Deterministic spatial epidemics and the asymptotic speed of propagation

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Overview

- Nonspatial spread
- Spatial spread
- Travelling waves
- Asymptotic speed of propagation

Nonspatial spread

$S(t)$	density of susceptibles at time t
$I(t)$	densitiy of infectives at time t
$i(t,\tau)$	density of infectives infected at time t with infection age τ

$$I(t) = \int_0^\infty i(t,\tau)d\tau$$

$$\frac{dS}{dt}(t) = -S(t)\int_0^\infty i(t,\tau)A(\tau)d\tau$$

$$i(t,\tau) = i(t-\tau, 0)$$

$$i(t,0) = -\frac{dS}{dt}(t)$$

Initial value problem

$$
\begin{aligned}
S(0) &= S_0 \\
i(0,\tau) &= i_0(\tau), \quad 0 \le \tau \le \infty
\end{aligned}
$$

$$
\begin{aligned}
\frac{dS}{dt}(t) &= -S(t)\left(\int_0^t i(t-\tau,0)A(\tau)d\tau + \int_0^\infty i_0(\tau)A(t+\tau)d\tau\right) \\
&= S(t)\left(\int_0^t \frac{dS}{dt}(t-\tau)A(\tau)d\tau - \int_0^\infty i_0(\tau)A(t+\tau)d\tau\right)
\end{aligned}
$$

Initial value problem

$\Rightarrow$

$$u(t) = S_0 \int_0^t g(u(t-\tau))A(\tau)d\tau + f(t)$$

with

$$u(t) = -\ln \frac{S(t)}{S_0}$$
$$g(y) = 1 - e^{-y}$$
$$f(t) = \int_0^t \int_0^\infty i_0(\tau)A(s+\tau)d\tau ds$$

Spatial spread

$x \in \Omega \subset \mathbb{R}^n$ closed.

$S(t,x)$ density of susceptibles at time t and position x

$I(t,x)$ density of infectives at time t and position x

$i(t,\tau,x)$ density of infectives with infection age τ at time t and position x

$$I(t,x) = \int_0^\infty i(t,\tau,x)d\tau.$$

Spatial spread

Consider $B(t,x)$ the rate at which susceptibles at time t and position x become infectious.

$A(\tau,x,\xi)$: infectivity at x due to one infective with age τ at position ξ. Then

$$B(t,x) = \int_0^\infty \int_\Omega i(t,\tau,\xi)A(\tau,x,\xi)d\xi d\tau$$

Spatial spread

$$\frac{\partial S}{\partial t}(t,x) = -S(t,x)\int_0^\infty \int_\Omega i(t,\tau,\xi)A(\tau,x,\xi)d\xi d\tau$$
$$i(t,0,x) = -\frac{\partial S}{\partial t}(t,x)$$
$$i(t,\tau,x) = i(t-\tau,0,x)$$

$\Rightarrow$

$$\frac{\partial S}{\partial t}(t,x) = S(t,x)\left(\int_0^\infty \int_\Omega \frac{\partial S}{\partial t}(t-\tau,\xi)A(\tau,x,\xi)d\xi d\tau\right)$$

Initial value problem

$$i(0,\tau,x) = i_0(\tau,x)$$
$$S(0,x) = S_0(x)$$

$$\frac{\partial S}{\partial t}(t,x) = S(t,x)\left(\int_0^t \int_\Omega \frac{\partial S}{\partial t}(t-\tau,\xi)A(\tau,x,\xi)d\xi d\tau - h(t,x)\right)$$

with

$$h(t,x) = \int_0^\infty \int_\Omega i_0(\tau,\xi)A(t+\tau,x,\xi)d\xi d\tau$$

Integral equation

Integrate the initial value problem w.r.t. $t \Rightarrow$

$$u(t,x) = \int_0^t \int_\Omega g(u(t-\tau,\xi))S_0(\xi)A(\tau,x,\xi)d\xi d\tau + f(t,x)$$

where

$$u(t,x) = -\ln \frac{S(t,x)}{S_0(x)}$$

$$g(y) = 1 - e^{-y}$$

$$f(t,x) = \int_0^t h(\tau,x)d\tau$$

Simplify

Simplifying assumptions

- $\Omega = \mathbb{R}^n$
- $A(\tau, x, \xi) = H(\tau)V(x - \xi)$
- $S_0(\xi) = S_0$

