)\right]$$

- Move to higher stimulus level
- Equilibrium: $s(x_1) = s(x_2)$
- Equilibrium length: $a_0 - \epsilon\frac{s(x_1)+s(x_2)}{w(a_0)}$
- Stable equilibrium: $w(a_0) < 0$
- Stimulus peak width $> a_0$

Move to peak



Double peak



Content

Results

- Classefication equilibria
- Stability equilibria
- 5 types of behavior
- Behavior to small stimulus
- Nummerical examples

Questions

Questions?