Study

$$u(t,x) = \int_0^t \int_{\mathbb{R}^n} g(u(t-\tau,\xi))H(\tau)V(x-\xi)d\xi d\tau + f(t,x),$$

$t \geq 0$, $x \in \mathbb{R}^n$

Traveling wave solutions

$$u(t,x) = w(x \cdot \nu + ct)$$

Study

$$u(t,x) = \int_0^\infty \int_{\mathbb{R}^n} g(u(t-\tau,\xi))H(\tau)V(x-\xi)d\xi d\tau$$

w has to satisfy

$$w(y) = \int_{-\infty}^\infty g(w(\eta))\tilde{V}_c(y-\eta)d\eta,$$

with

$$\tilde{V}_c(y-\eta) = \int_0^\infty \int_{\mathbb{R}^{n-1}} V(y-\eta-c\tau, x_2, \ldots, x_n)H(\tau)d\tau dx_2 \cdots dx_n$$

Minimal wave speed c_0

Characteristic equation

$$L_c(\lambda) = 1$$

Define

$$c_0 = \inf\{c > 0\colon L_c(\lambda) = 1 \text{ for some } \lambda\}$$

$c > c_0$	Existence of waves
	Uniqueness of waves (modulo translation)
$0 < c < c_0$	Nonexistence of waves

Asymptotic speed of propagation

Intuitively

- Run fast enough: leave epidemic behind
- Too slow: surrounded by epidemic

Asymptotic speed of propagation c^*

- For any $c > c^*$:

$$\lim_{t\to\infty} \sup\{u(t,x) \colon |x| > ct\} = 0$$

- For any $0 < c < c^*$:

$$\liminf_{t\to\infty} \min\{u(t,x) \colon |x| \leq ct\} \geq p$$

c_0 is the asymptotic speed of propagation

Part 1: For any $c > c_0$: $\limsup_{t\to\infty}\{u(t,x) \colon |x| > ct\} = 0$

$$u(t,x) = \int_0^t \int_{\mathbb{R}^n} g(u(t-\tau,\xi))H(\tau)V(x-\xi)d\xi d\tau + f(t,x),$$

$t \geq 0$, $x \in \mathbb{R}^n$.

- Nondecreasing sequence u_n with limit u
- Estimate of u_n in terms of $L_c(\lambda)$
- Estimate of u in terms of $L_c(\lambda)$.

c_0 is the asymptotic speed of propagation

Part 2: For any $0 < c < c_0$: $\liminf_{t\to\infty} \min\{u(t,x) \colon |x| \leq ct\} \geq p$

- Construction of a subsolution
- Comparison principle

Epidemic model

$$u(t,x) = S_0 \int_0^t \int_\Omega \tilde{g}(u(t-\tau,\xi))H(\tau)V(x-\xi)d\xi d\tau + f(t,x)$$

Recall

$$u(t,x) = -\ln \frac{S(t,x)}{S_0}$$

Take $g(y) = \alpha\tilde{g}(y)$ with

$$\alpha = S_0 \int_0^\infty H(\tau)d\tau \int_{\mathbb{R}^n} V(x)dx$$

Threshold behaviour

- $\alpha < 1$
- $\alpha > 1$

Multiple types

References

- O. Diekmann, “Thresholds and travelling waves for the geographical spread of infection”, *J. Math. Biol.* 1978; 6:109-130.
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- L. Rass, J. Radcliffe, *Spatial deterministic epidemics*, American Mathematical Society, 2003.